2023: A SPACE *ODE*-SSEY

EJ MERCER, NATHAN SCHILL, DALLIN SEYFRIED, BRIGG TRENDLER

ABSTRACT. In this paper, we explore ODE-based models for the laws of gravity. We begin with an ODE form of Newton's Law of Gravity, and derive additional models based on variants of that original system. By considering various factors like variable gravity, different spatial norms, and different relationships to radius, we explore mathematical implications of this model. Through doing this, we develop a greater understanding of the key features of this model, as well as its potential applications.

1. Background/Motivation

"The solar system is a reminder of our place in the universe, a tiny blue dot in a vast expanse of stars."—Stephen Hawking

Newton's law of gravity states that every particle attracts every other particle in the universe with a force proportional to the product of their masses and inversely proportional to the square of the distance between their centers of mass. This law has served as the cornerstone for much of planetary modeling research since the late 1600's [1].

Our project aims to take this fundamental physical principle and explore how other factors would alter the behavior of the Earth's trajectory from a theoretical standpoint. In particular, we seek to explore variants of the inverse relation between gravity and distance, different propulsion scenarios, orbits in non-Euclidean space, and evaluating gravity as a function of time.

1.1. **Related Work.** We have from [2] some previous thought given to different variants of gravity's relationship with distance (an example is gravity being directly proportional to the square of the distance). With regards to propulsion, in 1925, Walter Hohmann discovered a very intriguing trans-orbital maneuver [3], consisting of applying a velocity to get into a transfer orbit, and then another velocity to re-enter a stable orbit (at a different radius).

2. Modeling

The base model which we will be using is derived from the following equation (the acceleration portion of Newton's Law of Gravitation), where

Date: November 21, 2023.

$\vec{x} = (x, y)$ is the position of the Earth:

$$\ddot{\vec{x}} = -\frac{\vec{x}}{||\vec{x}||_2}\frac{m_{sun}}{||\vec{x}||_2^2}G \tag{1}$$

This is converted to a first order system by letting $z = (x, y, \dot{x}, \dot{y})^T$ to produce:

$$\dot{\vec{z}} = \frac{d}{dt}\begin{bmatrix} x & y & \dot{x} & \dot{y} \end{bmatrix}^T = \begin{bmatrix} \dot{x} & \dot{y} & -\frac{x m_{sun}}{||(x,y)^T||_2^3}G & -\frac{y m_{sun}}{||(x,y)^T||_2^3}G \end{bmatrix}^T$$

where G is the gravitational constant and m_{sun} is the mass of the sun. Starting with this model, we make the following modifications:

2.1. **Exponents on radius.** In the true formula of force due to gravity, the force is inversely proportional to the *square* of the distance between the bodies. Call this exponent on the distance g_{exp}. This section explores the effect of changing $g_{exp} = 2$ to other values.

2.1.1. *Velocity for orbit of constant radius.* To calculate the velocity of a stable orbit, we use the formula for the centripetal force of a body (the earth) swinging around another (the sun), which is $F_c = \frac{m_{earth}v^2}{r}$ where v is the tangent velocity. On the other hand, the force of gravity is $F_g = \frac{Gm_{earth}m_{sun}}{r^{g_{exp}}}$ where r is the distance between the bodies. To keep the earth at a constant distance from the sun, we need these forces to be equal, i.e., $F_g = F_c$. Solving, we find the equation for v_{escape}, shown below.

2.1.2. *Escape velocity.* From [5] we find the derivation for the velocity required to escape the sun's gravity. Replacing the exponent 2 on the radius r with g_{exp}, we follow the same steps and obtain v_{const}.

$$v_{escape} = \sqrt{\frac{2Gm_{sun}}{(g_{exp}-1)r_0^{g_{exp}-1}}}, \quad v_{const} = \sqrt{\frac{Gm_{sun}}{r_0^{g_{exp}-1}}}$$

(r_0 is the initial distance of earth from the sun)

2.2. **Propulsion.**

2.2.1. *Booster Rockets.* Imagine a scenario in which we have recently discovered that the sun will collapse into a black hole in a known amount of time. With the support of extraterrestrial beings, we've attached booster rockets to our lovely blue marble so we can gradually increase the tangential velocity to escape the Sun's orbit. Assuming the booster rockets apply constant tangential force with infinite fuel supply and have no mass (or detrimental effect to the Earth's inhabitants), we can compute the total force needed to escape the Sun's orbit by taking $F_{total} = m_e a_{total}$ where $a_{total} = -\frac{\vec{x}}{||\vec{x}||_2}\frac{m_{sun}}{||\vec{x}||_2^2}G + a_{boost}$. We calculate a_{boost} to be $a_{boost} = (v_{esc} - v_i)/T$ where v_{esc}, v_i, and T denote the escape velocity, initial velocity, and total time before the Sun's explosion in seconds, respectively. This gives us the acceleration needed to be applied at each time step in order to gain enough velocity to escape the Sun's orbit.

2.2.2. *Hohmann Transfer Orbit.* Building off of the booster rocket model, we can update our model to handle reducing the distance of the Earth to the Sun while still maintaining a stable orbit. For spacecraft, this maneuver is known as the Hohmann Transfer Orbit. Combining what we know about the Hohmann Transfer Orbit from [6] with our model (1) yields

$$\Delta v_i = \sqrt{\frac{Gm_{sun}}{||\vec{x}_i||}}\left(\sqrt{\frac{2||\vec{x}_i||}{||\vec{x}_0|| + ||\vec{x}_1||}} - 1\right),$$

where $||\vec{x}_0||$ and $||\vec{x}_1||$ represent the initial and final radii, respectively. We can then boost our tangential velocity $\dot{\vec{x_0}}$ with our initial change in velocity Δv_0 to switch to the transfer orbit. Once we have hit the desired radius, we can then decelerate with another change in velocity Δv_1 to give us our final tangential velocity.

2.3. **Norms.** What would happen if Newton's Laws weren't observed in Euclidean space under the two-norm, but instead arose from some generic p-norm like the one-norm? Under the two-norm, orbits behave in near-circular or elliptical orbits, so we assume that the radius from the Sun to Earth is constant. With the one-norm, we would expect a diamond shape (as that is the shape of the one-norm unit ball in $\mathbb{R}^2$). If the dimensions of G stay the same, via dimensional analysis we can determine the equations do not change significantly. This gave us the relation:

$$\ddot{\vec{x}} = -\frac{\vec{x}}{||\vec{x}||_1}\frac{m_{sun}}{||\vec{x}||_1^2}G$$

Now, suppose we modify the dimensions of the gravitational constant G so that the units became L^2/MT^2 instead of L^3/MT^2. This allows us to drop the exponent on the one-norm of the position vector. The reasoning here is that $||x||_2^2 = \sqrt{x^2 + ...}^2 = x^2 + ...$, and $||x||_1 = (|x| + ...)^1 = |x| + ...$ already has a similar form on its own. This gives us a slightly altered version:

$$\ddot{\vec{x}} = -\frac{\vec{x}}{||\vec{x}||_1}\frac{m_{sun}}{||\vec{x}||_1^1}G$$

2.4. **Gravity varying with time.** Another possibility we explore is the effects of gravity varying as a function of time. The derivation for this model is fairly simple, as it just involves replacing the constant G in (1) with a function. Initially, we had G as a function of time, $G(t)$, in years. However, we realized this did not lend itself to an easy intuition for what gravity was doing as the time progressed—since the definition of a 'year' is one orbit of the Earth, and changing gravity changes that orbit. So we also modeled $G(\theta)$, where θ represents the angle between the Earth and the sun. This allows us to explore asymmetrical gravity 'fields' emitting from the sun; for our exploration we will try various arbitrarily-chosen, positive sinusoidal functions.

3. Results

3.1. Exponents on radius. Figure 1 demonstrates the impact of keeping initial velocity and radius constant while varying the exponent g_{exp} on the distance. (Note the initial velocity and radius are kept as those required to keep a stable orbit with $g_{exp} = 2$.) Observe that for $g_{exp} < 2$, the orbit is elliptical and the earth is at its farthest from the sun at time 0. For $g_{exp} > 2$, the orbit may be elliptical with the earth closest to the sun at time 0, or if the exponent is large enough, the trajectory escapes the sun's gravity.

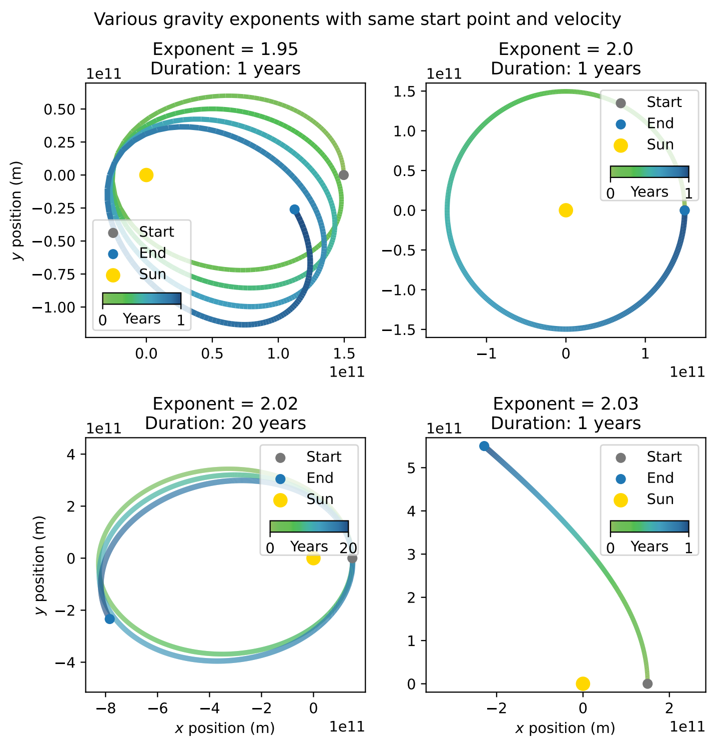
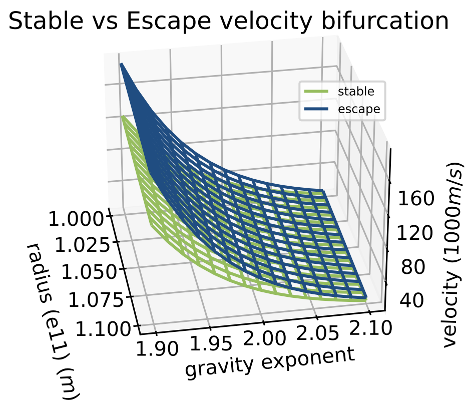


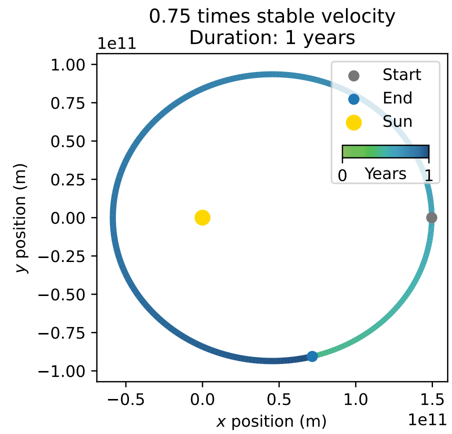
FIGURE 1. For values of g_{exp} other than 2, the orbit tends to drift, unless the exponent is too large, in which case the start velocity is sufficient for escape.

The escape velocity v_{escape} essentially gives a bifurcation point with respect to starting radius and g_{exp}. Initial velocities below this threshold yield orbits (though not circular), while those above this threshold make the sun an unstable equilibrium with respect to position. See Figure 2A.

Referring again to Figure 2A, we find something odd. Given an exponent $g_{exp} \neq 2$, when the initial velocity is not equal to that required to maintain



(A)



(B)

FIGURE 2. Plot A is of v_{const} and v_{escape} as functions of r_0 and g_{exp}. Plot A shows the elliptical orbit which results if $g_{exp} = 2$ and the initial velocity is lower than the required velocity for a circular orbit.

a circular orbit for that exponent, the orbits are not only elliptical but drift, whereas when $g_{exp} = 2$ the ellipse is maintained for all time (see Figure 2B). The reasons for this difference are not clear to us. Simulations not shown verify that for $g_{exp} \neq 2$, the corresponding v_{stable} yield circular, non-drifting orbits. This phenomenon is known as apsidal precession (see [7]).

3.2. **Propulsion.**

3.2.1. *Booster Rockets.* Using our model described above, we looked at three scenarios, one in which the Sun collapses after 1 year, one after 10 years, and one after 100 years. Increasing the time length higher than approximately 300 years of simulation resulted in large calculation times.

In Figure 3, we observe that after about 1 year of increased acceleration, or boost, the earth successfully escapes the Sun's orbit as shown in years 2 and 3.

In Figures 4 and 5, we observe some unexpected behavior. Instead of escaping at the 10 and 100 year marks, respectively, the earth appears to orbit for several years beyond the expected time frame before it finally escapes the Sun's orbit.

3.2.2. *Hohmann Transfer Orbit.* We applied the Hohmann equations to Earth in Figure 6 with the goal to cut its orbital radius in half. In this figure we have the outer and inner circles representing the initial and final stable orbits respectively. The ellipse connecting them is the path the Hohmann Transfer Orbit takes to reduce its orbit down to that of the inner circle.

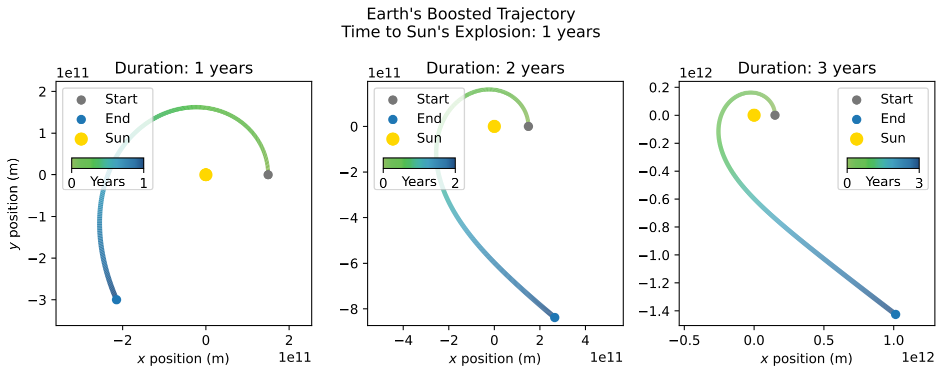


FIGURE 3. Trajectory of the Earth given 1 year before the Sun implodes

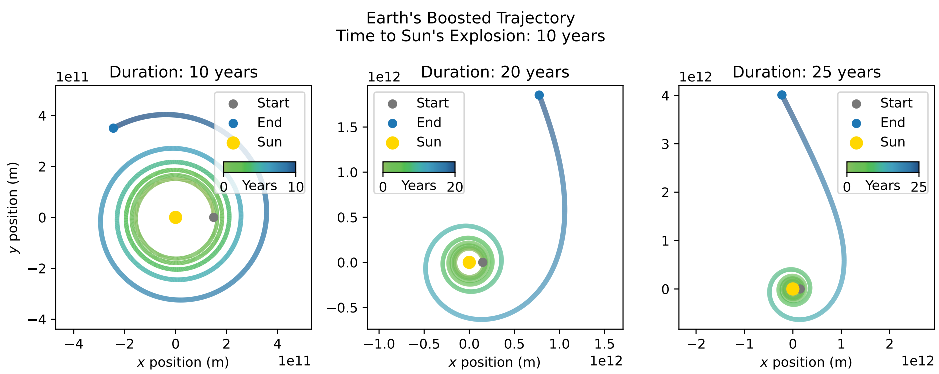


FIGURE 4. Trajectory of the Earth given 10 years before the Sun implodes

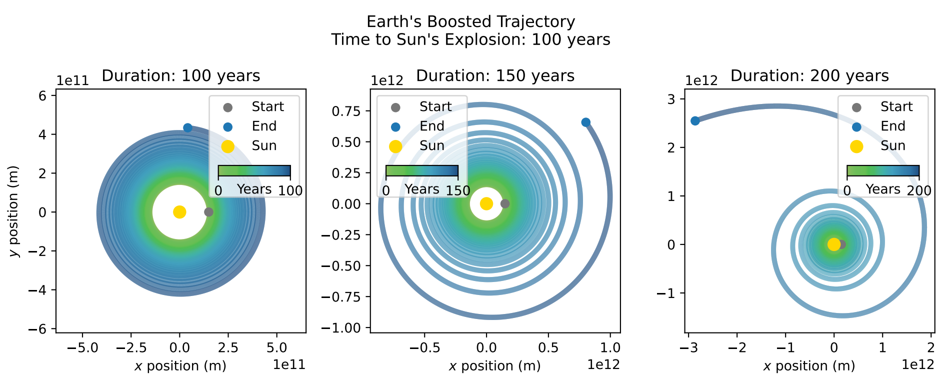


FIGURE 5. Trajectory of the Earth given 100 years before the Sun implodes

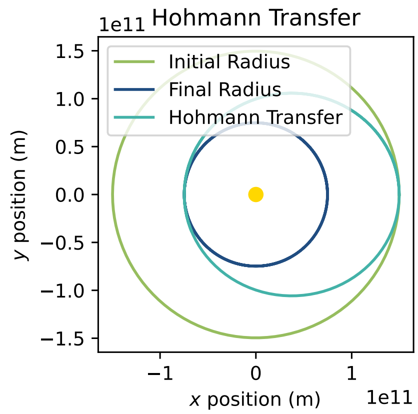


FIGURE 6. Hohmann Transfer Orbit from Earth's current orbit to half the radius

3.3. **Norms.** Figure 7 shows two plots of our two models for the one-norm. In the first plot our orbit takes on an elliptical shape bringing us a little too close to the sun for comfort. In the second we see an incredibly fast orbit taking only about 15 minutes to make multiple star shaped orbits.

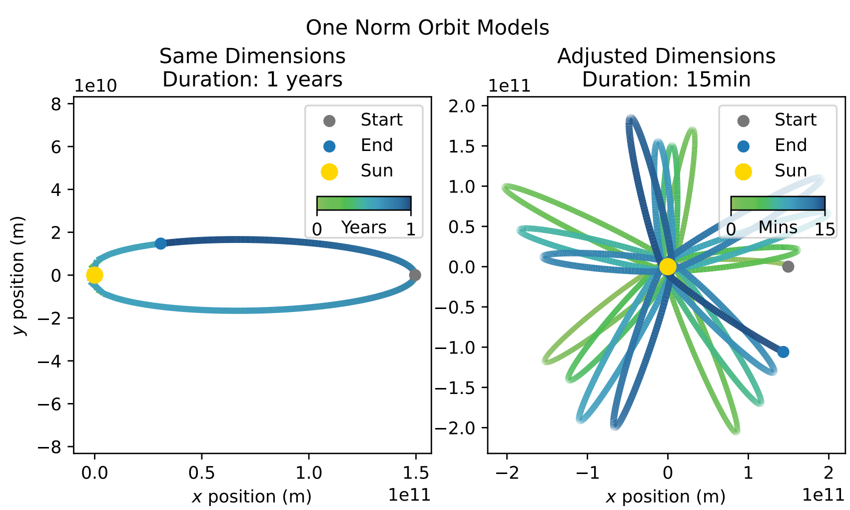


FIGURE 7. One-norm orbit using a change in norms

Both of these figures fail to form the familiar diamond-shaped unit sphere under the one-norm, meaning that our model does not yet correctly model stable orbits under the one-norm.

3.4. **Gravity varying with time.** This model had some interesting results. We found that if the function G depended on time, then the orbit would almost never be stable, and Earth would drift away from the sun. (Figure 8 shows one instance of this.) On the other hand, if G depended on theta, then the orbit would be stable but asymmetrical (with asymmetries corresponding to how G was defined), as suggested by Figure 9.

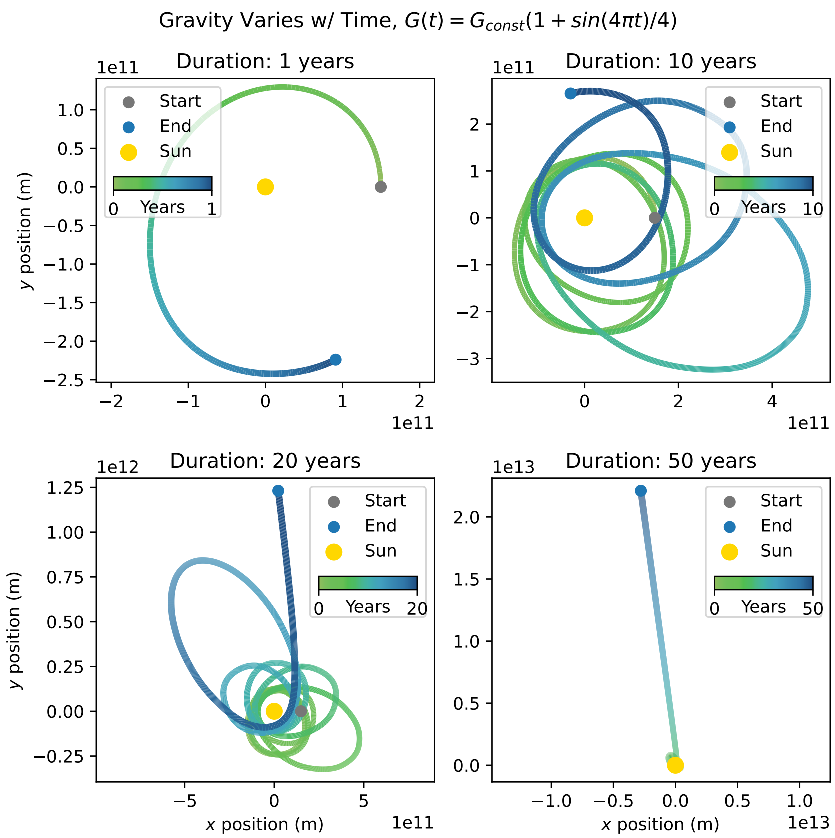


FIGURE 8. Trajectories if gravity varies as a function of time

4. Analysis/Conclusions

4.1. **Exponents on radius.** We found that this model breaks down at the origin due to division by zero. (This is true of Newton's law of gravity as well, but two objects cannot occupy the same space, so it is not an issue.) We attempted to linearize the system at the origin as if it were an equilibrium, but this failed for the same reason. Something unique about this system is that as we approach the origin, acceleration (and thus velocity) approaches infinity, but the origin is the only thing we could consider to be a 'stable' position.

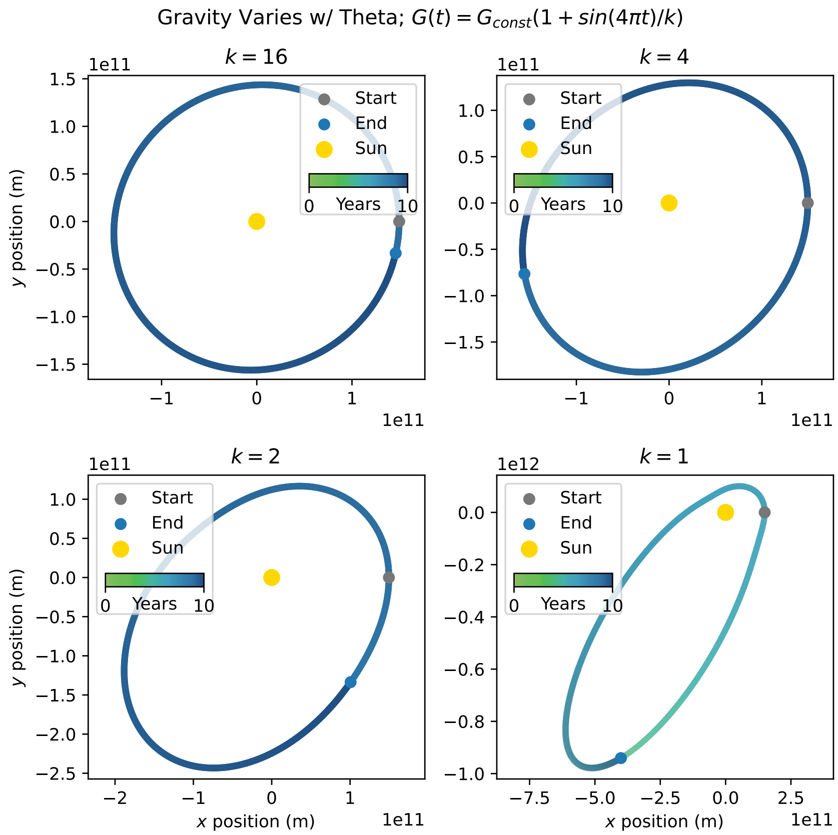


FIGURE 9. Trajectories if gravity varies with the Earth's angle θ to the sun.

With more time, it would be interesting to further investigate how different exponents on the radius influence the precession of the Earth's elliptical orbit.

4.2. **Propulsion.**

4.2.1. *Booster Rockets.* One potential reason for the inconsistent behavior described in section 3.2.1 is that our model does not accurately account for the change in the distance from the Earth to the Sun as the constant tangential force is applied. As the Earth increases in tangential velocity, it begins to move further from the Sun and the force the Earth experiences due to gravity decreases.

We also observed that the speed of the Earth increases initially, and then decreases as the earth moves away from the sun. As of right now, we are unsure of what is causing this behavior. In the future, we wish to explore more in depth what could explain the reduction of speed of the Earth's orbit and how adjusting the escape velocity as the Earth moves away from the Sun affects the outcome of the simulation.

4.2.2. *Hohmann Transfer Orbit.* We were very satisfied with the accuracy of this model; it correctly depicted a change in orbits.

4.3. **Norms.** After analyzing the results of our model operating under a different norm it appears that we got closer to approximating it using our modified dimensional equation than we did by keeping the dimensions constant. Both models still utterly fail in taking the diamond shape which would give constant one-norm radius. The Earth is forming an elliptical orbit in one of the plots, and a very erratic star shaped orbit in the other.

One problem we ran into is that from [4] the one-norms is not unitarily invariant (i.e. if multiplied by an orthogonal matrix like a rotation matrix, the length under the one-norm may scale up or down). This means that as the Earth orbits around the Sun, its one-norm radius may change. We could also look into experimenting with the gravitational constant and building geometric intuition for how Newton's Laws would work under different norms. We learned much about dimensional analysis in creating variants of models. It's a powerful tool that helped us identify potential places that we could change and still have a dimension-wise sensible model.

4.4. **Gravity varying with time.** This model led to some interesting intuitions about gravity. With gravity varying with time, there is not a way to produce a stable orbit, unless the corresponding function happens to have periodicity equal to some multiple of the period of the orbit. If this is not the case, the 'amount' of gravity experienced will be different with each orbit, resulting in orbits of different lengths. This variance in lengths accumulates over time, resulting in the orbit 'drifting', and the planet eventually escaping the field of the sun.

On the other hand, the second definition of the gravity function will not affect the stability of an orbit, so long as the function is always positive. (If it is negative, the sun would be briefly repelling the Earth, which would likely result in escapement.) The angular momentum of the Earth means that each point in the orbit has a unique angle. Further, the periodicity of the gravity function is guaranteed to match the periodicity of the orbit, and so the planet will always be experiencing the same gravity at a given point in its orbit. Further work could be done to determine exactly what conditions on these orbit functions result in stable orbits, and how they affect the associated escape velocities.

References

[1] *Gravity & Mechanics - NASA Science.* (n.d.). https://science.nasa.gov/learn/basicsofspaceflight/chapter33

[2] *If the gravitational force were inversely proportional to distance (rather than distance squared), will celestial bodies fall into each other?* (n.d.). Physics Stack Exchange. https://physics.stackexchange.com/questions/612075/ifthe-gravitationalforcewereinverselyproportionalto-distanceratherthan

[3] *Trajectories* - NASA Science. (n.d.). https://science.nasa.gov/learn/basicsof-spaceflight/chapter41/

[4] *Are matrix p-norms unitary invariant?* (n.d.). Mathematics Stack Exchange. https://math.stackexchange.com/questions/468799/arematrixpnorms-unitaryinvariant

[5] Wikipedia contributors. (2023, November 25). *Escape velocity.* In Wikipedia, The Free Encyclopedia. Retrieved 04:02, December 8, 2023, from https://en.wikipedia.org/wiki/Escape_velocity

[6] Wikipedia contributors. (2023, November 30). *Hohmann transfer orbit.* In Wikipedia, The Free Encyclopedia. Retrieved 04:07, December 8, 2023, from https://en.wikipedia.org/wiki/Hohmann_transfer_orbit

[7] Wikipedia contributors. (2023, September 23). *Apsidal precession.* In Wikipedia, The Free Encyclopedia. Retrieved 03:56, December 8, 2023, from https://en.wikipedia.org/wiki/Apsidal_precession