Statistics and Finance: An Introduction - Solutions

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Chapter 2

Useful formulae:

1. $E[X] = \sum_{i=1}^{N} x_i P_i$
2. $E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$
3. $Var(X) = \sigma_X^2 = E[(X - E[X])^2]$
4. $E[x^T \boldsymbol{X}] = \omega^T E[\boldsymbol{X}]$
5. $Var[x^T \boldsymbol{X}] = \omega^T COV(\boldsymbol{X})\omega$
6. $Cov(X, Y) = \sigma_{XY} = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$
7. $K = \frac{E[(X-E[X])^4}{\sigma^4}$

Problem 1

Given E[X] = 1, E[Y] = 1, Var[X] = 2, Var[y] = 3, Cov[X, Y] = 1

(a)

$$\begin{aligned} E[0.1X + 0.9Y] =& 0.1E[X] + 0.9E[Y] \\ &= 0.1(1) + 0.9(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} Var[0.1X + 0.9Y] &= (0.1)^2 Var[X] + (2)(0.1)(0.9)Cov[X, Y] + (0.9)^2 Var[Y] \\ &= (0.1)^2(2) + (2)(0.1)(0.9)(1) + (0.9)^2(3) \\ &= 2.6282 \end{aligned}$$

(b)

$$\begin{aligned} Var[\omega X + (1-\omega)Y] &= \omega^2 Var[X] + (1-\omega)^2 Var[Y] + 2\omega(1-\omega)Cov[X, Y] \\ &= 2\omega^2 + 3(1 - 2\omega + \omega^2) + 2(\omega - \omega^2) \\ &= 3\omega^2 - 4\omega + 3 \end{aligned}$$

Now, differentiating $Var[\omega X + (1-\omega)Y]$ with respect to ω, we get:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\omega}(Var[\omega X + (1-\omega)Y]) &= \frac{\mathrm{d}}{\mathrm{d}\omega}(3\omega^2 - 4\omega + 3) \\ &= 6\omega - 4 \end{aligned}$$

Evaluating the critical point, we get:

$$6\omega - 4 = 0$$
$$\rightarrow \omega = \frac{2}{3}$$

This value denotes the weights $\omega, (1-\omega)$ that minimize the variation in this two-stock portfolio. This is important because variation is proportional to risk.

Problem 2

(a)

$$\begin{aligned}
Cov[X_1 + X_2, Y_1 + Y_2] &= E[(X_1 + X_2)(Y_1 + Y_2)] - E[X_1 + X_2]E[Y_1 + Y_2] \\
&= E[X_1Y_1 + X_1Y_2 + X_2Y_1 + X_2Y_2] - E[X_1 + X_2]E[Y_1 + Y_2] \\
&= E[X_1Y_1] + E[X_1Y_2] + E[X_2Y_1] + E[X_2Y_2] - (E[X_1] + E[X_2])(E[Y_1] + E[Y_2]) \\
&= (E[X_1Y_1] - E[X_1]E[Y_1]) + (E[X_1Y_2] - E[X_1]E[Y_2])... \\
&... + (E[X_2Y_1] - E[X_2]E[Y_1]) + (E[X_2Y_2] - E[X_2]E[Y_2]) \\
&= Cov(X_1, Y_1) + Cov(X_1, Y_2) + Cov(X_2, Y_1) + Cov(X_2, Y_2)
\end{aligned}$$

(b)

$$\begin{aligned}
Cov[\alpha_1 X + Y, \alpha_2 X + Z] &= E[(\alpha_1 X_1 + Y)(\alpha_2 X_2 + Z)] - E[\alpha_1 X_1 + Y]E[\alpha_2 X_2 + Z] \\
&= \alpha_1\alpha_2 Cov[X_1, X_2] + \alpha_1 Cov[X_1, Z] + \alpha_2 Cov(Y, X_2) + Cov(Y, Z) \\
&= \alpha_1\alpha_2 Cov[X_1, X_2]
\end{aligned}$$

(c)
The number of cross terms is proportional to the outer product.

$$Cov[\omega_1 X_1 + \ldots + \omega_n X_n, \sigma_1 Y_1 + \ldots + \sigma_m Y_m] = \sum_{i=1}^{n}\sum_{j=1}^{m} \omega_i \sigma_j Cov[X_i, Y_j]$$

Problem 3

We need to evaluate the critical point of $L(\theta)$ and solve for σ^2. Since the log function is increasing, we can equivalently evaluate the critical point of $\log L(\theta)$. For simplicity, let $\sigma^2 = \nu$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\nu}(\log L(\theta)) &= \frac{\mathrm{d}}{\mathrm{d}\nu}(-\frac{n}{2}[\log \nu + \log 2\pi] - \frac{1}{2\nu}\sum_{i=1}^{n}(Y_i - \mu)^2) = 0 \\
&= -\frac{n}{2\nu} + \frac{1}{2\nu^2}\sum_{i=1}^{n}(Y_i - \mu)^2 = 0 \\
&= \frac{1}{2\nu}(-n + \frac{1}{\nu}\sum_{i=1}^{n}(Y_i - \mu)^2) = 0 \\
&\rightarrow -n + \frac{1}{\nu}\sum_{i=1}^{n}(Y_i - \mu)^2 = 0 \\
&\rightarrow \nu = n^{-1}\sum_{i=1}^{n}(Y_i - \mu)^2
\end{aligned}$$

Problem 4

This system can be represented as:

$$\begin{bmatrix} 1 & E[X] \\ E[X] & E[X^2] \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} E[Y] \\ E[XY] \end{bmatrix}$$

Solving, we get:

$$\begin{aligned}
\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} &= \frac{1}{E[X^2] - E[X]^2} \begin{bmatrix} E[X^2] & -E[X] \\ -E[X] & 1 \end{bmatrix} \begin{bmatrix} E[Y] \\ E[XY] \end{bmatrix} \\
&= \frac{1}{\sigma_X^2} \begin{bmatrix} E[X^2] & -E[X] \\ -E[X] & 1 \end{bmatrix} \begin{bmatrix} E[Y] \\ E[XY] \end{bmatrix} \\
&= \frac{1}{\sigma_X^2} \begin{bmatrix} E[X^2]E[Y] - E[X]E[XY] \\ -E[X]E[Y] + E[XY] \end{bmatrix} \\
&= \frac{1}{\sigma_X^2} \begin{bmatrix} E[X^2]E[Y] - E[X]E[XY] \\ \sigma_{XY} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\rightarrow \beta_0 &= \frac{E[X^2]E[Y] - E[X]E[XY]}{\sigma_X^2} \\
&= \frac{E[X^2]E[Y] + E[Y] - E[Y] - E[X]E[XY]}{\sigma_X^2} \\
&= \frac{E[X^2]E[Y] + E[Y] - E[Y] - E[X]E[XY]}{\sigma_X^2}
\end{aligned}$$

$$\rightarrow \beta_1 = \frac{\sigma_{XY}}{\sigma_X^2}$$

Alternatively, using substitution, we have:

$$\beta_0 = E[Y] - \beta_1 E[X]$$
$$(E[Y] - \beta_1 E[X])E[X] + \beta_1 E[X^2] = E[XY]$$
$$\beta_1(E[X^2] - E[X]^2) = E[XY] - E[X]E[Y]$$
$$\beta_1 = \frac{\sigma_{XY}}{\sigma_X^2}$$
$$\beta_0 = E[Y] - \frac{\sigma_{XY}}{\sigma_X^2} E[X]$$

Problem 5

$$\begin{aligned}
E[\omega^T \boldsymbol{X}] &= E[\omega_1 X_1 + \ldots + \omega_N X_N] \\
&= E[\omega_1 X_1] + \ldots + E[\omega_N X_N] \\
&= \omega_1 E[X_1] + \ldots + \omega_N E[X_N] \\
&= \omega_T E[\boldsymbol{X}]
\end{aligned}$$

Note:

$$\begin{aligned}
Var[\omega_1 X_1 + \omega_2 X_2] &= \omega_1^2 Var[X_1] + 2\omega_1\omega_2 Cov[X_1, X_2] + \omega_2^2 Var[X_2] \\
&= \omega_1^2 Cov[X_1, X_1] + 2\omega_1\omega_2 Cov[X_1, X_2] + \omega_2^2 Voc[X_2, X_2]
\end{aligned}$$

$$
\begin{aligned}
Var[\omega^T \boldsymbol{X}] &= \sum_{i=1}^{N}[\omega_i\omega_1 Cov(X_i, X_1) + \ldots + \omega_i\omega_N Cov(X_i, X_N)] \\
&= \omega_1\omega_1 Cov(X_1, X_1) + \ldots + \omega_1\omega_N Cov(X_1, X_N) + \ldots + \omega_N\omega_1 Cov(X_N, X_1) + \ldots + \omega_N\omega_N Cov(X_N, X_N) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i\omega_j Cov(X_i, X_j) \\
&= \sum_{i=1}^{N}[\omega_i\omega_1 Cov(X_i, X_1) + \ldots + \omega_i\omega_N Cov(X_i, X_N)] \\
&= \omega_1\omega_1 Cov(X_1, X_1) + \ldots + \omega_1\omega_N Cov(X_1, X_N) + \ldots + \omega_N\omega_1 Cov(X_N, X_1) + \ldots + \omega_N\omega_N Cov(X_N, X_N) \\
&= \omega_1^2 var(X_1) + \ldots + \omega_1\omega_N Cov(X_1, X_N) + \ldots + \omega_N\omega_1 Cov(X_N, X_1) + \ldots + \omega_N^2 Var(X_N) \\
&= \omega^T COV(\boldsymbol{X})\omega
\end{aligned}
$$

Problem 6

First statement:

$$
\begin{aligned}
\log L(\hat{Y}, \sigma_{ML}^2) &= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma_{ML}^2) - \frac{n}{2\sum_{i=1}^{n}(Y_i - \hat{Y})^2}\sum_{i=1}^{n}(Y_i - \hat{Y})^2 \\
&= \frac{n}{2}(\log(2\pi) + \log(\sigma_{ML}^2) + 1)
\end{aligned}
$$

Second statement:
In problem 3, we showed:

$$\sigma_{\mu,ML}^2 = n^{-1}\sum_{i=1}^{n}(Y_i - \mu)^2$$

Then:

$$\sigma_{0,ML}^2 = n^{-1}\sum_{i=1}^{n} Y_i^2$$

Problem 7

(a)

$$
\begin{aligned}
E[X - E[X]] &= E[X] - E[E[X]] \\
&= E[X] - E[X] \\
&= 0
\end{aligned}
$$

(b)

Being uncorrelated is the same as having zero covariance. Then:

$$
\begin{aligned}
Cov[X, Y] &= E[XY] - E[X]E[Y] \\
\text{(by independence (2.39))} &= E[X]E[Y] - E[X]E[Y] \\
&= 0
\end{aligned}
$$

Problem 8

(1)

$$
\begin{aligned}
E[Y - \hat{Y}] &= E[Y - E[Y] - \frac{\sigma_{XY}}{\sigma_X^2}(X - E[X])] \\
&= E[Y] - E[Y] - \frac{\sigma_{XY}}{\sigma_X^2}(E[X] - E[X]) \\
&= 0
\end{aligned}
$$

(b)

Note: I believe when the author writes $E\{Y - \hat{Y}\}^2$ he means $E[(Y - \hat{Y})^2]$ and not $E[Y - \hat{Y}]^2$. This confused me a bit ....

This requires substituting $\hat{Y} = E[Y] + \frac{\sigma_{XY}}{\sigma_X^2}(X - E[X])$ For simplicity, let $\mu = \frac{\sigma_{XY}}{\sigma_X^2}(X - E[X])$ Clean Version:

$$
\begin{aligned}
E[(Y - \hat{Y})^2] &= E[(Y - E[Y] - \mu)^2] \\
&= E[Y^2 + E[Y]^2 + \mu^2 - 2YE[Y] - 2Y\mu - 2E[Y]\mu] \\
&= E[Y^2] + E[Y]^2 + E[\mu^2] - 2E[Y]^2 - 2E[Y\mu] - 2E[Y]E[\mu] \\
(\text{applying } E[\mu] = 0) &= E[Y^2] - E[Y]^2 + E[\mu^2] - 2E[Y\mu] \\
&= \sigma_Y^2 + (\frac{\sigma_{XY}}{\sigma_X^2})^2 E[X - E[X]] - 2E[Y\frac{\sigma_{XY}}{\sigma_X^2}(X - E[X])] \\
&= \sigma_Y^2 + (\frac{\sigma_{XY}}{\sigma_X^2})^2(\sigma_X^2) - 2\frac{\sigma_{XY}}{\sigma_X^2}(E[XY] - E[X]E[Y]) \\
&= \sigma_Y^2 + (\frac{\sigma_{XY}}{\sigma_X^2})^2(\sigma_X^2) - 2\frac{\sigma_{XY}}{\sigma_X^2}(\sigma_{XY}) \\
&= \sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2}
\end{aligned}
$$

Problem 9

$$
\begin{aligned}
Cov(X, Y) &= Cov(X, X^2) \\
&= E[X^3] - E[X]E[X^2] \\
&= 0 - 0 * E[X^2] \\
&= 0
\end{aligned}
$$

Problem 10

Page 55 gives us the posterior density, so just solve for θ:

$$
f(\theta|3) = \frac{6\theta^4(1-\theta)}{\int 6\theta^4(1-\theta)d\theta} = 30\theta^4(1-\theta)
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta} f(\theta|3) &= 120\theta^3 - 150\theta^4 = 0 \\
\rightarrow \theta &= \frac{4}{5}
\end{aligned}
$$

Problem 11

The following formulas are needed to solve this problem:

1. $f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$b (Normal distribution density function)
2. $\int x^n e^{ax} = \frac{x^n e^{ax}}{a}\mathrm{d}x - \frac{n}{a}\int x^{n-1}e^{ax}\mathrm{d}x$ (relationship from integral table)

First, here is the derivation for $K[N(0,\sigma^2)]$:

$$K[N(0,\sigma^2)] = \frac{\mu_4}{\sigma^4}$$

where μ_4 is the fourth central moment:

$$\begin{aligned}\mu_4 &= E[(X-\mu)^4] \\ &= \int_{-\infty}^{\infty}(x-\mu)^4 f(x)dx\end{aligned}$$

Then:

$$\begin{aligned}\mu_4(N(0,\sigma^2)) &= \int_{-\infty}^{\infty} x^4 f(x)dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x^4 e^{\frac{-x^2}{2\sigma^2}}dx \\ (y = x^2, dy = 2xdx) &= \frac{1}{\sqrt{8\pi\sigma^2}}\int_{-\infty}^{\infty} y^{\frac{3}{2}} e^{\frac{-y}{2\sigma^2}}dy \\ &= \frac{1}{\sqrt{8\pi\sigma^2}}(\frac{y^{\frac{3}{2}}e^{\frac{-y}{2\sigma^2}}}{\frac{-1}{2\sigma^2}}|_{-\infty}^{\infty} + \frac{(3)(2\sigma^2)}{2}\int_{-\infty}^{\infty} y^{\frac{1}{2}}e^{\frac{-y}{2\sigma^2}}dy) \\ &= \frac{1}{\sqrt{8\pi\sigma^2}}(\frac{y^{\frac{3}{2}}e^{\frac{-y}{2\sigma^2}}}{\frac{-1}{2\sigma^2}}|_{-\infty}^{\infty} + \frac{(3)(2\sigma^2)}{2}[\frac{y^{\frac{1}{2}}e^{\frac{-y}{2\sigma^2}}}{\frac{-1}{2\sigma^2}}|_{-\infty}^{\infty} + \sigma^2\int_{-\infty}^{\infty} y^{\frac{-1}{2}}e^{\frac{-y}{2\sigma^2}}dy]) \\ &= \frac{1}{\sqrt{8\pi\sigma^2}}(\frac{y^{\frac{3}{2}}e^{\frac{-y}{2\sigma^2}}}{\frac{-1}{2\sigma^2}}|_{-\infty}^{\infty} + \frac{(3)(2\sigma^2)}{2}[\frac{y^{\frac{1}{2}}e^{\frac{-y}{2\sigma^2}}}{\frac{-1}{2\sigma^2}}|_{-\infty}^{\infty} + 2\sigma^2\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}dx]) \\ &= \frac{1}{\sqrt{8\pi\sigma^2}}(0 + 3\sigma^2[0 + 2\sigma^2\sqrt{2\pi\sigma^2}]) \\ &= 3\sigma^4\end{aligned}$$

Finally:

$$\begin{aligned}K[N(0,\sigma^2)] &= \frac{\mu_4}{\sigma^4} \\ &= \frac{3\sigma^4}{\sigma^4} \\ &= 3\end{aligned}$$

Based on the derivation above, any normal distribution with $\mu = 0$ will have a kurtosis of exactly 3. This finding further generalizes for any μ, which is why excess kurtosis is calculated as $K - 3$.

By the linearity of integration, we can easily extend this to mixed normal distributions:

$$\mu(pN_1(0,\sigma_1^4) + (1-p)N_2(0,\sigma_2^4)) = 3(p\sigma_1^2 + (1-p)\sigma_2^2)$$

$$\sigma^4(pN_1(0,\sigma_1^2) + (1-p)N_2(0,\sigma_2^2)) = (p\sigma_1^2 + (1-p)\sigma_2^2)^2$$

$$K[pN_1(0,\sigma_1^2) + (1-p)N_2(0,\sigma_2^2)] = \frac{3(p\sigma_1^2 + (1-p)\sigma_2^2)}{(p\sigma_1^2 + (1-p)\sigma_2^2)^2}$$

This formula can be used to solve problem 11.

(a)

$$K[0.95N(0,1) + (0.05)N(0,10)] = \frac{3(1 + (0.05)100)}{(1 + (0.05)(10))^2} = \frac{76}{9}$$

(b)

$$K[pN(0,1) + (1-p)N(0,\sigma^2)] = \frac{3(p + (1-p)\sigma^4)}{(p + (1-p)(\sigma^2))^2}$$

(c)

Either choose very small p and very small σ, or choose $p \approx 1$ and use very large σ. In essence, heavily weight the distribution with much smaller σ, and the smaller weight of the higher σ distribution will bloat kurtosis. The kurtosis function behavior inverts at $p = 0.5$, and it seems the max kurtosis is symmetric across $p = 0.5$

Remark:

This problem illustrates that while a univariate normal distribution has skewness of 0 and kurtosis of 3, the same is not true for a normal mixture distribution!

Problem 12

Chapter 3

Useful formulae:

1. $R_t = \frac{P_t + D_t}{P_{t-1}} - 1$

Problem 1

(a)

$$R_2 = \frac{56 + 0.2}{51} - 1 \approx 0.10196$$

(b)

$$R_4(3) = (\frac{58 + 0.25}{53})(\frac{53 + 0.25}{56})(\frac{56 + 0.2}{51}) - 1 \approx (1.09906)(0.950893)(1.10196) - 1 \approx 0.15165$$

(c)

$$r3 = \log(1 + R_3) = \log 1 + \frac{53 + 0.25}{56} \approx 0.2902$$

Problem 2

This formula for this problem was derived on page 81.

(a)

If r_t is a log return, then the kth period log return can be written as:

$$r_t(k) \sim N(k\mu, k\sigma^2)$$

Then, given that $\ldots, r_{-1}, r_0, r_1, \ldots$ are $N(0.1, 0.6)$,

$$r_t(3) \sim N(0.3, 1.8)$$

Chapter 4

Problem 1

Let's convert this AR(1) model into it's regression form:

$$\begin{aligned} Y_t &= 5 - 0.7Y_{t-1} + \epsilon_t \\ &= 2.94(1 + 0.7) - 0.7Y_{t+1} + \epsilon_t \end{aligned}$$

From this form, we get:
a) Yes, weights are not explosive. e.g. $\phi^h \to 0$ b) ≈ 2.94
c) $\approx 4/(1 - 0.49)\ = 7.84$
d) $\gamma(h) = \frac{2(-0.7)^{|h|}}{0.51}$
e) $\rho(h) = (-0.7)^{|h|}$

Problem 2

a) $\gamma(0) = \frac{2}{1-0.3^2} \approx 2.198$
b) $\gamma(2) = \frac{2*0.3^2}{1-0.3^2} \approx 0.1978$
c) Use equation 2.51:

$$\begin{aligned} Var[0.5Y_1 + 0.5Y_2] &= (0.5^2)(2.198) + (2)(0.5^2)(0.1978) + (0.5^2)(2.198) \\ &= 1.1979 \end{aligned}$$

Problem 3

$$\begin{aligned} Y_{n+1} &= \mu + \phi_1(Y_n - \mu) + \phi_2(Y_{n-1} - \mu) + \phi_3(Y_{n-2} - \mu) + \epsilon_{n+1} \\ &= 102 + (0.5)(99 - 102) + (0.2)(102 - 102) + (0.1)(104 - 102) \\ &= 100.7 \end{aligned}$$

$$\begin{aligned} Y_{n+2} &= \mu + \phi_1(Y_{n+1} - \mu) + \phi_2(Y_n - \mu) + \phi_3(Y_{n-1} - \mu) + \epsilon_{n+2} \\ &= 102 + (0.5)(100.7 - 99) + (0.2)(99 - 102) + (0.1)(102 - 102) \\ &= 102.25 \end{aligned}$$

Problem 4

ARIMA(0, 1, 0) is a time series in which the first difference is ARMA(0, 0), which is white noise. A random walk exhibits this property:

$$\begin{aligned}
Y_t &= Y_{t-1} + \epsilon_t \\
\Delta Y_t &= Y_t - Y_{t-1} \\
&= Y_{t-1} + \epsilon_t - Y_{t-1} \\
&= \epsilon_t
\end{aligned}$$

Therefore, a random walk is ARIMA(0,1,0).

Problem 5

$$Y_t = \mu + \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2}$$

$$\begin{aligned}
\gamma(h) &= Var[Y_t, Y_{t-h}] \\
&= E[Y_t Y_{t-h}] - E[Y_t]E[Y_{t-h}] \\
&= E[(\mu + \epsilon_t - \theta_1\epsilon_{t-1} - \theta_2\epsilon_{t-2})(\mu + \epsilon_{t-h} - \theta_1\epsilon_{t-1-h} - \theta_2\epsilon_{t-2-h})] - \mu^2 \\
\gamma(0) &= \mu^2 + E[(\epsilon_t)(\epsilon_t)] + E[(\theta_1\epsilon_{t-1})(\theta_1\epsilon_{t-1})] + E[(\theta_2\epsilon_{t-2})(\theta_2\epsilon_{t-2})] - \mu^2 \\
&= (1 + \theta_1^2 + \theta_2^2)\sigma^2 \\
\gamma(1) &= \mu^2 - E[(\theta_1\epsilon_{t-1})(\epsilon_{t-1})] + E[(\theta_2\epsilon_{t-2})(\theta_1\epsilon_{t-2})] - \mu^2 \\
&= (\theta_1 + \theta_1\theta_2)\sigma^2 \\
\gamma(2) &= -E[(\theta_2\epsilon_{t-2})(\epsilon_{t-2})] \\
&= \theta_2\sigma^2 \\
\gamma(h > 2) &= 0
\end{aligned}$$

Problem 6

(a)

$$(Y_t - \mu) = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t$$

$$\begin{aligned}
\gamma(h) &= Cov[\phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t, Y_{t-k}] \\
&= E[(\phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t)(Y_{t-k})] - E[\phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t]E[Y_{t-k}] \\
&= E[(\phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu))(Y_{t-k})] - E[\phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu)]E[Y_{t-k}]
\end{aligned}$$

(b)

$$\begin{aligned}
\begin{bmatrix} 1 & \rho(1)) \\ \rho(1) & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} &= \begin{bmatrix} \phi_1 + \phi_2\rho(1)) \\ \phi_1\rho(1) + \phi_2 \end{bmatrix} \\
&= \begin{bmatrix} \phi_1\rho(0) + \phi_2\rho(-1)) \\ \phi_1\rho(1) + \phi_2\rho(0) \end{bmatrix} \\
&= \begin{bmatrix} \rho(1) \\ \rho(2) \end{bmatrix}
\end{aligned}$$

(c)

$$\theta_1 = \frac{8}{21}$$
$$\theta_2 = \frac{1}{21}$$
$$\rho(3) \approx 0.0952$$

Problem 7

Since ϵ_i are iid, $E[\epsilon_i \epsilon_j] = E[\epsilon_i]E[\epsilon_j] = 0$ if $i1 = j$.

$$\begin{aligned}
Cov[\sum_{i=0}^{\infty} \epsilon_{t-i}\phi^i, \sum_{j=0}^{\infty} \epsilon_{t+h-j}\phi^j] &= E[(\sum_{i=0}^{\infty} \epsilon_{t-i}\phi^i)(\sum_{j=0}^{\infty} \epsilon_{t+h-j}\phi^j)] + E[\sum_{i=0}^{\infty} \epsilon_{t-i}\phi^i]E[\sum_{j=0}^{\infty} \epsilon_{t+h-j}\phi^j] \\
&= E[(\sum_{i=0}^{\infty} \epsilon_{t-i}\phi^i)(\sum_{j=0}^{\infty} \epsilon_{t+h-j}\phi^j)] \\
\text{(outer prod., only h-diagonal terms !}= 0) &= E[\sum_{i=0}^{\infty} \epsilon_{t-i}^2 \phi^i \phi^{i+h}] \\
&= \frac{\sigma^2 \phi^{|h|}}{1-\phi^2}
\end{aligned}$$

Problem 8

$$\begin{aligned}
\Delta\omega_t &= \omega_t - \omega_{t-1} \\
&= (\omega_{t_0} + Y_{t_0} + Y_{t_0+1} + \ldots + Y_t) - (\omega_{t_0} + Y_{t_0} + Y_{t_0+1} + \ldots + Y_{t-1}) \\
&= Y_t
\end{aligned}$$

Chapter 5

Problem 1

Let ω be the amount to invest in asset A, while $1-\omega$ is to be invested in asset B. Then:

$$\begin{aligned}
E[R] &= \omega\mu_A + (1-\omega)\mu_B \\
\sigma_R^2 &= \omega^2\sigma_A^2 + (1-\omega)\sigma_B^2 + 2\omega(1-\omega)\rho_{AB}\sigma_A\sigma_B
\end{aligned}$$

0.0.1 (a)

$$\begin{aligned}
E[R] &= \omega\mu_A + (1-\omega)\mu_B \\
0.03 &= 0.02\omega + 0.05(1-\omega) \\
\rightarrow \omega &= \frac{2}{3}
\end{aligned}$$

0.0.2 (b)

$$\begin{aligned}\sigma_R^2 &= \omega^2\sigma_A^2 + (1-\omega)\sigma_B^2 + 2\omega(1-\omega)\rho_{AB}\sigma_A\sigma_B \\ 5\% &= \omega^2 6\% + (1-\omega)^2 11\% + 2\omega(1-\omega)(0.1)\sqrt{6\%}\sqrt{11\%} \\ &\rightarrow (17-2\phi)\omega^2 + (2\phi-22)\omega + 6 = 0 \\ &\rightarrow (15.375)\omega^2 + (-20.375)\omega + 6 = 0 \\ \omega &= (0.441701, 0.883502)\end{aligned}$$

First portfolio:

$$\begin{aligned}E[R] &= 2\%(0.441701) + 5\%(1-0.441701) \\ &= 3.67\%\end{aligned}$$

Second portfolio:

$$\begin{aligned}E[R] &= 2\%(0.883502) + 5\%(1-0.883502) \\ &= 2.379\%\end{aligned}$$

The portfolio with $\omega = 0.441701$ lies on the efficient frontier. It is a superior portfolio to the one with $\omega = 0.883502$

Problem 2

The tangency portfolio is 60%C and 40%D. We have:

$$\begin{aligned}E[R_T] &= 5\% \\ \sigma_[R_T] &= 7\% \\ \mu_f &= 2\% \\ \sigma_R &= \omega\sigma_T\end{aligned}$$

Then:

$$\begin{aligned}\sigma_R &= \omega\sigma_T \\ 5\% &= \omega 7\% \\ &\rightarrow \omega = \frac{5}{7}\end{aligned}$$

Allocations:
C: $\frac{5}{7}(60\%)$
D: $\frac{5}{7}(40\%)$
Risk-free asset: $\frac{2}{7}(100\%)$

Problem 3

0.0.3 (a)

$$\omega = 0.87931$$

0.0.4 (b)

$$\omega_j = \frac{P_j n_j}{\sum_{i=1}^{N} P_i * n_i}$$

Problem 4

We have:

$$
\begin{aligned}
net: R_t &= \frac{P_t}{P_{t-1}} - 1 \\
gross: &\frac{P_t}{P_{t-1}} \\
log: r_t &= \log(1 + R_t)
\end{aligned}
$$

Use equation derived form problem 3, substitute for all ω:

$$
\begin{aligned}
R_p &= \omega 1 R_1 + \ldots + \omega_n R_n \\
&= \frac{1}{\sum_N^{i=1} P_{it} n_i} (P_{1t} n_1 (\frac{P_{1t+1}}{P_{1t}} - 1) + \ldots + P_{nt} n_n (\frac{P_{nt+1}}{P_{nt}} - 1)) \\
&= \frac{1}{\sum_N^{i=1} P_{it} n_i} (\sum_N^{i=1} P_{it+1} n_i - \sum_N^{i=1} P_{it} n_i) \\
&= \frac{Port_{t+1}}{Port_t} - 1 \\
&= R_p^{net}
\end{aligned}
$$

$$
\begin{aligned}
R_p &= \omega 1 R_1 + \ldots + \omega_n R_n \\
&= \frac{1}{\sum_N^{i=1} P_{it} n_i} (P_{1t} n_1 (\frac{P_{1t+1}}{P_{1t}}) + \ldots + P_{nt} n_n (\frac{P_{nt+1}}{P_{nt}})) \\
&= \frac{1}{\sum_N^{i=1} P_{it} n_i} (\sum_N^{i=1} P_{it+1} n_i) \\
&= \frac{Port_{t+1}}{Port_t} \\
&= R_p^{gross}
\end{aligned}
$$

Problem 5

$$\Omega = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$\Omega^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - (\rho_{12}\sigma_1\sigma_2)^2} \begin{bmatrix} \sigma_2^2 & -\rho_{12}\sigma_1\sigma_2 \\ -\rho_{12}\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

$$\Omega^{-1} = \frac{1}{(\sigma_1^2\sigma_2^2)(1-\rho_{12}^2)} \begin{bmatrix} \sigma_2^2 & -\rho_{12}\sigma_1\sigma_2 \\ -\rho_{12}\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

$$\Omega^{-1} = \frac{1}{1-\rho_{12}^2} \begin{bmatrix} \sigma_1^{-2} & -\rho_{12}\sigma_1^{-1}\sigma_2^{-1} \\ -\rho_{12}\sigma_1^{-1}\sigma_2^{-1} & \sigma_2^{-2} \end{bmatrix}$$

Chapter 6

Problem 1

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_i + \epsilon_i \\
\epsilon_i &\sim N(0, 0.6) \\
\beta_0 &= 2 \\
\beta_1 &= 1
\end{aligned}
$$

(a)

$$
\begin{aligned}
E[Y_i|X_i = 1] &= E[\beta_0 + \beta_1 X_i + \epsilon_i] \\
&= 2 + 1 + 0 \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
Var[Y|X_i = 1] &= [\beta_0 + \beta_1 X_i + \epsilon_i] \\
&= Var[N(0, 0.6)] \\
&= 0.6^2
\end{aligned}
$$

(a)

$$X_i \sim N(1, 0.4)$$

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_i + \epsilon_i \\
&\sim N(2 + 1, 0.4 + 0.6) \\
&\sim N(3, 1)
\end{aligned}
$$

Problem 2

To make this more simple and skip a few steps, we can directly substitute $\mu = \beta_0 + \beta_1 X_i$ into the log form MLE equation:

$$\log(L(\theta)) = -\frac{n}{2}(\log(\sigma^2) + \log(2\pi)) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_i)^2$$

Then, the estimates for β_0 and β_1 will be those that minimize:

$$\sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_i)^2$$

Which is exactly the same as 6.3, so we should get the same answer. The calculus works out like this.

$$
\begin{aligned}
\frac{\partial}{\partial \beta_0} \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_i)^2 &= -2 \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_i) \\
\frac{\partial}{\partial \beta_1} \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_i)^2 &= -2 \sum_{i=1}^{n} X_i (Y_i - \beta_0 - \beta_1 X_i)
\end{aligned}
$$

Then, setting the partial derivatives to zero, we get two equations and two unknowns:

$$
\begin{aligned}
-2 \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_i) &= 0 \\
-2 \sum_{i=1}^{n} X_i (Y_i - \beta_0 - \beta_1 X_i) &= 0
\end{aligned}
$$

Solving for β_0:

$$
\begin{aligned}
&\sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_i) = 0 \\
&\sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} \beta_0 - \sum_{i=1}^{n} \beta_1 X_i = 0 \\
&\sum_{i=1}^{n} Y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} X_i = 0 \\
&\beta_0 = \frac{1}{n} \sum_{i=1}^{n} Y_i - \frac{1}{n} \beta_1 \sum_{i=1}^{n} X_i \\
&\beta_0 = \bar{Y} - \beta_1 \bar{X}
\end{aligned}
$$

Solving for β_1:

$$
\begin{aligned}
&\sum_{i=1}^{n} X_i (Y_i - \beta_0 - \beta_1 X_i) = 0 \\
&\sum_{i=1}^{n} X_i (Y_i - \bar{Y} + \beta_1 \bar{X} - \beta_1 X_i) = 0 \\
&\sum_{i=1}^{n} X_i (Y_i - \bar{Y} - \beta_1 (X_i - \bar{X})) = 0 \\
&\sum_{i=1}^{n} X_i (Y_i - \bar{Y}) - \sum_{i=1}^{n} X_i (\beta_1 (X_i - \bar{X})) = 0 \\
&\sum_{i=1}^{n} X_i (Y_i - \bar{Y}) = \beta_1 \sum_{i=1}^{n} X_i (X_i - \bar{X}) \\
\rightarrow \beta_1 &= \frac{\sum_{i=1}^{n} X_i (Y_i - \bar{Y})}{\sum_{i=1}^{n} X_i (X_i - \bar{X})} \\
&= \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (X_i - \bar{X})^2}
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
Var[\hat{\beta}_1] &= Var[\omega^T Y] \\
&= \sum_{i=1}^{n} \omega_i^2 Var[Y_i] \\
&= \sum_{i=1}^{n} \omega_i^2 \sigma_\epsilon^2 \\
&= \sigma_\epsilon^2 \sum_{i=1}^{n} \omega_i^2 \\
&= \sigma_\epsilon^2 \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{(\sum_{j=1}^{N}(X_j - \bar{X})^2)^2} \\
&= \frac{\sigma_\epsilon^2}{(\sum_{j=1}^{N}(X_j - \bar{X})^2)^2} \sum_{i=1}^{n}(X_i - \bar{X})^2 \\
&= \frac{\sigma_\epsilon^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \\
&= \frac{\sigma_\epsilon^2}{(n-1)s_X^2}
\end{aligned}
$$

Problem 5

(a)

Using $R^2 = r_{Y\hat{Y}}^2$:

$$R^2 = r_{Y\hat{Y}}^2 = 0.5^2 = 0.25$$

(b)

$$
\begin{aligned}
\text{residual error SS} &= (1 - R^2)\text{total SS} \\
&= (1 - 0.25)100 \\
&= 75
\end{aligned}
$$

(c)

$$
\begin{aligned}
\text{regression SS} &= R^2 * \text{total SS} \\
&= (0.25)100 \\
&= 25
\end{aligned}
$$

(d)

Problem 6

$$
\begin{aligned}
\sigma_{\hat{\epsilon},5} &= \frac{SSE(5)}{n-p-1} \\
&= \frac{10}{66-5-1} \\
&= 0.167
\end{aligned}
$$

R^2 is not a good statistic for comparing models of different sizes, so let's just compute the C_p values:

$$\begin{aligned} C_5 &= \frac{SSE(5)}{\hat{\sigma_{\epsilon,5}}} - n + 2(p+1) \\ &= \frac{10.0}{0.167} - 66 + 2(5+1) \\ &= 6 \end{aligned}$$

$$\begin{aligned} C_4 &= \frac{SSE(4)}{\hat{\sigma_{\epsilon,5}}} - n + 2(p+1) \\ &= \frac{10.2}{0.167} - 66 + 2(4+1) \\ &= 5.2 \end{aligned}$$

$$\begin{aligned} C_3 &= \frac{SSE(3)}{\hat{\sigma_{\epsilon,5}}} - n + 2(p+1) \\ &= \frac{12.0}{0.167} - 66 + 2(3+1) \\ &= 14 \end{aligned}$$

C_3 is substantially greater than $p + 1 = 6$, so we can consider this model to be underfit.

Problem 7

Yes, the p-values are quite high, meaning we cannot reject the null hypothesis.

Problem 8

This problem is similar to problem 2. Here, the estimates for β_0 and β_1 will be those that minimize:

$$\sum_{i=1}^{n}(Y_i - -\beta_1 X_i)^2$$

Now, we only have to differentiate with respect to β_1 and solve. The calculus works out like this:

$$\frac{\partial}{\partial \beta_1}\sum_{i=1}^{n}(Y_i - \beta_1 X_i)^2 = -2\sum_{i=1}^{n} X_i(Y_i - \beta_1 X_i)$$

Solving for β_1:

$$\begin{aligned} &\sum_{i=1}^{n} X_i(Y_i - \beta_1 X_i) = 0 \\ &\sum_{i=1}^{n} X_i Y_i - \beta_1 \sum_{i=1}^{n} X_i^2 = 0 \\ &\to \beta_1 = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2} \end{aligned}$$

Problem 9

Chapter 7

Problem 1

no explicit form for computing beta for a portfolio, other than the linear combo from security betas, so I believe (7.10) is valid.

$$\begin{aligned}
\beta_P &= \frac{\mu_P - \mu_f}{\mu_M - \mu_f} \\
&= \frac{16 - 6}{11 - 6} \\
&= 2
\end{aligned}$$

Extra: show this equation arises from linear combo form.

$$\begin{aligned}
\beta_P &= \sum_{i=1}^{N} \omega_i \beta_i \\
&= \omega_1 \frac{\mu_1 - \mu_f}{\mu_M - \mu_f} + \ldots + \omega_N \frac{\mu_N - \mu_f}{\mu_M - \mu_f} \\
\beta_P(\mu_M - \mu_f) &= \omega_1(\mu_1 - \mu_f) + \ldots + \omega_N(\mu_1 - \mu_f) \\
\beta_P(\mu_M - \mu_f) &= \sum_{i=1}^{N} \omega_i \mu_i - \mu_f \\
\beta_P &= \frac{\sum_{i=1}^{N} \omega_i \mu_i - \mu_f}{\mu_M - \mu_f} \\
\beta_P &= \frac{\mu_P - \mu_f}{\mu_M - \mu_f}
\end{aligned}$$

Problem 2

(a)

Refer to Figure 7.1 and page 229

$$\begin{aligned}
\mu_P &= \omega \mu_M + (1 - \omega)\mu_f \\
0.11 &= \omega 0.14 + (1 - \omega)0.07 \\
\rightarrow \omega &= \frac{4}{7}
\end{aligned}$$

(b)

$$\begin{aligned}
\sigma_P &= \sqrt{\omega^2 \sigma_M^2} \\
\sigma_P &= (\frac{4}{7})(0.12) \\
&= 0.06857
\end{aligned}$$

Problem 3

(a)

$$\mu_R = \frac{(\mu_M - \mu_f)\sigma_R}{\sigma_M} + \mu_f$$
$$= \frac{(0.10 - 0.04)0.05}{0.12} + 0.04$$
$$= 0.65$$

(b)

$$\beta_A = \frac{\sigma_{AM}}{\sigma_M^2}$$
$$\beta_A = \frac{0.004}{0.12^2}$$
$$= 0.277$$

Section 7.4.1

(i)

$$R_P = \mu_f + \sum_{j=1}^{N} \beta_j \omega_j (\mu_M - \mu_f) + \sum_{j=1}^{N} \omega_j \epsilon_j$$
$$= 0.04 + 0.5(1.5 + 1.8)(0.10 - 0.04) + 0$$
$$= 0.139$$

(ii)

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma_{\epsilon P}^2$$
$$= 0.5(1.5 + 1.8)(0.12)^2 + (0.5)^2(0.08^2 + 0.10^2)$$
$$= 0.02786$$
$$\rightarrow \sigma_P \approx 0.16691$$

Problem 4

To evaluate $\sigma_{j,M}$, use (2.54), noting that $X_1 = R_j$ (a linear combo of just one security) and $X_2 = \sum_N^{i=1} \omega_{i,M} R_i$ (the market portfolio).

Then:

$$
\begin{aligned}
\sigma_{j,M} &= Cov(\omega_1^T X, \omega_2^T X) \\
&= \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}^T COV(X) \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{j1} \\ \vdots \\ \sigma_{jN} \end{bmatrix}^T \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} \\
&= \sum_{i=1}^{N} \omega_{i,M} \sigma_{i,j}
\end{aligned}
$$

Problem 5

False

Problem 6

We have $\mu_f = 5\%$, $\mu_M = 14\%$, $\sigma_M = 15\%$ and $\sigma_{AM} = 165\%$.

(a)

$$
\begin{aligned}
\beta_A &= \frac{\sigma_{AM}}{\sigma_M^2} \\
&= \frac{165}{15^2} \\
&= 0.733
\end{aligned}
$$

(b)

Use a reformulation of (7.10):

$$
\begin{aligned}
\mu_A &= \beta_A(\mu_M - \mu_f) + \mu_f \\
&= (0.733)(9\%) + 5\% \\
&= 11.6\%
\end{aligned}
$$

(c)

$$
\begin{aligned}
\sigma_A^2 &= \beta_A^2 \sigma_M^2 + \sigma_{\epsilon,A}^2 \\
220\% &= (0.733)(15\%) + \sigma_{\epsilon,A}^2 \\
\rightarrow \sigma_{\epsilon,A}^2 &= 55\%
\end{aligned}
$$

so the percentage due to market risk would be $\frac{(220-55)}{220} * 100\% = 75\%$

Problem 7

(a)

$$\begin{aligned}
\beta_P &= \sum_{i=1}^{n} \omega_i \beta_i \\
&= \frac{1}{3}(0.9 + 0.7 + 0.6) \\
&= 0.733
\end{aligned}$$

(this is the same number as 6.a, author must like it?)

(b)

$$\begin{aligned}
\sigma^2_{\epsilon,P} &= \frac{\bar{\sigma_e}^2}{N} \\
&= \frac{\frac{1}{3}(0.010 + 0.015 + 0.012)}{3} \\
&= 0.00411
\end{aligned}$$

(c)

Problem 8

$$\begin{aligned}
\sigma_{j,M} &= Cov(\omega_1^T X, \omega_2^T X) \\
&= \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}^T COV(X) \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} \\
&= \begin{bmatrix} \beta_{1,j} \\ \beta_{2,j} \end{bmatrix}^T \begin{bmatrix} Var(F_1) & Cov(F_1, F_2) \\ Cov(F_1, F_2) & Var(F_2) \end{bmatrix} \begin{bmatrix} \beta_{1,j'} \\ \beta_{2,j'} \end{bmatrix} \\
&= \begin{bmatrix} \beta_{1,j} Var(F_1) + \beta_{2,j} Cov(F_1, F2) \\ \beta_{1,j} Cov(F_1, F2) + \beta_{2,j} Var(F_2) \end{bmatrix}^T \begin{bmatrix} \beta_{1,j'} \\ \beta_{2,j'} \end{bmatrix} \\
&= \beta_{1,j}\beta_{1,j'} Var(F_1) + \beta_{2,j}\beta_{1,j'} Cov(F_1, F_2) + \beta_{1,j}\beta_{2,j'} Cov(F_1, F2) + \beta_{2,j}\beta_{2,j'} Var(F_2)
\end{aligned}$$

Chapter 12

This chapter builds past ARMA models and investigates time series models with non-constant variance. Recall Chapter 6 Problem 9 discussion. We delve into non-constant conditional variance (conditional heteroscedasticity).

Problem 1

$$
\begin{aligned}
E(|Z|) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} |z| e^{-z^2/2} \\
&= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} z e^{-z^2/2} \\
&= \frac{2}{\sqrt{2\pi}} (-e^{-x^2/2})|_0^\infty \\
&= \sqrt{\frac{2}{\pi}}(0+1) \\
&= \sqrt{\frac{2}{\pi}}
\end{aligned}
$$

Remark: finding expectation of a distribution is finding the moment along the x axis, similar to centroid problems in physics.

Problem 2

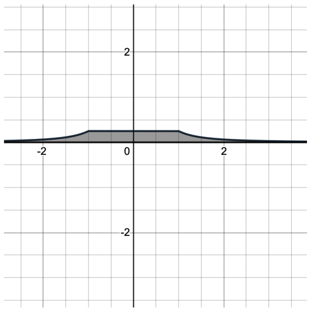

Figure 1: Piecewise density function

$$
\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)dx &= \frac{1}{4}(\int_{-\infty}^{-1} \frac{1}{x^2}dx + \int_{0}^{-1} dx + \int_{1}^{0} dx + \int_{1}^{\infty} \frac{1}{x^2}dx) \\
&= \frac{1}{4}(-\frac{1}{x}|_{-\infty}^{-1} + x|_{-1}^{1} - \frac{1}{x}|_1^\infty) \\
&= \frac{1}{4}(1+2+1) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{0} x f_X(x)dx &= \frac{1}{4}(\int_{-\infty}^{-1} \frac{1}{x}dx + \int_{-1}^{0} xdx) \\
&= \frac{1}{4}(\ln|x||_{-\infty}^{-1} + \frac{x^2}{2}|_{-1}^{0}) \\
&= \frac{1}{4}(\ln(1) - \ln(\infty) + + \frac{x^2}{2}|_{-1}^{0}) \\
&= -\infty
\end{aligned}
$$

$$\begin{aligned}\int_0^\infty x f_X(x)dx &= \frac{1}{4}(\int_0^1 xdx + \int_1^\infty \frac{1}{x}dx) \\ &= \frac{1}{4}(\frac{x^2}{2}|_0^1 + \ln|x||_\infty^1) \\ &= \frac{1}{4}(\frac{x^2}{2}|_0^1 + \ln(\infty) - \ln(1)) \\ &= \infty\end{aligned}$$

Problem 3

We can rewrite this process to form of (4.4) and (4.5)

$$\begin{aligned}u_t &= 3 + 0.7u_{t-1} + a_t \\ &= 10(1-0.7) + 0.7u_{t-1} + a_t \\ &= 10 + \Sigma_{h=0}^\infty(0.7^h a_{t-h})\end{aligned}$$

(a)

$$E[10 + \Sigma_{h=0}^\infty(0.7^h a_{t-h})] = 10$$

(b)

For variance, use (2.55), (4.11), and expectation of (12.6) (for unconditional marginal variance of a).

$$\begin{aligned}Var[10 + \Sigma_{h=0}^\infty(0.7^h a_{t-h})] &= \Sigma_{h=0}^\infty(0.7^{2h}\gamma_a(0)) \\ &= \Sigma_{h=0}^\infty(0.7^{2h}\frac{1}{1-0.5}) \\ &= 2\Sigma_{h=0}^\infty 0.7^{2h} \\ &= \frac{2}{1-0.7^2}\end{aligned}$$