Stochastic Calculus for Finance I Solutions

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1 Chapter 1 Exercises

Exercise 1.1

Given:
1. $X_o = 0$
2. $X_1 = \Delta_o S_1 + (1+r)(X_o - \Delta_o S_o)$
3. $0 < d < 1 + r < u$
4. $u = \frac{S_1(H)}{S_o}$
5. $d = \frac{S_1(T)}{S_o}$
6. Assume $\Delta_o! = 0$ (no position)

Starting with 2.

$$X_1 = \Delta_o S_1 + (1+r)(X_o - \Delta_o S_o)$$

Using 1.

$$= \Delta_o S_1 + (1+r)(-\Delta_o S_o)$$

$$= \Delta_o(S_1 - S_o(1+r))$$

We have
8. $X_1 = \Delta_o(S_1 - S_o(1+r))$
9. $S_1(H) = uS_o$
10. $S_1(H) = dS_o$

Combining (8.) and (9.), we get:

$$X_1 = \Delta_o(uS_o - S_o(1+r))$$

$$= \Delta S_o(u - (1+r))$$

Combining (8.) and (10.), we get:

$$X_1 = \Delta S_o(d - (1+r))$$

We have:
11. $X_1 = \Delta S_o(u - (1+r))$

12. $X_1 = \Delta S_o(d-(1+r))$

Note that both (11.) and (12.) occur with positive probability

Proof by cases:
Case 1: $\Delta_o < 0$ (short position)
Using (11.) and (3.): $X_1 < 0$ with positive probability
Using (12.) and (3.): $X_1 > 0$ with positive probability

Case 2: $\Delta_o > 0$ (long position)
Using (11.) and (3.): $X_1 > 0$ with positive probability
Using (12.) and (3.): $X_1 < 0$ with positive probability

Exercise 1.2

1. $X_1 = \Delta_o S_1 + \Gamma_o(S_1-5)^+ - \frac{5}{4}(4\Delta_o + 1.2\Gamma_o)$
2. $S_1(H) = 8$
3. $S_1(T) = 2$

Starting with (1.):

$$X_1 = \Delta_o S_1 + \Gamma_o(S_1-5)^+ - \frac{5}{4}(4\Delta_o + 1.2\Gamma_o)$$

$$= \Delta_o S_1 + \Gamma_o(S_1-5)^+ - 5\Delta_o - \frac{5}{4}1.2\Gamma_o$$

$$= \Delta_o(S_1-5) + \Gamma_o((S_1-5)^+ - 1.5)$$

Then, the two possible outcomes are:

$$H : X_1 = 3\Delta_o + 1.5\Gamma_o$$

$$T : X_1 = -3\Delta_o - 1.5\Gamma_o$$

In other words:

$$X_1(H) = -X_1(T)$$

If one outcome results in profit, the other outcome results in loss of the same magnitude. Since H and T both have positive probabilities of occurring, arbitrage is not possible.

Exercise 1.3

1. $V_o = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]$
2. $\tilde{p} = \frac{1+r-d}{u-d}$
3. $\tilde{q} = \frac{u-1-r}{u-d}$
4. $S_1(H) = S_o u$

5. $S_1(T) = S_o d$

Combining all given equations:

$$\begin{aligned} V_o &= \frac{1}{1+r}[\frac{1+r-d}{u-d}S_o u + \frac{u-1-r}{u-d}S_o d] \\ &= \frac{S_o}{1+r}[\frac{u(1+r-d)+d(u-1-r)}{u-d}] \\ &= \frac{S_o}{1+r}[\frac{u(1+r)-d(1+r)}{u-d}] \\ &= \frac{S_o}{1+r}[\frac{(u-d)(1+r)}{u-d}] \\ &= S_o \end{aligned}$$

Of course, if the option pays off exactly the stock price, then the cost of the option should also be exactly the stock price.

Exercise 1.4

The logic is the same as on page 13.
Use (1.2.14) and $S_{n+1} = S_n d$ to find $X_{n+1}(\omega_1\omega_2\ldots\omega_3 T)$:

$$X_{n+1}(\omega_1\omega_2\ldots\omega_3 T) = \Delta_n(\omega_1\omega_2\ldots\omega_3)dS_n(\omega_1\omega_2\ldots\omega_3)$$
$$+(1+r)(X_n(\omega_1\omega_2\ldots\omega_3) - \Delta_n(\omega_1\omega_2\ldots\omega_3)S_n(\omega_1\omega_2\ldots\omega_3))$$

Suppressing $\omega_1\omega_2\ldots\omega_3$, we get:

$$\begin{aligned} X_{n+1}(T) &= \Delta_n dS_n + (1+r)(X_n - \Delta_n S_n) \\ &= \Delta_n S_n(d-(1+r)) + (1+r)X_n \end{aligned}$$

Likewise, suppressing $\omega_1\omega_2\ldots\omega_3$, we get:

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}$$

Combining these two, we get:

$$\begin{aligned} X_{n+1}(T) &= (1+r)X_n + \frac{(V_{n+1}(H) - V_{n+1}(T))(d-(1+r))}{u-d} \\ &= (1+r)X_n + (V_{n+1}(H) - V_{n+1}(T))(-\tilde{p}) \\ &= (1+r)X_n + \tilde{p}V_{n+1}(T) - \tilde{p}V_{n+1}(H) \end{aligned}$$

Applying the induction hypothesis:

$$X_{n+1}(T) = (1+r)V_n + \tilde{p}V_{n+1}(T) - \tilde{p}V_{n+1}(H)$$

Using the definition of V_n (1.2.16):

$$\begin{aligned} X_{n+1}(T) &= \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) + \tilde{p}V_{n+1}(T) - \tilde{p}V_{n+1}(H) \\ &= V_{n+1}(T)(\tilde{p}+\tilde{q}) \\ &= V_{n+1}(T) \end{aligned}$$

Exercise 1.5

1. $S_o = 4$
2. $u = 2$
3. $d = \frac{1}{2}$ 4. $r = \frac{1}{4}$

At time 1, we have a portfolio of value 2.24. We invest:

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}$$

$$= \frac{3.2 - 2.4}{4(2^2 - 1)}$$

$$= \frac{1}{15}$$

Our portfolio value for time 2 (X_2) is computed as:

$$X_2 = (1.25) * (2.24 - \frac{1}{15}8) + \frac{1}{15}S_2$$

If the price goes up, we get:

$$X_2(HH) = 3.2$$

If the price goes down, we get:

$$X_2(HT) = 2.4$$

Assuming the price went down, we invest:

$$\Delta_2(HT) = \frac{V_3(HTH) - V_3(HTT)}{S_2(HTH) - S_2(HTT)}$$

$$= \frac{0 - 6}{8 - 2}$$

$$= -1$$

Our portfolio value for time 3 (X_3) is computed as:

$$X_3 = (1.25) * (2.4 + 4) - S_3$$

If the price goes up, we get:

$$X_2(HTH) = 6$$

If the price goes down, we get:

$$X_2(HTT) = 0$$

Exercise 1.6

The investor will need to replicate a position which negates the outcome of the long call position:

$$(1+r)(X_o - \Delta S_o) + \Delta_o S_1 = -(S_1 - K)^+$$

Then, the system of two linear equations, for both outcomes H and T, should be solved for X_o and Δ_o:

$$1.25(X_o - 4\Delta_o) + 8\Delta_o = -3$$

$$1.25(X_o - 4\Delta_o) + 2\Delta_o = 0$$

The solution is:

$$X_o = -1.2$$

$$\Delta_o = -0.5$$

This solution requires the short sale of 0.5 shares of stock to acquire an income of 2. Then, investing 2 - 1.2 = 0.8 of this into the money market will be necessary to negate the long call position. The other 1.2 can be invested into the money market to yield a final portfolio value of 1.5, which was the desired outcome.

Note: the solution above is consistent with discussions at the bottom of page 4 and the top of page 7. In essence, we sell back the 0.5 shares we bought and equalize our debt to the money market to return back to our original position (before we bought the call) of having 1.2 cash. Then, we simply invest the 1.2 into the money market, and the coin toss outcome becomes irrelevant.

Exercise 1.7

Similar to Exercises 1.6, the solution requires the trader to reverse to position of the original replicating portfolio: Sell short 0.1733 shares of the stock to acquire 0.6933 dollars. Then, investing 0.6933 - 1.376 = -0.6827 (borrowing) to the money market negates the long call position. With the borrowed cash plus the short sale cash, we get $(1+r)^3(1.376) = 2.6875$. This allows us to earn interst, regardless of the coin tosses.

Exercise 1.8

Exercise 1.9

2 Chapter 2 Exercises