Sheet Stitching Problem Definition

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Abstract

This write-up contains the problem definition of the Sheet Stitching Problem as a graph problem and possible functions to optimize for.

1 Introduction

The herculanuem papyri collection consists of a few hundred still unopened and charred scrolls that were burried by the vesuvius eruption in 79 AD. To read the papyri, they are scanned and a 3D reconstruction is created with computed tomography. Then the scroll is digitally unrolled and the ink made visible with AI. This write-up focuses on a specific step in the ThaumatoAnakalyptor unrolling pipeline. This specific step, called "Sheet Stitching" might be encountered by other, future automatic segmentation pipelines as well.

2 ThaumatoAnakalyptor Pipeline

The pipeline of automatic scroll CT papyrus surface extraction has multiple steps. First, extract 3D points of the surface from the raw scan V. Second, use PointCloud instance segmentation to extract small, instances of points belonging to the same local surface. Create a graph from the instance segmentations and solve it to extract the global sheet connectivity. Finally mesh the segmentation and digitally unroll the mesh.

You can find the complete automatic segmentation pipeline ThaumatoAnakalyptor at https://github.com/schillij95/ThaumatoAnakalyptor.

3 Instance Segmentation

Pointcloud Instance segmentation describes the step of extracting groups of point from the PointCloud which belong to the same sheet. The scroll volume V of size X, Y, Z is divided into overlapping subvolumes S of size 50, 50, 50. For $s_{xyz} \in S$, $s_{xyz} \subseteq V$ the top-right-front corner is at position (x, y, z) in the scroll volume.

$$\forall s_{xyz} \in S, \quad \forall i \in \{x, y, z\}, \quad i \in \mathbb{Z}^+ \quad \wedge \\ i \mod 25 \equiv 0 \quad \wedge \quad x \leq X, y \leq Y, z \leq Z$$

The instance segmentation is performed on each subvolume. For all $s_{xyz} \in S$ containing the PointCloud PC_{xyz}, the instance segmentation model Mask3D creates instances set I_{xyz}. $\forall i \in I_{xyz}$: $i \subseteq PC_{xyz}$ and $\forall i, j \in I_{xyz}, i \neq j : i \cap j = \emptyset$.

For an average scroll volume V, this process creates around 10 million instances.

4 Translation to Graph

Let's number the instance patches by integer p:

$$I_{xyz} = \{i_p \mid p = 0, 1, 2, \ldots, n-1\}$$

All nodes can now be defined by:

$$N = \{(x, y, z, p) \mid s_{xyz} \in S, \, p \in I_{xyz}\}$$

Edges connect nodes from neighboring subvolumes:

$$E = \{((x_0, y_0, z_0, p_0), (x_1, y_1, z_1, p_1)) \mid \\ |x_0 - x_1|, |y_0 - y_1|, |z_0 - z_1| \leq 25\}$$

Let's define the directed, weighted, winding angle graph as $G = (N, E, w, k)$.

We define the winding angle difference function $k : E \rightarrow \mathbb{R}$ as an estimation for the angular rotation that results of traversing the papyrus surface from the first to the second node of the edge with respect to a center axis (called umbilicus).

The estimation of the winding angle difference k is accurate up to a multiple $o \in \mathbb{Z}$ of 360 to the actual winding angle difference k^*, since the position of the instance with respect to the umbilicus is known:

$$k^*(e) = k(e) + o \cdot 360$$

k is antisymmetric:

$$k((n_0, n_1)) = -k((n_1, n_0))$$

The weight function $w : E \to \mathbb{R}^+$ estimates if the assignment $k(e)$ is true.

w is symmetric:

$$w((n_0, n_1)) = w((n_1, n_0))$$

The resulting graph will have edges with conflicting information about the winding angle difference within the graph.

5 Sheet Stitching

We seek to find a winding angle assignment function $f : N \to \mathbb{R}$, which assigns each node a winding angle.

Following are some different ways to think about the problem. There is no single best way so far. Thanks to Francesco for 5.2 and Giorgio for 5.3.

Be sure to check out how the histogram of your solution looks as another method of inspecting and tracking your progress.

5.1 Desired Graph Node Assignment Characteristics

A good measure of the accuracy of the found solution $\tilde{f}$ could be to optimize for:

$$f^* = \arg\max_{f \in F} \sum_{e \in E} w(e) \cdot c(e)$$

where

$$c((n_0, n_1)) = \begin{cases} 1, & \text{if } f(n_1) - f(n_0) = k((n_0, n_1)) \\ 0, & \text{otherwise} \end{cases}$$

Which translates to the least number of wrongly estimated edge winding angle differences.

5.2 Approximate message passing

As an alternative approach, we can define a probability measure over the function $f \equiv (f(1), \dots, f(N))$ as

$$P(f) = \frac{1}{Z} \prod_{e \in E} \psi(f_{\partial e}), \tag{1}$$

where $f_{\partial e} = (f(n_0), f(n_1))$ with $e = (n_0, n_1)$. The factor $\psi(f_{\partial e})$ should describe the winding number difference $k(e)$ and the confidence $w(e)$. Assuming $w(e) > 0$, a possible choice is

$$\psi(f_{\partial e}) = w(e) e^{-a w(e) |\ell|}, \tag{2}$$

if there exist an integer ℓ such that

$$f(n_1) - f(n_0) = k((n_0, n_1)) + 2\pi\ell. \tag{3}$$

and $\psi(f_{\partial e}) = 0$ otherwise. Here a is a parameter to be determined, e.g., via cross-validation.

5.3 Archimedean Spiral approximation

Let us limit to 2D and consider for simplicity an Archimedean spiral in the plane described by the curve $r(\theta) = b\theta$ where b is a real coefficient and θ the winding angle around the pole. An Archidean spiral has the very special property for the arclength ds of couple of points with the same θ up to an integer factor of 2π:

$$ds(p_1, p_2) = 2\pi l b \tag{4}$$

if $\Delta\theta = \theta_2 - \theta_1 = 2\pi n$ and $l \in \mathbb{Z}$

Let $f : N \to \mathbb{R}$ be the estimate of the arclength along an Archimedean spiral that starts at the umbelicus and reaches the barycenter of the instance n.

Let what we called so far $k((n_1, n_2))$ be actually the arclength between points $ds(n_1, n_2)$.

The arclength between two general points along an Archimedean spiral is given by

$$ds(n_1, n_2) = \frac{b}{2} \left(\Delta\theta \sqrt{1 + \Delta\theta^2} + \log \left(\Delta\theta + \sqrt{1 + \Delta\theta^2} \right) \right). \tag{5}$$

We notice that, since we can measure (r_1, α_1) and (r_2, α_2) from the data, where α_i is just the planar angle and r_i is the distance from the umbilicus, we can estimate b by solving the equation:

$$r_2 - r_1 = b(\alpha_2 - \alpha_1) \tag{6}$$

which has the solution:

$$b = \frac{r_2 - r_1}{\alpha_2 - \alpha_1} \tag{7}$$

. or

$$b = \frac{\Delta r}{\Delta \alpha}. \tag{8}$$

If n_1 and n_2 are consecutive instances, we can approximate $\Delta\theta \approx \Delta\alpha$ and say that an estimate of ds is

$$f(n_2) - f(n_1) \approx ds(n_1, n_2) + 2\pi l b \approx \tag{9}$$

$$\frac{b}{2}\left(\Delta\alpha\sqrt{1+\Delta\alpha^2} + \log\left(\Delta\alpha + \sqrt{1+\Delta\alpha^2}\right)\right) + 2\pi l b. \tag{10}$$

The last equation can be simplified inserting the expression for b,

$$\frac{\Delta r}{2}\left(\sqrt{1+\Delta\alpha^2} + \frac{\log\left(\Delta\alpha + \sqrt{1+\Delta\alpha^2}\right) + 2l\pi}{\Delta\alpha}\right). \tag{11}$$

Notice that when $l = 0$ and $\Delta\alpha \to 0$, $f(n_2) - f(n_1) \to \Delta r$.