1 Introduction

We implement the method of harmonic balance using exponential periodic basis functions instead of trigonometric. The sources [2, 1, 3] develop and implement this method using trigonometric basis functions (i.e., sines and cosines). They are used extensively to guide the development presented here.

Let $(\cdot)^*$ denote the conjugate transpose of a vector or matrix.

Assume the state $\mathbf{x}(t) : \mathbb{R} \to \mathbb{R}^n$ is governed by

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} + \mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{\mathrm{ext}}(t, \omega). \tag{1}$$

Note that n is the number of degrees of freedom; M, C, and K are the mass, damping, and stiffness matrices, respectively; $\mathbf{f}_{\mathrm{nl}}$ is the internal, potentially nonlinear force; $\mathbf{f}_{\mathrm{ext}}$ is the external force; and ω is the fundamental frequency of the external force.

Then $\mathbf{x}$ is periodic with period $2\pi/\omega$, so we can estimate it as a finite sum of exponential periodic basis functions:

$$\mathbf{x}(t, \omega) \approx \sum_{k=-N_H}^{N_H} \mathbf{a}_k \exp(ik\omega t). \tag{2}$$

We can write the forces similarly:

$$\mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}}, \omega) \approx \sum_{k=-N_H}^{N_H} \mathbf{c}_k \exp(ik\omega t) \tag{3}$$

$$\mathbf{f}_{\mathrm{ext}}(t, \omega) \approx \sum_{k=-N_H}^{N_H} \mathbf{d}_k \exp(ik\omega t). \tag{4}$$

2 Projecting the governing equation

Collect the Fourier coefficients:

$$\mathbf{z}(\omega) \coloneqq \begin{bmatrix} \mathbf{a}_{-N_H}^\top & \cdots & \mathbf{a}_{-1}^\top & \mathbf{a}_0^\top & \mathbf{a}_1^\top & \cdots & \mathbf{a}_{N_H}^\top \end{bmatrix}^\top \in \mathbb{C}^{(2N_H+1)n \times 1} \tag{5}$$

$$\mathbf{b}_{\mathrm{nl}}(\mathbf{z}) \coloneqq \begin{bmatrix} \mathbf{c}_{-N_H}^\top & \cdots & \mathbf{c}_{-1}^\top & \mathbf{c}_0^\top & \mathbf{c}_1^\top & \cdots & \mathbf{c}_{N_H}^\top \end{bmatrix}^\top \tag{6}$$

$$\mathbf{b}_{\mathrm{ext}}(\omega) \coloneqq \begin{bmatrix} \mathbf{d}_{-N_H}^\top & \cdots & \mathbf{d}_{-1}^\top & \mathbf{d}_0^\top & \mathbf{d}_1^\top & \cdots & \mathbf{d}_{N_H}^\top \end{bmatrix}^\top. \tag{7}$$

Also let $\mathbf{b}(\mathbf{z}, \omega) = \mathbf{b}_{\mathrm{ext}}(\omega) - \mathbf{b}_{\mathrm{nl}}(\mathbf{z})$ denote the Fourier coefficients of the force $\mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}}, \omega) = \mathbf{f}_{\mathrm{ext}}(t, \omega) - \mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}}, \omega)$.

Collect the exponential periodic basis functions

$$Q(t, \omega) \coloneqq \begin{bmatrix} e^{-iN_H\omega t} & \cdots & e^{-i\omega t} & 1 & e^{i\omega t} & \cdots & e^{iN_H\omega t} \end{bmatrix} \in \mathbb{C}^{1 \times (2N_H+1)} \tag{8}$$

so that

$$\mathbf{x}(t, \omega) = (Q(t, \omega) \otimes I_n)\mathbf{z}(\omega) \tag{9}$$

$$\mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}}, \omega) = (Q(t, \omega) \otimes I_n)\mathbf{b}_{\mathrm{nl}}(\mathbf{z}) \tag{10}$$

$$\mathbf{f}_{\mathrm{ext}}(t, \omega) = (Q(t, \omega) \otimes I_n)\mathbf{b}_{\mathrm{ext}}(\omega) \tag{11}$$

$$\mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}}, \omega) = (Q(t, \omega) \otimes I_n)\mathbf{b}(\mathbf{z}, \omega) \tag{12}$$

where $\otimes$ denotes the Kronecker product. Expanding the Kronecker product to make sure we're not pulling our own leg,

$$\begin{aligned} &(Q(t, \omega) \otimes I_n) \\ &= \begin{bmatrix} e^{-iN_H\omega t} I_n & \cdots & e^{-i\omega t} I_n & I_n & e^{i\omega t} I_n & e^{iN_H\omega t} I_n \end{bmatrix} \in \mathbb{C}^{(2N_H+1)n \times n} \end{aligned}$$

so

$$
\begin{aligned}
&(Q(t,\omega)\otimes I_n)\mathbf{z}(\omega)\\
&= \begin{bmatrix} e^{-iN_H\omega t}I_n & \cdots & e^{-i\omega t}I_n & I_n & e^{i\omega t}I_n & e^{iN_H\omega t}I_n \end{bmatrix} \begin{bmatrix} \mathbf{a}_{-N_H} \\ \vdots \\ \mathbf{a}_{-1} \\ \mathbf{a}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{N_H} \end{bmatrix}\\
&= \mathbf{x}(t,\omega).
\end{aligned}
$$

2.1 Differentiating

Suppose we want to write the Fourier series for $\dot{\mathbf{x}}$? Observe,

$$
\begin{aligned}
\dot{\mathbf{x}}(t,\omega) &= \frac{d}{dt}\mathbf{x}(t,\omega) = \frac{d}{dt}(Q(t,\omega)\otimes I_n)\mathbf{z}(\omega)\\
&= \left(\dot{Q}(t,\omega)\otimes I_n\right)\mathbf{z}(\omega).
\end{aligned}
$$

Letting

$$
\widetilde{\nabla} \coloneqq \begin{bmatrix} -iN_H & \cdots -i & 0 & i & \cdots & iN_H \end{bmatrix} \tag{13}
$$

$$
\nabla \coloneqq \operatorname{diag}\widetilde{\nabla} \tag{14}
$$

we find that

$$
\begin{aligned}
\dot{Q}(t,\omega) &= \omega \begin{bmatrix} -iN_He^{-iN_H\omega t} & \cdots & -ie^{-i\omega t} & 0 & ie^{i\omega t} & \cdots & iN_He^{iN_H\omega t} \end{bmatrix}\\
&= \omega Q(t,\omega)\odot\widetilde{\nabla}\\
&= \omega Q(t,\omega)\nabla.
\end{aligned}
$$

Likewise,

$$
\ddot{Q}(t,\omega) = \omega^2 Q(t,\omega)\nabla^2.
$$

This yields the identities

$$
\mathbf{x}(t,\omega) = (Q(t,\omega)\otimes I_n)\mathbf{z}(\omega) \tag{15}
$$

$$
\dot{\mathbf{x}}(t,\omega) = \omega(Q(t,\omega)\nabla\otimes I_n)\mathbf{z}(\omega) \tag{16}
$$

$$
\ddot{\mathbf{x}}(t,\omega) = \omega^2\left(Q(t,\omega)\nabla^2\otimes I_n\right)\mathbf{z}(\omega). \tag{17}
$$

2.2 Performing the projection

First, we rewrite (1) using (15), (16), and (17), beginning with the first and using the mixed-product property of the Kronecker product:

$$
\begin{aligned}
&M\ddot{\mathbf{x}}\\
&= M\omega^2\left(Q\nabla^2\otimes I_n\right)\mathbf{z}\\
&= \omega^2(1\otimes M)\left(Q\nabla^2\otimes I_n\right)\mathbf{z}\\
&= \omega^2\left[\left(1Q\nabla^2\right)\otimes(MI_n)\right]\mathbf{z}\\
&= \omega^2\left[\left(Q\nabla^2\right)\otimes M\right]\mathbf{z}.
\end{aligned} \tag{18}
$$

The same computations show

$$C\dot{\mathbf{x}} = \omega[(Q\nabla) \otimes C]\mathbf{z} \tag{19}$$

$$K\mathbf{x} = [Q \otimes K]\mathbf{z}. \tag{20}$$

Now the system equation (1) becomes (suppressing Q's dependence on ω)

$$\left[\omega^2 \big(Q(t)\nabla^2\big) \otimes M + \omega(Q(t)\nabla) \otimes C + Q(t) \otimes K\right]\mathbf{z}(\omega) = (Q(t) \otimes I_n)\mathbf{b}(\mathbf{z}). \tag{21}$$

Next, we project the system equation from the time domain (1) to the frequency domain via Fourier–Galerkin projection (I think). Define the inner product on functions $f : [0, 2\pi/\omega] \to \mathbb{C}$:

$$\langle f, g\rangle := \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \overline{f(t)} g(t) dt. \tag{22}$$

Now let $-N_H \leq k, \ell \leq N_H$, and write $\hat{k} := k + N_H$ and $\hat{\ell} := \ell + N_H$. Then

$$\begin{aligned}
&\frac{\omega}{2\pi}\left(\int_0^{2\pi/\omega} Q(t)^* Q(t) dt\right)_{\hat{k}\hat{\ell}} \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} (Q(t)^*Q(t))_{\hat{k}\hat{\ell}} dt \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \exp(-ik\omega t)\exp(i\ell\omega t) dt \\
&= \langle \exp(ik\omega t),\ \exp(i\ell\omega t)\rangle \\
&= \delta_{k\ell}
\end{aligned}$$

since the basis functions are orthonormal in this inner product. Then

$$\frac{\omega}{2\pi}\left(\int_0^{2\pi/\omega} Q(t)^* Q(t) dt\right) = I_{2N_H+1}.$$

Now we integrate (21) against $Q(t)^*$ to obtain an algebraic system of equations in the frequency domain:

$$\begin{aligned}
&\frac{\omega}{2\pi}\int_0^{2\pi/\omega} Q(t)^* \left[\omega^2 \big(Q(t)\nabla^2\big) \otimes M\right] dt \\
&= \omega^2 \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[Q(t)^* \otimes 1\right]\left[\big(Q(t)\nabla^2\big) \otimes M\right] dt \\
&= \omega^2 \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[Q(t)^* \big(Q(t)\nabla^2\big)\right] \otimes \left[1M\right] dt \\
&= \omega^2 \left[\frac{\omega}{2\pi}\int_0^{2\pi/\omega} Q(t)^* Q(t)\, dt\ \nabla^2\right] \otimes M \\
&= \omega^2 \left[I_{2N_H+1}\ \nabla^2\right] \otimes M \\
&= \omega^2 \nabla^2 \otimes M.
\end{aligned}$$

Following the same process for the other terms in (21), we at last obtain

$$\left[\omega^2\nabla^2 \otimes M + \omega\nabla \otimes C + I_{2N_H+1} \otimes K\right]\mathbf{z} = \mathbf{b}(\mathbf{z}) \tag{23}$$

and define $A(\omega) := \left[\omega^2\nabla^2 \otimes M + \omega\nabla \otimes C + I_{2N_H+1} \otimes K\right]$.

3 Computing the nonlinear force

How do we compute the Fourier coefficients of the nonlinear force, $\mathbf{b}(\mathbf{z})$? One method is the alternating frequency/time method (AFT), in which we take the frequency domain solution $\mathbf{z}$, send it to the time domain to obtain $\mathbf{x}$ and $\dot{\mathbf{x}}$, pass these through the nonlinear force to obtain $\mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}})$, and finally send this back to the frequency domain to obtain $\mathbf{b}_{\mathrm{nl}}$.

We also use a fancy technique called trigonometric collocation in which we construct discrete Fourier transform (DFT) matrices using extra time samples DFT so that we can capture information from higher frequencies resulting from the nonlinearity. (For example, consider passing $e^{i\omega t}$ through a cubic nonlinearity $y \mapsto y^3$, yielding $e^{i3\omega t}$.)

3.1 Computing time from frequency

Assume we use N samples evenly spaced in one period, $t_j = \frac{2\pi}{\omega}\frac{j}{N}$, $j = 0, 1, \ldots, N-1$. Define

$$\mathbf{q}(k) := \begin{bmatrix} e^{ik\omega t_0} \\ e^{ik\omega t_1} \\ \vdots \\ e^{ik\omega t_{N-1}} \end{bmatrix} \tag{24}$$

$$\Gamma := \begin{bmatrix} I_n \otimes \mathbf{q}(-N_H) & \cdots & I_n \otimes \mathbf{q}(0) & I_n \otimes \mathbf{q}(1) & \cdots & I_n \mathbf{q}(N_H) \end{bmatrix} \tag{25}$$

Now

$$\Gamma \mathbf{z} = (I_n \otimes \mathbf{q}(-N_H))\mathbf{a}_{-N_H} + \cdots + (I_n \otimes \mathbf{q}(N_H))\mathbf{a}_{N_H}.$$

Examining a single summand, we find

$$\begin{aligned} [I_n \otimes \mathbf{q}(k)]\mathbf{a}_k &= \begin{bmatrix} \mathbf{q}(k) & & \\ & \ddots & \\ & & \mathbf{q}(k) \end{bmatrix} \begin{bmatrix} a_{k,0} \\ \vdots \\ a_{k,n-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{q}(k) a_{k,0} \\ \vdots \\ \mathbf{q}(k) a_{k,n-1} \end{bmatrix} \end{aligned}$$

where $a_{k,d}$ is the kth Fourier coefficient of the dth degree of freedom of $\mathbf{x}$. Then

$$\Gamma \mathbf{z} = \begin{bmatrix} \sum_{k=-N_H}^{N_H} \mathbf{q}(k) a_{k,0} \\ \vdots \\ \sum_{k=-N_H}^{N_H} \mathbf{q}(k) a_{k,n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \vdots \\ \mathbf{x}_{n-1} \end{bmatrix}$$

where $\mathbf{x}_d = \begin{bmatrix} x_d(t_0) & x_d(t_1) & \cdots & x_d(t_{N-1}) \end{bmatrix}$ contains the values of the dth degree of freedom of $\mathbf{x}(t)$ at $t = t_0, \ldots, t_{N-1}$. Write

$$\mathbf{x}_s := \Gamma \mathbf{z}$$

to denote this sampling of $\mathbf{x}$ at N points in time.

3.2 Computing frequency from time

We now guess at the form of $\Gamma^\dagger$, the pseudo-inverse of Γ, and check that it in fact computes $\mathbf{z} = \Gamma^\dagger \mathbf{x}_s$ We try

$$\Gamma^\dagger := \frac{1}{N} \begin{bmatrix} I_n \otimes \mathbf{q}(-N_H)^* \\ \vdots \\ I_n \otimes \mathbf{q}(N_H)^* \end{bmatrix}. \tag{26}$$

First note

$$\begin{aligned}
\frac{1}{N}(I_n \otimes \mathbf{q}(k)^*)\mathbf{x}_s &= \frac{1}{N}\begin{bmatrix} \mathbf{q}(k)^* & & & \\ & \mathbf{q}(k)^* & & \\ & & \ddots & \\ & & & \mathbf{q}(k)^* \end{bmatrix}\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{n-1} \end{bmatrix} \\
&= \frac{1}{N}\begin{bmatrix} \mathbf{q}(k)^*\mathbf{x}_0 \\ \mathbf{q}(k)^*\mathbf{x}_1 \\ \ddots \\ \mathbf{q}(k)^*\mathbf{x}_{n-1}. \end{bmatrix}
\end{aligned}$$

Now observe that

$$\begin{aligned}
\frac{1}{N}\mathbf{q}(k)^*\mathbf{x}_d &= \frac{1}{N}\begin{bmatrix} e^{-ik\omega t_0} & e^{-ik\omega t_1} & \cdots & e^{-ik\omega t_{N-1}} \end{bmatrix}\begin{bmatrix} x_d(t_0) \\ x_d(t_1) \\ \vdots \\ x_d(t_{N-1}) \end{bmatrix} \\
&= \frac{1}{N}\sum_{j=0}^{N-1} e^{-ik\omega t_j} x_d(t_j) \\
&\approx \frac{\omega}{2\pi}\int_0^{2\pi/\omega} e^{-ik\omega t} x_d(t)\, dt \\
&= a_{k,d},
\end{aligned}$$

the kth Fourier coefficient of the dth degree of freedom of $\mathbf{x}$. Then

$$\frac{1}{N}(I_n \otimes \mathbf{q}(k)^*)\mathbf{x}_s = \begin{bmatrix} a_{k,0} \\ a_{k,1} \\ \vdots \\ a_{k,n-1} \end{bmatrix} = \mathbf{a}_k$$

Now

$$\begin{aligned}
\Gamma^\dagger \mathbf{x}_s &= \frac{1}{N}\begin{bmatrix} I_n \otimes \mathbf{q}(-N_H)^* \\ \vdots \\ I_n \otimes \mathbf{q}(N_H)^* \end{bmatrix}\mathbf{x}_s \\
&= \begin{bmatrix} I_n \otimes \mathbf{q}(-N_H)^*\mathbf{x}_s \\ \vdots \\ I_n \otimes \mathbf{q}(N_H)^*\mathbf{x}_s \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{a}_{-N_H} \\ \vdots \\ \mathbf{a}_{N_H} \end{bmatrix} \\
&= \mathbf{z},
\end{aligned}$$

so we have $\mathbf{z} = \Gamma^\dagger \mathbf{x}_s$, as expected.

3.3 Verifying the pseudo-inverse

We now check that $\Gamma^\dagger\Gamma = I_{(2N_H+1)n}$ (assuming $N \geq 2N_H + 1$). Again using the mixed-product property of the Kronecker product, we note

$$\begin{aligned}
&[I_n \otimes \mathbf{q}(k)^*][I_n \otimes \mathbf{q}(\ell)] \\
&= I_n \otimes [\mathbf{q}(k)^*\mathbf{q}(\ell)].
\end{aligned}$$

Next observe that

$$\begin{aligned}
\frac{1}{N}\mathbf{q}(k)^*\mathbf{q}(\ell) &= \frac{1}{N}\sum_{j=0}^{N-1} e^{-ik\omega t_j} e^{i\ell\omega t_j} \\
&\approx \frac{\omega}{2\pi}\int_0^{2\pi/\omega} e^{-ik\omega t} e^{i\ell\omega t}\, dt \\
&= \langle e^{ik\omega t},\ e^{i\ell\omega t}\rangle \\
&= \delta_{k\ell}.
\end{aligned}$$

Then

$$\begin{aligned}
&\Gamma^\dagger\Gamma \\
&= \frac{1}{N}\begin{bmatrix} I_n \otimes \mathbf{q}(-N_H)^* \\ \vdots \\ I_n \otimes \mathbf{q}(N_H)^* \end{bmatrix} \begin{bmatrix} I_n \otimes \mathbf{q}(-N_H) & \cdots & I_n \mathbf{q}(N_H) \end{bmatrix} \\
&= \frac{1}{N}\begin{bmatrix} I_n \otimes [\mathbf{q}(-N_H)^*\mathbf{q}(-N_H)] & I_n \otimes [\mathbf{q}(-N_H)^*\mathbf{q}(-N_H+1)] & \cdots & I_n \otimes [\mathbf{q}(-N_H)^*\mathbf{q}(N_H)] \\ I_n \otimes [\mathbf{q}(-N_H+1)^*\mathbf{q}(-N_H)] & I_n \otimes [\mathbf{q}(-N_H+1)^*\mathbf{q}(-N_H+1)] & \cdots & \\ \vdots & & \ddots & \\ I_n \otimes [\mathbf{q}(N_H)^*\mathbf{q}(-N_H)] & \cdots & \cdots & I_n \otimes [\mathbf{q}(N_H)^*\mathbf{q}(N_H)] \end{bmatrix} \\
&\approx \begin{bmatrix} I_n \otimes \delta_{-N_H,-N_H} & I_n \otimes \delta_{-N_H,-N_H+1} & \cdots & I_n \otimes \delta_{-N_H,N_H} \\ I_n \otimes \delta_{-N_H+1,-N_H} & I_n \otimes \delta_{-N_H+1,-N_H+1} & \cdots & I_n \otimes \delta_{-N_H+1,N_H} \\ \vdots & & \ddots & \\ I_n \otimes \delta_{N_H,-N_H} & I_n \otimes \delta_{N_H,-N_H+1} & \cdots & I_n \otimes \delta_{N_H,N_H} \end{bmatrix} \\
&= I_{(2N_H+1)n}.
\end{aligned}$$

3.4 Application

Recall that given a function $y(t)$ with Fourier coefficients $\{\widehat{y}_k\}_{k=-N_H}^{N_H}$, the Fourier coefficients of $\dot{y}(t)$ are $\{ik\omega\widehat{y}_k\}_{k=-N_H}^{N_H}$. If we collect the Fourier coefficients of y into a vector $\widehat{\mathbf{y}} = \begin{bmatrix}\widehat{y}_{-N_H} & \cdots & \widehat{y}_{N_H}\end{bmatrix}^\top$, then using the definition of ∇ (14) we can write the Fourier coefficients of $\dot{y}$ as

$$\begin{aligned}
\widehat{\dot{\mathbf{y}}} &= \begin{bmatrix} i(-N_H)\omega\widehat{y}_{-N_H} & \cdots & iN_H\omega\widehat{y}_{N_H}\end{bmatrix}^\top \\
&= \omega\nabla \begin{bmatrix}\widehat{y}_{-N_H} & \cdots & \widehat{y}_{N_H}\end{bmatrix}^\top
\end{aligned}$$

With the Fourier coefficients $\mathbf{z} = \begin{bmatrix}\mathbf{a}_{-N_H} & \cdots & \mathbf{a}_{N_H}\end{bmatrix}$ for the multiple-degree of freedom system $\mathbf{x}(t) \in \mathbb{R}^n$, we can write the Fourier coefficients for $\dot{\mathbf{x}}(t)$ as

$$\begin{aligned}
(\omega\nabla \otimes I_n)\mathbf{z} &= \omega \begin{bmatrix} i(-N_H)I_n & & & \\ & i(-N_H+1)I_n & & \\ & & \ddots & \\ & & & iN_H I_n \end{bmatrix} \begin{bmatrix}\mathbf{a}_{-N_H} \\ \mathbf{a}_{-N_H+1} \\ \vdots \\ \mathbf{a}_{N_H}\end{bmatrix} \\
&= \omega \begin{bmatrix} i(-N_H)\mathbf{a}_{-N_H} & & & \\ & i(-N_H+1)\mathbf{a}_{-N_H+1} & & \\ & & \ddots & \\ & & & iN_H\mathbf{a}_{N_H} \end{bmatrix}.
\end{aligned}$$

Now we can write

$$\mathbf{x}_s = \Gamma\mathbf{z} \tag{27}$$

$$\dot{\mathbf{x}}_s = \omega\Gamma(\nabla \otimes I_n)\mathbf{z} \tag{28}$$

$$\ddot{\mathbf{x}}_s = \omega^2\Gamma(\nabla^2 \otimes I_n)\mathbf{z}. \tag{29}$$

Compare with (15), (16), (17).

Importantly, we can now compute the Fourier coefficients of the nonlinear force $\mathbf{f}_{\rm nl}(\mathbf{x}, \dot{\mathbf{x}})$:

$$\mathbf{b}_{\rm nl}(\mathbf{z}) = \Gamma^\dagger \mathbf{f}_{\rm nl}\big(\mathbf{x}_s,\ \dot{\mathbf{x}}_s\big) = \Gamma^\dagger \mathbf{f}_{\rm nl}\big(\Gamma \mathbf{z},\ \omega\Gamma(\nabla \otimes I_n)\mathbf{z}\big). \tag{30}$$

4 Solving the equation

Define the residual equation

$$\begin{aligned}\mathbf{R}(\mathbf{z}) &\coloneqq \big[\omega^2\nabla^2 \otimes M + \omega\nabla \otimes C + I_{2N_H+1} \otimes K\big]\mathbf{z} + \mathbf{b}_{\rm nl}(\mathbf{z}) - \mathbf{b}_{\rm ext} &(31)\\ &= A(\omega)\mathbf{z} + \mathbf{b}_{\rm nl}(\mathbf{z}) - \mathbf{b}_{\rm ext} &(32)\end{aligned}$$

measuring how well a solution $\mathbf{z}$ satisfies the dynamics. If we have $\mathbf{R}_\mathbf{z}(\mathbf{z}) \coloneqq d\mathbf{R}(\mathbf{z})/d\mathbf{z}$, we can find roots of this equation using the Newton–Raphson method (and related algorithms).

We compute

$$\mathbf{R}_\mathbf{z}(\mathbf{z}) = A(\omega) + \frac{d\mathbf{b}_{\rm nl}(\mathbf{z})}{d\mathbf{z}}.$$

Then using (27), (28), and (30),

$$\begin{aligned}\frac{d\mathbf{b}_{\rm nl}(\mathbf{z})}{d\mathbf{z}} &= \frac{d}{d\mathbf{z}}\Big[\Gamma^\dagger \mathbf{f}_{\rm nl}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\Big]\\ &= \Gamma^\dagger \frac{d}{d\mathbf{z}}\Big[\mathbf{f}_{\rm nl}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\Big]\\ &= \Gamma^\dagger\left[\frac{d}{d\mathbf{x}}\mathbf{f}_{\rm nl}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\frac{d\mathbf{x}_s}{d\mathbf{z}} + \frac{d}{d\dot{\mathbf{x}}}\mathbf{f}_{\rm nl}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\frac{d\dot{\mathbf{x}}_s}{d\mathbf{z}}\right]\\ &= \Gamma^\dagger\left[\frac{d}{d\mathbf{x}}\mathbf{f}_{\rm nl}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\Gamma + \omega\frac{d}{d\dot{\mathbf{x}}}\mathbf{f}_{\rm nl}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\Gamma(\nabla \otimes I_n)\right].\end{aligned}$$

5 Continuation

It is useful to compute harmonic balance solutions as a chosen parameter varies. For example, varying the fundamental forcing frequency and plotting the amplitude of the response yields a *nonlinear frequency response* (NLFR) *curve*. Two techniques which this code (attempts) to implement are *arclength continuation* (in section 5.1) and *pseudo-arclength continuation* (in 5.2).

5.1 Arclength continuation

Define

$$P(\mathbf{z}_{i+1}^k, \omega_{i+1}^k, s) \coloneqq \|\mathbf{z}_{i+1}^k - \mathbf{z}_i\|^2 + (\omega_{i+1}^k - \omega_i)^2 - s^2 \tag{33}$$

where the subscript denotes the ith solution along the NLFR curve and the superscript denotes the kth iteration of Newton–Raphson or another iterative solver. Finding roots $\mathbf{y}_{i+1} = \begin{bmatrix}\mathbf{z}_{i+1}^\top & \omega_{i+1}\end{bmatrix}^\top$ amounts to constraining $\mathbf{y}$ to a hypersphere of radius s centered at $\mathbf{y}_i$. Solutions are found using a predictor-corrector method. Predictions may be obtained using, for example, secant vectors:

$$\mathbf{y}_{i+1}^0 = \mathbf{y}_i + s_{i+1}\frac{\mathbf{y}_i - \mathbf{y}_{i-1}}{\|\mathbf{y}_i - \mathbf{y}_{i-1}\|}. \tag{34}$$

The prediction is corrected by augmenting the residual R

$$\mathbf{R}^{\rm aug}(\mathbf{y}, s) = \begin{bmatrix}\mathbf{R}(\mathbf{y})\\ P(\mathbf{y}, s)\end{bmatrix} \tag{35}$$

and then finding roots using, for example, Newton–Raphson:

$$\begin{bmatrix} \mathbf{R}_{\mathbf{z}}(\mathbf{y}_{i+1}^k) & \mathbf{R}_\omega(\mathbf{y}_{i+1}^k) \\ P_{\mathbf{z}}(\mathbf{y}_{i+1}^k, s) & P_\omega(\mathbf{y}_{i+1}^k, s) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z} \\ \Delta\omega \end{bmatrix} = \mathbf{R}^{\mathrm{aug}}(\mathbf{y}_{i+1}^k, s) \tag{36}$$

$$\mathbf{y}_{i+1}^{k+1} = \mathbf{y}_{i+1}^k + \begin{bmatrix} \Delta\mathbf{z} \\ \Delta\omega \end{bmatrix}. \tag{37}$$

Note

$$\begin{aligned} P_{\mathbf{z}}(\mathbf{y}_{i+1}^k, s) &= \left(\mathbf{z}_{i+1}^k - \mathbf{z}_i\right)^* \\ P_\omega(\mathbf{y}_{i+1}^k, s) &= 2\left(\omega_{i+1}^k - \omega_i\right) \\ \mathbf{R}_\omega(\mathbf{y}_{i+1}^k) &= \left(2\omega\nabla^2 \otimes M + \nabla \otimes C\right)\mathbf{z}_{i+1}^k. \end{aligned}$$

It is common to adjust the step size s using the number of solver iterations. One method is

$$s_{i+1} = s_i 2^{\frac{k_{\mathrm{optimal}} - k_i}{k_{\mathrm{optimal}}}} \tag{38}$$

where k_{optimal} is a hyperparameter chosen according to the solver and the problem.

5.2 Pseudo-arclength continuation

We'll reuse notation from the previous section, but note that these two methods (arclength continuation and pseudo-arclength continuation) are mutually exclusive and so the notation should be unambiguous upon deciding which method to use.

Define

$$P(\mathbf{z}_{i+1}^k, \omega_{i+1}^k) \coloneqq \mathbf{v}_i^*\left(\mathbf{y}_{i+1}^k - \mathbf{y}_{i+1}^0\right) \tag{39}$$

where the subscript denotes the ith solution along the NLFR curve; the superscript denotes the kth iteration of Newton–Raphson or another iterative solver; and $\mathbf{v}_i$ denotes the unit tangent vector to the curve at the ith solution. Finding roots $\mathbf{y}_{i+1} = \begin{bmatrix} \mathbf{z}_{i+1}^\top & \omega_{i+1} \end{bmatrix}^\top$ amounts to constraining $\mathbf{y}$ to a hyperplane perpendicular to the tangent vector $\mathbf{v}_i$ emanating from $\mathbf{y}_i$. The tangent vector $\mathbf{v} = \begin{bmatrix} \mathbf{v}_{\mathbf{z}}^\top & v_\omega \end{bmatrix}^\top$ is computed as

$$\begin{bmatrix} \mathbf{R}_{\mathbf{z}}(\mathbf{y}_i) & \mathbf{R}_\omega(\mathbf{y}_i) \\ \mathbf{v}_{i-1,\,\mathbf{z}}^* & v_{i-1,\,\omega} \end{bmatrix} \mathbf{v}_i = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}. \tag{40}$$

The first (vector) equation computes the tangent vector and the second (scalar) equation ensures $\mathbf{v}_i$ points in the same direction as $\mathbf{v}_{i-1}$ by having a positive inner product. The tangent vector is subsequently normalized to have unit length.

Solutions are found using a predictor-corrector method. Predictions may be obtained using the tangent vector:

$$\mathbf{y}_{i+1}^0 = \mathbf{y}_i + s_{i+1}\mathbf{v}_i. \tag{41}$$

The prediction is corrected by augmenting the residual R

$$\mathbf{R}^{\mathrm{aug}}(\mathbf{y}) = \begin{bmatrix} \mathbf{R}(\mathbf{y}) \\ P(\mathbf{y}) \end{bmatrix} \tag{42}$$

and then finding roots using, for example, Newton–Raphson:

$$\begin{bmatrix} \mathbf{R}_{\mathbf{z}}(\mathbf{y}_{i+1}^k) & \mathbf{R}_\omega(\mathbf{y}_{i+1}^k) \\ P_{\mathbf{z}}(\mathbf{y}_{i+1}^k, s) & P_\omega(\mathbf{y}_{i+1}^k, s) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z} \\ \Delta\omega \end{bmatrix} = \mathbf{R}^{\mathrm{aug}}(\mathbf{y}_{i+1}^k, s) \tag{43}$$

$$\mathbf{y}_{i+1}^{k+1} = \mathbf{y}_{i+1}^k + \begin{bmatrix} \Delta\mathbf{z} \\ \Delta\omega \end{bmatrix}. \tag{44}$$

Note

$$
\begin{aligned}
P_{\mathbf{z}}(\mathbf{y}_{i+1}^k, s) &= \mathbf{v}_{i,\,\mathbf{z}}^* \\
P_{\omega}(\mathbf{y}_{i+1}^k, s) &= v_{i,\,\omega} \\
\mathbf{R}_{\omega}(\mathbf{y}_{i+1}^k) &= \big(2\omega\nabla^2 \otimes M + \nabla \otimes C\big)\mathbf{z}_{i+1}^k.
\end{aligned}
$$

As mentioned in the previous section, it is common to adjust the step size s using the number of solver iterations. In addition to the method discussed there, another method is

$$
s_{i+1} = s_i \frac{k_{\text{optimal}}}{k_i} \tag{45}
$$

where k_{optimal} is again a hyperparameter.

6 It's getting real

Since $\mathbf{x}$ is real, we know $\mathbf{a}_{-k} = \overline{\mathbf{a}_k}$ and $\exp(-ik\omega t) = \overline{\exp(ik\omega t)}$, so we can write

$$
\begin{aligned}
\mathbf{x}(t) &\approx \sum_{k=-N_H}^{N_H} \mathbf{a}_k \exp(ik\omega t) \\
&= \mathbf{a}_0 + \sum_{k=1}^{N_H} \mathbf{a}_k \exp(ik\omega t) + \mathbf{a}_{-k} \exp(-ik\omega t) \\
&= \mathbf{a}_0 + \sum_{k=1}^{N_H} \mathbf{a}_k \exp(ik\omega t) + \overline{\mathbf{a}_k \exp(ik\omega t)} \\
&= \mathbf{a}_0 + 2\sum_{k=1}^{N_H} \Re[\mathbf{a}_k \exp(ik\omega t)].
\end{aligned}
$$

Alternatively, if we define

$$
\widehat{\mathbf{a}}_k := \begin{cases} \mathbf{a}_k/2 & k = 0, \\ \mathbf{a}_k & k \neq 0, \end{cases}
$$

then since $\mathbf{x}$ is real-valued, we know that $\mathbf{a}_0 \in \mathbb{R}$, and so we can write

$$
\begin{aligned}
\mathbf{x}(t) &\approx \sum_{k=0}^{N_H} \widehat{\mathbf{a}}_k \exp(ik\omega t) + \overline{\widehat{\mathbf{a}}_k \exp(ik\omega t)} \\
&= 2\sum_{k=0}^{N_H} \Re[\widehat{\mathbf{a}}_k \exp(ik\omega t)]
\end{aligned}
$$

We can write the forces similarly:

$$
\begin{aligned}
\mathbf{f}_{\text{nl}}(\mathbf{x}, \dot{\mathbf{x}}) &= \mathbf{c}_0 + 2\sum_{k=1}^{N_H} \Re[\mathbf{c}_k \exp(ik\omega t)] \\
&= 2\sum_{k=0}^{N_H} \Re[\widehat{\mathbf{c}}_k \exp(ik\omega t)] \\
\mathbf{f}_{\text{ext}}(t) &= \mathbf{d}_0 + 2\sum_{k=1}^{N_H} \Re[\mathbf{d}_k \exp(ik\omega t)] \\
&= 2\sum_{k=0}^{N_H} \Re\Big[\widehat{\mathbf{d}}_k \exp(ik\omega t)\Big].
\end{aligned}
$$

If we write $H \coloneqq \operatorname{diag}(1/2, 1, \ldots, 1)_{N_H+1} \otimes I_n$, then we can store the Fourier coefficients

$$
\begin{aligned}
\widehat{\mathbf{z}} &\coloneqq \begin{bmatrix} \widehat{\mathbf{a}}_0^\top & \widehat{\mathbf{a}}_1^\top & \cdots & \widehat{\mathbf{a}}_{N_H}^\top \end{bmatrix}^\top \in \mathbb{C}^{n(N_H+1)} \\
&= \begin{bmatrix} \mathbf{a}_0^\top/2 & \mathbf{a}_1^\top & \cdots & \mathbf{a}_{N_H}^\top \end{bmatrix}^\top = H\mathbf{z} \\
\widehat{\mathbf{b}}_{\text{nl}} &\coloneqq \begin{bmatrix} \widehat{\mathbf{c}}_0^\top & \widehat{\mathbf{c}}_1^\top & \cdots & \widehat{\mathbf{c}}_{N_H}^\top \end{bmatrix}^\top \\
&= \begin{bmatrix} \mathbf{c}_0^\top/2 & \mathbf{c}_1^\top & \cdots & \mathbf{c}_{N_H}^\top \end{bmatrix}^\top = H\mathbf{b}_{\text{nl}} \\
\widehat{\mathbf{b}}_{\text{ext}} &\coloneqq \begin{bmatrix} \widehat{\mathbf{d}}_0^\top & \widehat{\mathbf{d}}_1^\top & \cdots & \widehat{\mathbf{d}}_{N_H}^\top \end{bmatrix}^\top \\
&= \begin{bmatrix} \mathbf{d}_0^\top/2 & \mathbf{d}_1^\top & \cdots & \mathbf{d}_{N_H}^\top \end{bmatrix}^\top = H\mathbf{b}_{\text{ext}}
\end{aligned}
$$

and again write $\widehat{\mathbf{b}}(\mathbf{z}, \omega) = \widehat{\mathbf{b}}_{\text{ext}}(\omega) - \widehat{\mathbf{b}}_{\text{nl}}(\mathbf{z})$.

6.1 Projecting the governing equation

Then writing

$$
\widehat{Q}(t, \omega) \coloneqq \begin{bmatrix} 1 & e^{i\omega t} & \cdots & e^{iN_H\omega t} \end{bmatrix} \in \mathbb{C}^{1\times(N_H+1)} \tag{46}
$$

we have

$$
\begin{aligned}
\mathbf{x}(t, \omega) &= 2\Re\Big[(\widehat{Q} \otimes I_n)\widehat{\mathbf{z}}\Big] &(47)\\
\mathbf{f}_{\text{nl}}(\mathbf{x}, \dot{\mathbf{x}}, \omega) &= 2\Re\Big[(\widehat{Q} \otimes I_n)\widehat{\mathbf{b}}_{\text{nl}}\Big] &(48)\\
\mathbf{f}_{\text{ext}}(t, \omega) &= 2\Re\Big[(\widehat{Q} \otimes I_n)\widehat{\mathbf{b}}_{\text{ext}}\Big] &(49)\\
\mathbf{f}(t, \mathbf{x}, \dot{\mathbf{x}}, \omega) &= 2\Re\Big[(\widehat{Q} \otimes I_n)\widehat{\mathbf{b}}\Big]. &(50)
\end{aligned}
$$

We also have

$$
\begin{aligned}
\dot{\mathbf{x}}(t, \omega) &= 2\omega\Re\Big[(\widehat{Q}\nabla \otimes I_n)\widehat{\mathbf{z}}\Big] &(51)\\
\ddot{\mathbf{x}}(t, \omega) &= 2\omega^2\Re\Big[\Big(\widehat{Q}\nabla^2 \otimes I_n\Big)\widehat{\mathbf{z}}\Big]. &(52)
\end{aligned}
$$

We arrive at the system

$$
2\Re\Big[\Big(\omega^2\Big(\widehat{Q}(t)\widehat{\nabla}^2\Big) \otimes M + \omega\Big(\widehat{Q}(t)\widehat{\nabla}\Big) \otimes C + \widehat{Q}(t) \otimes K\Big)\widehat{\mathbf{z}} - \Big(\widehat{Q}(t) \otimes I_n\Big)\widehat{\mathbf{b}}\Big] = 0. \tag{53}
$$

We define

$$
\begin{aligned}
\widehat{\widetilde{\nabla}} &\coloneqq \begin{bmatrix} 0 & i & \cdots & iN_H \end{bmatrix} &(54)\\
\widehat{\nabla} &\coloneqq \operatorname{diag}\widehat{\widetilde{\nabla}}. &(55)
\end{aligned}
$$

We then perform Fourier–Galerkin projection on the system. We project the first term after rewriting the real operator as a sum of the term and its complex conjugate:

$$
\begin{aligned}
&\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \widehat{Q}(t)^* 2\Re\Big[\omega^2\Big(\widehat{Q}(t)\nabla^2\Big) \otimes M\Big]\,dt \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} Q(t)^*\Big[\omega^2\Big(\widehat{Q}(t)\nabla^2\Big) \otimes M + \overline{\omega^2\Big(\widehat{Q}(t)\nabla^2\Big) \otimes M}\Big]\,dt \\
&= \omega^2\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \Big(\widehat{Q}(t)^*\Big[\Big(\widehat{Q}(t)\nabla^2\Big) \otimes M\Big] + \widehat{Q}(t)^*\Big[\overline{\Big(\widehat{Q}(t)\nabla^2\Big)} \otimes M\Big]\Big)\,dt.
\end{aligned}
$$

The first summand works out to $\omega^2\widehat{\nabla}^2 \otimes M$, the only difference in the derivation from that in 2.2 being $\widehat{Q}$ instead of Q. For the second summand, we find

$$
\begin{aligned}
&\omega^2 \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \widehat{Q}(t)^* \left[\overline{\left(\widehat{Q}(t)\nabla^2\right)} \otimes M\right] dt \\
&= \omega^2 \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[\widehat{Q}(t)^* \otimes 1\right] \left[\overline{\left(\widehat{Q}(t)\nabla^2\right)} \otimes M\right] dt \\
&= \omega^2 \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[\widehat{Q}(t)^* \overline{\left(\widehat{Q}(t)\nabla^2\right)}\right] \otimes [1M]\, dt \\
&= \omega^2 \left[\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \widehat{Q}(t)^* \overline{\widehat{Q}(t)}\, dt\; \overline{\nabla^2}\right] \otimes M \\
&= \omega^2 \left[\operatorname{diag}(1, 0, \ldots, 0)_{N_H+1}\; \overline{\nabla^2}\right] \otimes M \\
&= 0_{N_H+1}.
\end{aligned}
$$

Note that each entry of the integral on the third-to-last line is of the form $\langle \exp(ik\omega t),\ \exp(-i\ell\omega t)\rangle = \delta_{k(-\ell)}$, and since $k, \ell \geq 0$, only the (0, 0) entry is 1. On the other hand $\overline{\nabla^2}_{0,0} = 0$, so the entire expression is zero.

Moving on to the other terms in (53), the non-complex conjugated terms work out as in 2.2. The complex conjugate of the second term, $\omega\left(\widehat{Q}(t)\widehat{\nabla}\right) \otimes C$, works out to zero like the first term. We now look at the last two terms.

$$
\begin{aligned}
&\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \widehat{Q}(t)^* \left[\overline{\widehat{Q}(t)} \otimes K\right] dt \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[\widehat{Q}(t)^* \otimes 1\right] \left[\overline{\widehat{Q}(t)} \otimes K\right] dt \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[\widehat{Q}(t)^* \overline{\widehat{Q}(t)}\right] \otimes [1K]\, dt \\
&= \left[\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \widehat{Q}(t)^* \overline{\widehat{Q}(t)}\, dt\right] \otimes K \\
&= \operatorname{diag}(1, 0, \ldots, 0)_{N_H+1} \otimes K.
\end{aligned}
$$

The final term (involving $\widehat{\mathbf{b}}$) becomes $\operatorname{diag}(1, 0, \ldots, 0)_{N_H+1} \otimes I_n$.

Summing the results from the non-complex conjugated terms and the conjugated terms, our algebraic system is

$$
\left(\omega^2\widehat{\nabla}^2 \otimes M + \omega\widehat{\nabla} \otimes C + \operatorname{diag}(2, 1, \ldots, 1)_{N_H+1} \otimes K\right)\widehat{\mathbf{z}} - (\operatorname{diag}(2, 1, \ldots, 1)_{N_H+1} \otimes I_n)\widehat{\mathbf{b}} = 0.
$$

However, we can simplify this in a fortunate (and perhaps surprising) way.

$$
\begin{aligned}
&(\operatorname{diag}(2, 1, \ldots, 1)_{N_H+1} \otimes I_n)\widehat{\mathbf{b}} \\
&= (\operatorname{diag}(2, 1, \ldots, 1)_{N_H+1} \otimes I_n)H\mathbf{b} \\
&= (\operatorname{diag}(2, 1, \ldots, 1)_{N_H+1} \otimes I_n)(\operatorname{diag}(1/2, 1, \ldots, 1)_{N_H+1} \otimes I_n)\mathbf{b} \\
&= \mathbf{b}
\end{aligned}
$$

$$
\begin{aligned}
&\left(\omega^2\widehat{\nabla}^2 \otimes M + \omega\widehat{\nabla} \otimes C + \operatorname{diag}(2,1,\dots,1)_{N_H+1} \otimes K\right)\widehat{\mathbf{z}} \\
&= \left(\omega^2\widehat{\nabla}^2 \otimes M + \omega\widehat{\nabla} \otimes C + \operatorname{diag}(2,1,\dots,1)_{N_H+1} \otimes K\right)H\mathbf{z} \\
&= \left(\omega^2\widehat{\nabla}^2 \otimes M + \omega\widehat{\nabla} \otimes C + \operatorname{diag}(2,1,\dots,1)_{N_H+1} \otimes K\right)(\operatorname{diag}(1/2,1,\dots,1)_{N_H+1} \otimes I_n)\mathbf{z} \\
&= \left[\omega^2\left(\operatorname{diag}(1/2,1,\dots,1)_{N_H+1}\widehat{\nabla}^2\right) \otimes M + \omega\left(\operatorname{diag}(1/2,1,\dots,1)_{N_H+1}\widehat{\nabla}\right) \otimes C + I_{N_H+1} \otimes K\right]\mathbf{z} \\
&= \left[\omega^2\widehat{\nabla}^2 \otimes M + \omega\widehat{\nabla} \otimes C + I_{N_H+1} \otimes K\right]\mathbf{z}.
\end{aligned}
$$

We define $\widehat{A}(\omega) := \left(\omega^2\widehat{\nabla}^2 \otimes M + \omega\widehat{\nabla} \otimes C + I_{N_H+1} \otimes K\right)$ and obtain the system

$$
\widehat{A}(\omega)\mathbf{z} = \mathbf{b}. \tag{56}
$$

Note the wonderful similarity to (23).

6.2 Computing the nonlinear force

We reuse

$$
\mathbf{q}(k) := \begin{bmatrix} e^{ik\omega t_0} \\ e^{ik\omega t_1} \\ \vdots \\ e^{ik\omega t_{N-1}} \end{bmatrix} \tag{57}
$$

and define

$$
\widehat{\Gamma} := \begin{bmatrix} I_n \otimes \mathbf{q}(0) & I_n \otimes \mathbf{q}(1) & \cdots & I_n \mathbf{q}(N_H) \end{bmatrix}. \tag{58}
$$

Then

$$
\begin{aligned}
2\Re\left[\widehat{\Gamma}\widehat{\mathbf{z}}\right] &= 2\Re[(I_n \otimes \mathbf{q}(0))\widehat{\mathbf{a}}_0 + \dots + (I_n \otimes \mathbf{q}(N_H))\widehat{\mathbf{a}}_{N_H}] \\
&= \mathbf{x}_s.
\end{aligned}
$$

Similarly, defining

$$
\widehat{\Gamma}^\dagger := \frac{1}{N}\begin{bmatrix} I_n \otimes \mathbf{q}(0)^* \\ \vdots \\ I_n \otimes \mathbf{q}(N_H)^* \end{bmatrix} \tag{59}
$$

leads to

$$
\begin{aligned}
\widehat{\Gamma}^\dagger \mathbf{x}_s &= \mathbf{z} \\
H\widehat{\Gamma}^\dagger \mathbf{x}_s &= \widehat{\mathbf{z}}.
\end{aligned}
$$

We then can write

$$
\mathbf{x}_s = 2\Re\left[\widehat{\Gamma}\widehat{\mathbf{z}}\right] \tag{60}
$$

$$
\dot{\mathbf{x}}_s = 2\omega\Re\left[\widehat{\Gamma}(\widehat{\nabla} \otimes I_n)\widehat{\mathbf{z}}\right] \tag{61}
$$

$$
\ddot{\mathbf{x}}_s = 2\omega^2\Re\left[\widehat{\Gamma}(\widehat{\nabla}^2 \otimes I_n)\widehat{\mathbf{z}}\right]. \tag{62}
$$

Additionally, we have

$$
\mathbf{b}_{\text{nl}}(\mathbf{z}) = \Gamma^\dagger\, \mathbf{f}_{\text{nl}}\big(\mathbf{x}_s,\ \dot{\mathbf{x}}_s\big). \tag{63}
$$

6.3 Solving the equation

Define the residual

$$\widehat{\mathbf{R}}(\mathbf{z}) := \widehat{A}(\omega)\mathbf{z} - \mathbf{b}. \tag{64}$$

We need to compute

$$\widehat{\mathbf{R}}_{\mathbf{z}}(\mathbf{z}) = \widehat{A}(\omega) + \frac{d\mathbf{b}_{\mathrm{nl}}(\mathbf{z})}{d\mathbf{z}}.$$

Observe,

$$\begin{aligned}
\frac{d\mathbf{b}_{\mathrm{nl}}(\mathbf{z})}{d\mathbf{z}} &= \frac{d}{d\mathbf{z}}\left[\Gamma^{\dagger}\mathbf{f}_{\mathrm{nl}}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\right] \\
&= \Gamma^{\dagger}\frac{d}{d\mathbf{z}}\left[\mathbf{f}_{\mathrm{nl}}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\right] \\
&= \Gamma^{\dagger}\left[\frac{d}{d\mathbf{x}}\mathbf{f}_{\mathrm{nl}}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\frac{d\mathbf{x}_s}{d\mathbf{z}} + \frac{d}{d\dot{\mathbf{x}}}\mathbf{f}_{\mathrm{nl}}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\frac{d\dot{\mathbf{x}}_s}{d\mathbf{z}}\right] \\
&= \Gamma^{\dagger}\left[\frac{d}{d\mathbf{x}}\mathbf{f}_{\mathrm{nl}}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\widehat{\Gamma} + \omega\frac{d}{d\dot{\mathbf{x}}}\mathbf{f}_{\mathrm{nl}}(\mathbf{x}_s,\ \dot{\mathbf{x}}_s)\widehat{\Gamma}(\widehat{\nabla}\otimes I_n)\right].
\end{aligned}$$

6.4 Continuation

I believe continuation should be basically the same as in 5.1 and 5.2. However, I haven't gotten neither arclength nor pseudo-arclength continuation to continue past a fold bifurcation in the NLFR curve. And I haven't implemented either method without the real assumption, so I don't have a successful baseline to compare against in either case.

References

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