CPSC 406, Term I, 2016-17
Assignment 4, due Tuesday, November 8

Answer the **first question** as well as **two out of the last three questions**.

Please show all your work: provide a hardcopy of the entire assignment (including plots and programs); in addition, e-mail your MATLAB programs to `cs406@ugrad.cs.ubc.ca`. When e-mailing your programs, include your name and student ID in the message's title.

Do not e-mail a complete assignment: only the programs.

1. Consider the least squares problem

$$\min_{\mathbf{x}} \|\mathbf{b} - A\mathbf{x}\|_2,$$

where the columns of the $m \times n$ matrix A are linearly dependent or almost so. Consider the *regularization* approach that replaces the normal equations by the modified, better conditioned system

$$(A^T A + \gamma I)\mathbf{x}_\gamma = A^T \mathbf{b},$$

where $\gamma > 0$ is a parameter. This approach "rivals" the truncated SVD (TSVD) method that we saw in Chapter II3.

(a) Show that $\kappa_2^2(A) \geq \kappa_2(A^T A + \gamma I)$.

(b) Reformulate the equations for $\mathbf{x}_\gamma$ as a linear least squares problem.

(c) Show that $\|\mathbf{x}_\gamma\|_2 \leq \|\mathbf{x}\|_2$.

(d) Find a bound for the relative error $\frac{\|\mathbf{x}-\mathbf{x}_\gamma\|_2}{\|\mathbf{x}\|_2}$ in terms of either the largest or the smallest singular value of the matrix A.

State a sufficient condition on the value of γ that would guarantee that the relative error is bounded below a given value ε.

(e) Write a short program to solve the 5×4 problem of the Example on slide 30 of chapII3 regularized as above, using MATLAB's "backslash". Try $\gamma = 10^{-j}$ for $j = 0, 3, 6$ and 12. For each γ, calculate the ℓ_2-norms of the residual, $\|A\mathbf{x}_\gamma - \mathbf{b}\|$, and the solution, $\|\mathbf{x}_\gamma\|$. Compare to the results for $\gamma = 0$ and to those using TSVD. What are your conclusions?

(f) For large ill-conditioned least squares problems, what is a potential advantage of the regularization with γ presented here over minimum norm TSVD?

2. Prove the following **theorem**:

   Let A be $m \times n$, $m < n$, with full row rank m. Further, let $B = R^T R$ where R is a nonsingular $n \times n$ matrix.

   Then the limit solution of the problem

   $$(A^T A + \gamma B)\mathbf{x}_\gamma = A^T \mathbf{b}$$

   as $\gamma \downarrow 0$ exists and is unique (even though $A^T A$ is singular). It is given by

   $$\mathbf{x}^* = -B^{-1} A^T \mathbf{y} = B^{-1} A^T \left(A B^{-1} A^T\right)^{-1} \mathbf{b}.$$

   Furthermore, this $\mathbf{x}^*$ is the unique solution of the problem

   $$\min_{\mathbf{x}} \frac{1}{2} \mathbf{x}^T B \mathbf{x}, \quad \text{s.t. } A\mathbf{x} = \mathbf{b}.$$

3. (a) Convert the following linear program to standard primal form:

   $$\begin{aligned} \min_{\mathbf{z}}\ & f = z_1 - 5z_2 - 6z_3 \\ \text{s.t. } & 4z_1 - 2z_2 + 6z_3 \geq 5 \\ & 3z_1 + 4z_2 - 9z_3 = 3 \\ & 7z_1 + 3z_2 + 5z_3 \leq 9 \\ & z_1 \geq -2. \end{aligned}$$

   (b) Solve the problem using any means you like (but tell us what you have used). Report the values of the optimal $\mathbf{z}^*$ and $f(\mathbf{z}^*)$.

4. (a) Please read the writeup `sparseI`. (Sorry, no points for this item. Note also that the last part of this writeup is reproduced in Section 9b.4.)

   (b) Consider the problem

   $$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|_p \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b}, \end{aligned}$$

   where A is $m \times n$, $m \leq n$.
   Describe an instance where solving this with $p = 2$ actually makes more sense than with $p = 1$.

   (c) For $p = 2$ the problem can be solved as described in the writeup `sparseI` using SVD. But it can also be solved using the techniques of constrained optimization:

      i. Explain why it is justified to replace the objective function by $\frac{1}{2}\|\mathbf{x}\|_2^2$.
      ii. Form the KKT conditions of the resulting constrained optimization problem, obtaining a linear system of equations.
      iii. Devise an example and solve it using the method just developed as well as the one using SVD from `sparseI`. Compare and discuss.