حل عددی گرمادهی بهینه میدان دمایی در محیط تصادفی مدل سازی شده با استفاده از کنترل مرزی

علی محمودی نهرانی

دکتری ریاضی کاربردی – نویسنده مسئول

ایران، گیلان، دانشگاه گیلان، دانشکده علوم ریاضی

nehrani@gmail.com

محمد کیانپور

دانشیار ریاضی کاربردی

ایران، گیلان، دانشگاه گیلان، دانشکده علوم ریاضی

m.keyanpour@gmail.com

چکیده

در مقاله حاضر مساله گرما دهی بهینه یک میدان دمایی در یک محیط تصادفی با استفاده از کنترل بهینه مرزی فرمولبندی شده و به عددی حل شده است. در مدل سازی فیزیکی از معادلات با مشتقات جزئی با پارامترهای تصادفی به عنوان قید استفاده شده است. کنترل‌ها که مدل المنت های گرمایی هستند، به صورت شرایط دیریکله به معادله اعمال شده اند و به صورت توابع تصادفی در نظر گرفته شده اند. در گسسته سازی عددی ورودی و پارامتر تصادفی با استفاده از توسیع کارهونن–لوئو بسط داده شده و به مساله اعمال شده اند. برای گسسته سازی عددی از روش گالرکین تصادفی و با استفاده از چند جمله ای های آشوب تعمیم یافته استفاده شده است. بهینه سازی عددی با استفاده از روش گرادیان انجام شده است. مسئله به صورت کامل پیاده سازی شده و برای نشان دادن کارایی روش مثال‌های عددی ارائه شده و نتایج عددی با استفاده از شکل نمایش داده شده‌اند.

کلمات کلیدی

کنترل بهینه مرزی، معادله با مشتقات جزئی تصادفی، تصویرسازی تصادفی، روش گرادیان

Numerical Solution of Optimal Heating of Temperature Field in Uncertain Environment Modelled Using Boundary Control

A. M. Nehrani*[1], and M. Keyanpour[2]
[1,2] Depatment of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

Abstract. In the present paper, optimal heating of temperature field which is modelled as boundary optimal control problem, is considered in random environment and then is solved numerically. In physical modelling, partial differential equation with stochastic input and stochastic parameter is applied as constraint of optimal control problem. Control that represents heating element, is implemented as Dirichlet boundary condition. In numerical quantification, stochastic input and parameter are approximated via Karhunen-Loéve expansion and inserted to the problem. In effect, for numerical discretization of the the problem stochastic Galerkin method is applied with generalized polynomial chaos. Numerical optimization is performed via gradient method. The problem is fully implemented and in order to show applicability of the method, numerical examples are solved and numerical results are represented through figures.

Keywords: Boundary optimal control, Stochastic partial differential equation, Stochastic Quantification, Gradient method.

I. Introduction

BOUNDARY of systems is always play a substantial role in identification, manipulation and control of them. In physical systems mostly, boundary is accessible or easier to be accessed for manipulation and making effects. Optimal heating of temperature field is one of the important problems in industry and has wide range of applications in physics, engineering, medicine and etc. In many industrial processes, one needs to adjust the thermal state of the system by manipulating temperature at the boundaries of the system. For more applied examples we refer the reader to [1], [2], [3]. Temperature field problem in deterministic case is introduced by McKinney and Savitsky [4]. Mathematically these problems can be modeled as a boundary optimal control problem. In boundary optimal control problems, control is implemented on the boundary as a Dirichlet or Neumann condition to the Partial Differential Equation (PDE) and derives the state. Also there are lots of problems can be modelled as optimal heating problem. For detailed discussions about numerical treatments of boundary control problems with PDE constraints we refer to [5] and [6].

Recently researches in engineering, headed to consider problems with stochastic parameters, in between we can mention to [7], [8]. The safety factor is the traditional engineering approach to manage uncertainty. However, safety factors are often heuristically defined and do not take direct account of uncertainties. This can lead to an overly conservative solution that fails in the uncertain environment. Thus, there has been recently a growing interest in applying probabilistic methods that take a more direct account of uncertainties to protect against failure as well as to reduce conservatism. There are two types of uncertainties to be considered: inherent uncertainty and model-form uncertainty [9], [10]. Inherit uncertainty is classified as objective and irreducible uncertainty with sufficient information on uncertain input data, whereas model form uncertainty stems from stochastic behaviour of the environment.

A non-statistical approach, called polynomial chaos expansion (PCE), based on Wiener-Hermite polynomial chaos expansion [11] was introduced by Ghanem and Spanos for uncertainty quantification in PDE models [12]. PCE, expresses stochastic solutions as orthogonal polynomials of the input random parameters. Further PCEs were generalized so that any set of complete bases can be a viable choice instead of globally smooth basis polynomials [13], [14], [15].

Transient heat conduction with uncertain parameters is modeled and numerically solved via PCE in [16]. They considered media with random heat conductivity and capacity.

In the present paper, the model is obtained from deterministic versions of optimal heating problem by including uncertainty in input function and material

*Corresponding author
e-mail: nehrani@gmail.com.

parameter (heat conductivity). The idea of modeling is that uncertainties appeared in heat conductivity, heat capacity, source terms, boundary and initial conditions or some combinations. An example if inherit uncertainty is stochastic behaviour of the environment which enforces stochastic input to the field. Such problem raises e.g., in industries, medicine, physics, biology and etc. Another source of uncertainty is modeled by heat conductivity of the material, which raise up when impurity effects conductivity of the temperature field. It has been shown that the uncertainty in heat conductivity has substantial influence on the temperature prediction [17].

The main theoretical underpinning of present paper is to consider the problem of optimal heating with uncertainties, exert PCEs for uncertainty quantification and apply optimal control methods for solving the problem. As most of the times it is not possible to control whole of the environment, the control is considered as Dirichlet boundary condition. Therefore we reached to a boundary optimal control problem constrained with stochastic PDE.

Once one has chosen an approximation space of the random process of interest, a solution within that space can be found by solving the stochastic PDE of interest in the weak form. Because of its analogy with the classic Galerkin method as employed in the finite element method [12], this methodology is often referred to as generalized polynomial chaos stochastic Galerkin (gPC-SG) method [19]. Random parameters are approximated by applying Karhunen-Loéve expansion. In order to consider uniform and normal distributions (commonly appeared in practice) gPC-SG method is applied for discretizing the stochastic PDE. It has been shown that PCEs for proper chaos expansion, the solution converges exponentially [16]. In this paper as we are dealing with correlated random data gPC-SG is very effective comparing with other techniques [19], [16].

For numerical treatment, first stochasticity is parametrized by which the problem leads to a higher order deterministic linear system. For numerical treatment of the problem the optimize-then-discretize strategy is applied. Lagrangian functional is constructed and optimality system is invoked by applying Karush-Kuhn-Tucker (KKT) conditions.

Matrix formulation of the optimality system and gradient computation is presented. The higher order optimality system is solved applying Gradient based techniques. Numerical examples are chosen so that the effect of uncertainty (specially in conductivity and inputs) be apparent on the final results. Numerical results are compared with Monte Carlo Sampling (MCS) method and results are presented in Figures.

The present paper is organized as follows: in section II problem is stated in section III, variational formulation is derived. In section IV, the optimality system is obtained by introducing Lagrangian, numerical optimization is implemented in section V and numerical examples are presented in section VI. Finally conclusion is derived in section VII.

II. Problem Statement

In this section we consider the problem of determining the optimal heating of the rectangular plate $D \subset \mathbb{R}^2$. It is placed in an environment that enforces stochastic temperature on the field (e.g. plate). This effect is modeled as stochastic source function. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space where $\mathcal{F}$ is sigma algebra on events space $\Omega \subset \mathbb{R}^n$ and $\mathcal{P} : \Omega \to [0, 1]$ is a probability measure while integer n denotes dimension of stochastic space. We assume that $\kappa, f : \Omega \times D \to \mathbb{R}$ are $\mathcal{P}$-measurable second order random fields.

In the stochastic space $\Omega \subset \mathbb{R}$, general form of the heat exchange equation in stochastic environment can be stated as

$$\nabla.(\kappa \nabla y(\omega; x)) = f(\omega; x), \qquad \omega \in \Omega,\ x \in D, \quad (1)$$

$$y(\omega; x) = u(x), \qquad \omega \in \Omega,\ x \in \partial D_u, \quad (2)$$

where $\kappa \in L^2(\Omega; L^\infty(D))$ is conductivity of the thermal field (e.g. plate) models the real problem situation. $f \in L^2(\Omega; L^2(D))$ models the environmental effects on plate distributively, $y \in \mathcal{H}^1_\Omega(D) := L^2(\Omega; H^1(D))$ is the temperature of the plate or state of the system. The heating-cooling device is stated on the controled boundary $\partial D_u \subset \partial D$.

The spaces $L^2(\Omega; L^p(D))$ for $1 < p \leq \infty$ are Bochner that inherit most of the properties of Banach and Hilbert spaces on $\Omega \times D$ and $L^\infty(D)$ is the space of essentially bounded measurable functions on D. For more information we refer the reader to [18] and references therein.

Optimal heating problem is indeed an inverse problem that tries to trace a desired temperature $y_d \in L^2(D)$ on the plate D. The performance criteria that suits for this purpose can be stated as

$$J(y, u) = \frac{1}{2} \int_\Omega \int_D (y - y_d)^2 dx \mu(d\omega) + \frac{\gamma}{2} \int_{\partial D_u} u^2 ds, \quad (3)$$

where real constant $\gamma > 0$ is called regularization parameter.

With this formulation, optimal heating problem is transformed to an optimal control problem constrained by stochastic PDE. The goal is to minimize (3) subject to (1) and boundary condition (2). The problems is pursued in the following sections.

III. Variational Formulation

In this section we are going to study the solution spaces and express the assumptions that are required for the existence and uniqueness of the boundary optimal control in problem (1)-(3). The solution of the optimal control problem i.e. obtaining state and control functions are performed in Bochner spaces $\mathcal{H}_\Omega^1(D)$ which are built on Hilbert space $H^1(D)\otimes L^2(\Omega)$ and equipped with inner product

$$\langle y, v\rangle_{\mathcal{H}_\Omega^1} := \int_\Omega \langle y, v\rangle_{H^1}\mu(d\omega),$$

and the induced norm on this space. Admissible control u is considered on the trace space, $\mathcal{U}_{ad} \subset L^2(\partial D_u)$. We define the following trace space

$$\mathcal{H}_u^{1/2}(\partial D_u) := \{w : \gamma_u(w) = u, w \in \mathcal{H}_\Omega^1(D)\},$$

which is a Hilbert space equipped with norm

$$\|w\|_{\mathcal{H}_\Omega^{1/2}} := \inf\left\{\|w\|_{\mathcal{H}_\Omega^1(D)} : \gamma_u(w) = u\right\},$$

where γ_u, is the trace operator.

In order to guaranty the existence of the state variable y (temperature of the plate) for given control u (heating strategy), and source function f (environmental effects), it is needed to state some assumptions over the stochastic PDE;

Assumption 1. There exists constants $a^+, a^- \in \mathbb{R}^+$ so that for all $(\omega; x) \in \Omega\times D$ we have

$$0 < a^- \leq \kappa(\omega; x) \leq a^+ < +\infty.$$

Assumption 2. The control function $u \in \mathcal{U}_{ad}$ is Lipschitz continuous function, and the boundary ∂D_u is Lipschitz continuous.

Noting that Assumption 2 is necessary for the regularity of the solution.

For the weak formulation of the stochastic elliptic PDE (1), we chose test function space $\mathcal{V} := L^2(\Omega; H_0^1(D))$, then the week form is constructed as follows; for all $v \in \mathcal{V}$ find $y \in \mathcal{H}_\Omega^1(D)$ that solves

$$\int_\Omega\int_D \kappa(\omega; x)\nabla y.\nabla v dx\mu(d\omega) = \int_\Omega\int_D f.v dx\mu(d\omega).$$

For variational form representation let us define bilinear form $\mathcal{A} : \mathcal{H}_\Omega^1(D)\times\mathcal{V} \to \mathbb{R}$ as

$$\mathcal{A}(y, v) := \int_\Omega\int_D \kappa(\omega; x)\nabla y.\nabla v dx\mu(d\omega),$$

and linear form $l(.) : \mathcal{V} \to \mathbb{R}$ with

$$l(v) := \int_\Omega\int_D f.v dx.$$

With Assumptions 1 and for given $f \in L^2(\Omega; L^2(D))$, it is easy to check that Lax-Milgiram conditions are hold (see Lemma 2.29 in [18]), then applying Lax-Milgiram theorem analogous to Theorem 2.30 in [18], one can verify that the variational problem

$$\mathcal{A}(y, v) = l(v), \qquad \forall v \in \mathcal{V}. \tag{4}$$

has a unique solution $y \in \mathcal{V}$.

Remark 1. The Dirichlet boundary condition (2) is not considered in the weak formulation, this helps us to make simple formulation for the problem.

The Dirichlet boundary condition consists of the boundary control of the system and is implemented as constraint in optimality system with Lagrange multiplier as presented in the next section.

IV. Optimality System

To drive the optimality system we introduce Lagrangian functional $\mathcal{L} : \mathcal{H}_\Omega^1\times\mathcal{U}\times\mathcal{H}_\Omega^{-1}\times\mathcal{U}_\Omega^{-1} \to \mathbb{R}$ as

$$\begin{aligned}\mathcal{L}(y, u, p, \lambda) &= J(y, u) + \mathcal{A}(y, p) - l(p)\\ &+ \int_\Omega\int_{\partial D_u} \lambda(y - u)ds\mu(d\omega),\end{aligned} \tag{5}$$

assuming that Lagrangian multipliers are existing, the optimality system is obtained by computing the derivatives of Lagrangian with respect to independent variables y, u, p and λ.

Partial derivative of $\mathcal{L}$ over Lagrange variable λ gives the Dirichlet boundary condition (2), and with respect to Lagrange variable p gives the state equation (1). Computing $\mathcal{L}_y$ in direction $\delta y \in \mathcal{H}_\Omega^1(D)$ gives

$$\begin{aligned}\mathcal{L}_y(y, u, p, \lambda)(\delta w) &= \int_\Omega\int_D (y - y_d)\delta y dx\mu(d\omega) + \mathcal{A}(\delta y, p)\\ &+ \int_\Omega\int_{\partial D_u} \lambda\delta y ds\mu(d\omega),\end{aligned} \tag{6}$$

that leads to the adjoint equation. The adjoint variables p and λ are computed by taking different values for δy and relations that obtained by integration and double integration by parts.

Partial derivative $\mathcal{L}_u$ is computed in the direction $\delta u \in \mathcal{U}_{ad}$ as follows:

$$\begin{aligned}\mathcal{L}_u(y, u, p, \lambda)(u - \delta u) &= \gamma\int_{\partial D_u} u(u - \delta u)ds\\ &+ \int_\Omega\int_{\partial D_u} \lambda(u - \delta u)ds\mu(d\omega).\end{aligned} \tag{7}$$

According to the relations (1), (2), (6) and (7) the optimality system is drown as follows:

$$\begin{cases} u \in \mathcal{U}_{ad}\\ \gamma\int_{\partial D_u} u(u - \delta u)ds + \int_\Omega\int_{\partial D_u} \lambda(u - \delta u)ds\mu(d\omega) \geq 0\\ \mathcal{A}(\delta y, p) + \int_\Omega\int_{\partial D_u} \lambda\delta y ds\mu(d\omega)\\ \qquad\qquad = -\int_\Omega\int_D (y - y_d)\delta y dx\mu(d\omega),\\ \mathcal{A}(u, \delta p) = l(\delta p), \forall\delta p \in \mathcal{H}_\Omega^{-1}(D),\\ y = u, \qquad \text{on}\Omega\times\partial D_u, \end{cases} \tag{8}$$

The optimality system (8) is first order necessary optimality condition obtained by defining Lagrangian (5) and existence assumption over Lagrange multipliers. For the existence of Lagrange multipliers, according to the analogy we refer the reader to [21].

The problem of existence of optimal solution in the case of distributed controls are considered in [21]. Noting that the convex objective function (3) is constrained by affine linear condition (1) and (2), in the convex state and control spaces, therefore there exists unique optimal solution for the optimization problem (1)-(3). Gradient based method converges to the optimal solution with efficient computations in convex optimization problems that motivates us to apply it for the stochastic optimization problem.

In the following section numerical discretization of the optimality system is introduced .

V. Numerical Optimization

In order to apply numerical techniques we first need to discretize the spatial space and quantify the uncertainty. In this section a very short review on the uncertainty quantification technique and then discretization of the optimality system is presented. The discretization helps us to describe numerical optimization algorithm at the final part of the section.

We divide the section into two parts; Uncertainty quantification and numerical optimization two discuss two separate numerical implementation parts. At the first part, the problem is discretized with uncertainty quantification and at the second part, the discretized problem is optimized numerically. obviously both parts are very technical and important in order to obtain proper results.

A. Uncertainty Quantification

For Uncertainty quantification gPC-SG is implemented along with approximating stochastic parameters with deterministic series. Random fields are approximated by Karhunen-Loéve expansion. The Karhunen-Loeve expansion has been proven to be a very useful tool in manipulating the random inputs and parameters in the stochastic PDEs [19]. Let $f(\omega; x)$ be a spatially varying random field over the spatial domain D with mean $\overline{f}(x)$ and covariance function $C_f(x_1,\ x_2)$. Then $f(\omega; x)$ can be represented as an infinite series [22], [23];

$$f(\omega; x) = \overline{f}(x) + \sum_{k=1}^{\infty} \sqrt{\lambda_k}\varphi_k(x)\xi_k(\omega),$$

where, λ_k and φ_k are the eigenvalues and corresponding eigenfunctions according to the given covariance function. In other words they solve the integral equations

$$\int_D C_f(x_1,\ x_2)\,\varphi_k dx_2 = \lambda_k\varphi_k(x_1), \qquad k = 1, 2, \ldots,$$

as we cannot execute real computations using the infinite series, the series is truncated to get finite number of terms. This truncation gives us an approximate representation of random field. In most cases covariance function of random fields are not known to use the preceding expansion, for more details about the computation of Karhunen-Loéve expansion we refer the reader to [24].

The following polynomial chaos expansion is presented for approximation state variable

$$y(\omega; x) = \sum_{k=1}^{\infty} y_k(x)\zeta_k(\omega),$$

where, $y_k(x)$ are deterministic coefficients and ζ_k for $k = 1, 2, \ldots$ is multi-dimensional orthogonal polynomials with the following properties

$$E[\zeta_0] = 1, \quad E[\zeta_k] = 0,\ k > 0, \quad E[\zeta_i\zeta_j] = h_i\delta_{ij}$$

In the following ξ is a vector of orthonormal random variables. By the Cameron-Martin theorem, the PCE of a random quantity converges in L^2 sense, i.e.

$$E\left[y(\omega; x) - \sum_{k=1}^{\infty} y_k(x)\,\zeta_k(\xi) \right] \xrightarrow{L^2} 0.$$

This approximation can be truncated into P terms in which P is determined with $P+1 = \frac{(N+K)!}{N!K!}$, where N is number of random variables and K is highest degree of polynomials used to represent y.

Assuming that we have finite dimensional noise, the measure $\rho(\omega)\,d\omega$ is applied instead of $\mu(d\omega)$ and stochastic space Ω is replaced by parametrized space Γ [19].

In order to represent the discretized form of the problem, we take the spaces $\mathrm{Q}_i \subset L^2_\mu(\Gamma_i)$ with dimension p_i for $i = 1, 2, \ldots, M$ and $\mathrm{X} \subset H^1_0(D)$ with dimension N. We assume that $\{\xi^i_n\}_{n=1}^{p_i}$ for $i = 1, 2, \ldots, M$ is bases of Q_i and $\{\varphi_i\}_{i=1}^{N}$ is basis of X. Noting that these bases are orthogonal. We also define the finite dimensional tensor product space

$$\mathrm{Q}_1 \otimes \cdots \otimes \mathrm{Q}_M \otimes \mathrm{X}, \tag{9}$$

as the space spanned by $\{\xi^i_{n_1}, \ldots, \xi^i_{n_M}, \varphi_i\}$ where $n_i \in \{1, \ldots, p_i\}$, $i = 1, \ldots, N$ and $\otimes$ denotes tensor product of matrices defined by $A \otimes B := [a_{ij} \times B]_{ij}$ for all entries a_{ij} of A. For simplicity we can replace indices with multi-index $\mathbf{n}$ whose kth component is $n_k \in \{1, \ldots, p_k\}$, and let $\mathtt{T}$ to denote the set of such multi indices. Now the basic functions for the tensor product space have the form

$$y_{\mathbf{n}i}(\omega; x) = \zeta_{\mathbf{n}}(\omega)\varphi_i(x),$$

where $\zeta_{\mathbf{n}} = \prod_{k=1}^{M} \xi^k_{n_k}(\omega_k)$. Furthermore we are looking for the solution $y^h \in \mathrm{Q}_1 \otimes \cdots \otimes \mathrm{Q}_M \otimes \mathrm{X}$ such that

$$\int_\Gamma \rho(\omega) \int_D \kappa(\omega; x)\nabla y^h(\omega; x)\,.\nabla y_{\mathbf{n}i}(\omega; x)\,dx d\omega = \int_\Gamma \rho(\omega) \int_D f(\omega; x)\ dx d\omega,$$

for all $\mathbf{n} \in \mathtt{T}$ and $i = 1, \ldots, N$, discretized state variable y^h is represented in terms of tensor product finite element basis functions as follows

$$y^h\left(\omega;x\right) = \sum_{m\in\mathtt{T}} \sum_{j=1}^{N} y_{mj}\zeta_m(\omega)\varphi_j(x).$$

The discretized control function is represented as

$$u^h\left(x\right) = \sum_{m\in\mathtt{T}} \sum_{j=1}^{N} u_{mj}\zeta^u_m\varphi_j(x),$$

where ζ^u_m is not an stochastic basis function, it can be considered as a projection for unifying matrix dimension. As mentioned earlier, random fields κ and f are approximated as

$$\begin{aligned}\kappa\left(\omega;x\right) &= \overline{\kappa}\left(x\right) + \sum_{i=1}^{M} \omega^{\kappa}_i \kappa_i\left(x\right),\\ f\left(\omega;x\right) &= \overline{f}\left(x\right) + \sum_{i=1}^{M} \omega^{f}_i f_i(x).\end{aligned}$$

Now we can state univariate version of the Galerkin discretization of the variational formulation (4) as

$$\begin{aligned}&\sum_{m=1}^{P}\sum_{j=1}^{N}\Big((K_0)_{ij}\int_\Gamma \rho(\omega)\zeta_n(\omega)\zeta_m(\omega)d\omega\\ &+\sum_{k=1}^{M}(K_k)_{ij}\int_\Gamma \rho(\omega)\zeta_n(\omega)\zeta_m(\omega)\omega_k d\omega\Big)v_{mj}\\ &=(\overline{F})_i\int_\Gamma \rho\left(\omega\right)\zeta_n\left(\omega\right)d\omega\\ &+\sum_{k=1}^{M}(F_k)_i\int_\Gamma \rho\left(\omega\right)\zeta_n\left(\omega\right)\omega_k d\omega.\end{aligned}$$

where $i = 1, 2, \ldots N$ and $n = 1, 2, \ldots P$ then

$$\begin{aligned}(K_k)_{ij} &= \int_{\mathrm{D}} \kappa_i\left(x\right)\nabla\varphi_i\left(x\right) . \nabla\varphi_j\left(x\right)dx,\\ (F_k)_i &= \int_{\mathrm{D}} f_k(x)\varphi_i\left(x\right)dx.\end{aligned}$$

We also can use the following relations for simplifying the equations;

$$\begin{aligned}&\int_\Gamma \rho\left(\omega\right)\zeta_n\left(\omega\right)\zeta_m\left(\omega\right)d\omega = \delta_{nm},\\ &\int_\Gamma \rho\left(\omega\right)\zeta_n\left(\omega\right)\zeta_m\left(\omega\right)\omega_k d\omega = C_{kn}\delta_{nm},\end{aligned}$$

finally we reach to the following discretized form

$$\begin{aligned}&\sum_{j=1}^{N}\left((K_0)_{ij} + \sum_{j=1}^{N} C_{kn}(K_k)_{ij}\right)v_{mj}\\ &=(\overline{F})_i\int_\Gamma \rho\left(\omega\right)\zeta_n\left(\omega\right)d\omega\\ &\quad+\sum_{k=1}^{M}(F_k)_i\int_\Gamma \rho\left(\omega\right)\zeta_n\left(\omega\right)\omega_k d\omega. \qquad (10)\end{aligned}$$

With tensor notation we can derive the following linear system:

$$\left(I\otimes K_0 + \sum_{k=1}^{M} C_k \otimes K_k\right)\overrightarrow{v} = F,$$

where we applied identity matrix $I \in \mathbb{R}^{P\times P}$, and vectors $\overrightarrow{v} = (\overrightarrow{v}_1, \ldots, v_P)^T$ and $F = (\overrightarrow{F}_1, \ldots, \overrightarrow{F}_P)^T$.

B. Numerical Optimization

The optimality system is derived and spatial-stochastic discretization is performed with gPC-SG. We present the discretization for the state equation. For numerical optimization, Gradient decent method is developed. One of the key features of the method is that decent direction is obtained by computing the gradients of the objective function [25]. One of the efficient approaches for computing the gradient is to apply adjoint equation (6);

$$K\overrightarrow{p} = F,$$

where $F = (\overrightarrow{F}_1, \ldots, \overrightarrow{F}_P)^T$, and in turn

$$(\overrightarrow{F}_k)_i = \overrightarrow{b}_i\delta_{1k} - \sum_{j=1}^{N} M_{ij}(\overrightarrow{v}_k)_j.$$

where again

$$\overrightarrow{b}_i = \int_{\mathrm{D}} v_d\left(x\right)\varphi_i\left(x\right)dx, \quad M_{ij} = \int_{\mathrm{D}} \varphi_i\left(x\right)\varphi_j\left(x\right)dx.$$

Then we can perform Gradient computation in (8) as

$$\begin{aligned}J_u(v,u)(u-\delta u) &= \int_{\partial D_u} u(u-\delta u)dx\\ &= \int_{\partial D_u} u(x)\phi_j(x)ds \approx \sum_{i=1}^{N}\overrightarrow{u}_i\int_{\partial D_u}\phi_i(x)\phi_j(x)ds\\ &= \sum_{i=1}^{N}\overrightarrow{u}_i {M^s}_{ij}.\end{aligned}$$

And for computing J_v we have

$$\begin{aligned}J_v(v,u)(\delta v) &= \int_\Gamma\int_D (v-y_d)\delta v dx\rho(\omega)d\omega\\ &= \int_\Gamma\int_D\sum_m\sum_j(v_{mj} - y_{dj}\mathcal{I}_m)\zeta_m\phi_j\zeta_k\phi_i ds\rho(\omega)d\omega\\ &\approx \sum_m\sum_j(v_{mj} - y_{dj}\mathcal{I}_m)\int_D\phi_i\phi_j dx\int_\Gamma\zeta_k\zeta_m\rho(\omega)d\omega\\ &= (\overrightarrow{v} - \overrightarrow{y}_d)M\otimes S.\end{aligned}$$

where

$$S_{mk} = \int_\Gamma \zeta_k\zeta_m\rho(\omega)d\omega, \quad m,k = 1,2,\ldots,P.$$

Remark 2. For unifying the dimensions of control and state vector, and also the desired state, we applied identical projections denoted by ζ^u_m and $\mathcal{I}_m$

respectively.

Numerical optimization method is summarized in the following steps:

1) *State and adjoint state*: set $k = 0$, compute the state and adjoint, and put the solutions into $\vec{v}_k$and $\vec{p}_k$
2) *Search direction*: search direction is computed as follows
$$d := -J_u\left(\vec{v}_k, \vec{u}_k\right) - J_v\left(\vec{v}_k, \vec{u}_k\right),$$
3) *Step-size control*: analogous to deterministic case the optimal step size $\bar{s}$ is determined by
$$\begin{aligned} & J\left(\vec{v}_k, \mathcal{P}_{\mathcal{U}_{ad}}\left(\vec{u}_k + \bar{s}\vec{u}'_k\right)\right) \\ & = \min_{s>0} J\left(\vec{v}_k, \mathcal{P}_{\mathcal{U}_{ad}}\left(\vec{u}_k + s\vec{u}'_k\right)\right), \end{aligned} \tag{11}$$

where $\mathcal{P}_{\mathcal{U}_{ad}}$ is the projection onto the space of admissible controls.

1) *Update control*: The control vector is updated according to
2) $\vec{u}_{k+1} := \mathcal{P}_{\mathcal{U}_{ad}}\left(\vec{u}_k + \bar{s}\vec{u}'_k\right)$, set $k := k+1$,
3) *Update state and adjoints*: Solve optimality system (8) for $\vec{u}_{k+1}$ and put the solutions into $\vec{v}_{k+1}$ and $\vec{p}_{k+1}$.

Remark 3. Iteration is continued until suitable stopping criteria is satisfied. In this paper we set stopping criteria as $\|u_{k+1} - u_k\|_{L^2} < \epsilon$.
Noting that direction can be computed using Armijo's rule or Wolf's method.

VI. Numerical Examples

Two numerical examples are presented in this section. The examples are designed in order to illustrate the applicability of the purposed method for solving optimal heating problems with stochastic data and the effect of random data on the state. In the examples, as discussed earlier, conductivity coefficient κ (Fig. 1) and source function f are considered to be taken as random fields. For simplicity, the desired state is get constant function $y_d = 0.2$, and regulator coefficient is taken $\gamma = .1$, in both examples. In both of examples stochastic space as $\Gamma = [0\ 1]$.

Example 1. In this example optimal boundary control problem (1)-(3) is considered with uniformly distributed $\kappa(.;x)$ (Fig. 1 (a)) and $f(.;x)$ (Fig. 2 (b)) for every $x \in D$. Spatial space is taken as $D = [0\ 2] \times [0\ 2]$, and controlled boundary is $\{(x_1, x_2) \in D\ :\ x_1 = 0\}$. Thermal conductivity is $\kappa = 1/2 + w$, and random input is taken as
$$f = \begin{cases} w+2 & x_1 \in [1\ 1.8],\ x_2 \in [0.2\ 0.8], \\ w & \text{elsewhere}, \end{cases}$$

where $w(\omega)$ is distributed uniformly on sample space and $\omega \in \Gamma$. Numerical method is implemented and computed optimal state which represents optimal distribution of heat on the plate, is represented in Fig. 2 (a) and random source f for Example 1 is shown in Fig. 2 (b). Optimal boundary control that denotes optimal heating strategy, is computed by implementing proposed method (gPC-SG uncertainty quantification) and MCS method. MCS is implemented with 1000 sampling points to reach the stopping criteria.
The results are illustrated in Fig. 3 (a). The optimal strategy u is obtained such that optimal state become as close as possible to y_d i.e. the temperature on the whole of the plate becomes 0.2 in norm 2 sense.
The value of objective function in both numerical methods (MCS and gPC-SG based methods) is reported in Fig. 3 (b).

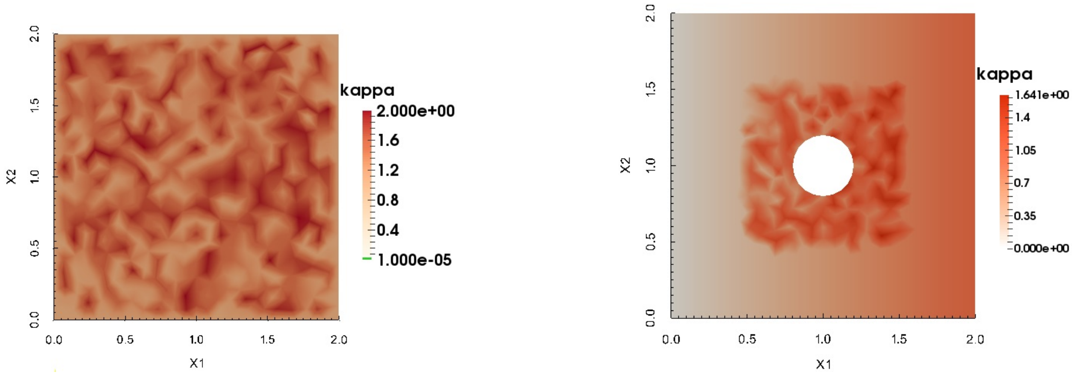
Example 2. Optimal boundary control problem (1)-(3) is considered with normally distributed random data
$$\kappa = \begin{cases} \frac{1}{2}x_1 + w & x_1 \in [0.5\ 1.5],\ x_2 \in [0.5\ 1.5] \\ \frac{1}{2}x_1 & \text{elsewhere} \end{cases}$$
and
$$f(.;x) = \begin{cases} w+2 & x_1 \in [1\ 1.8],\ x_2 \in [0.2\ 0.8] \\ w+1 & x_1 \in [1\ 1.8],\ x_2 \in [1.2\ 1.8] \\ w & \text{elsewhere} \end{cases}$$

where $w(\omega; x)$ is normal random field for $\omega \in \Gamma$ and $x \in D$. In this example spatial space has hole in the middle that is $D = [0\ 2] \times [0\ 2] \setminus C$ where $C = \{x \in D : (x_1 - 1)^2 + (x_2 - 1)^2 < 0.2\}$. Controlled boundary is same as previous example $\{(x_1, x_2) \in D\ :\ x_1 = 0\}$. State and boundary control are numerically computed with MCS and gPC-SG based methods for given κ (Fig. 1 (b)) and source function f (Fig. 4 (b)). Fig. 4 (a) represents state which computed with gPC-SG based method. Obtained boundary control with both methods are compared in Fig. 5 (a). The value of objective function for both MCS and gPC-SG based methods are comparatively presented in Fig. 5 (b).
In both examples, it can be seen that MCS based method makes smooth reduction in objective function while we see rapid reduction in value of objective function in gPC-SG based method. Thus we can conclude that as it was asserted PCE leads to fast convergence with respect to MCS based method.
Numerical examples are implemented in Matlab® 7.10 and figures are extracted applying ParaView [1] software.

VII. Conclusion

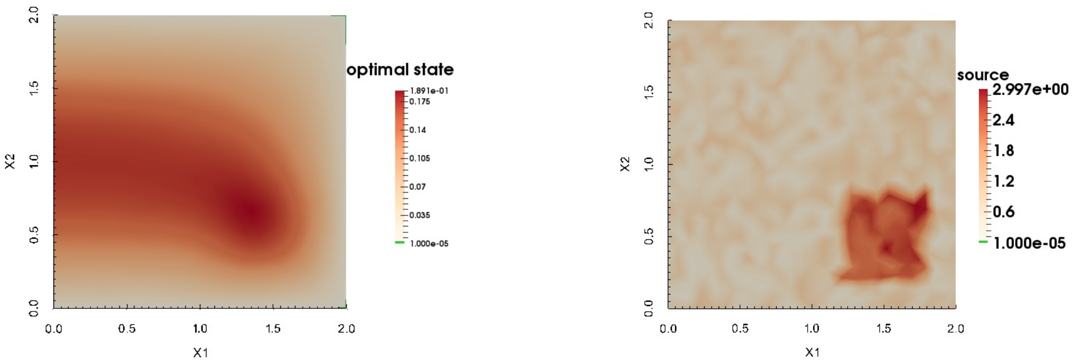
Optimal heating problem with uncertainties in thermal conductivity of the plate and stochastic behaviour of the environment is modelled as boundary optimal control of stochastic PDE. Parameter uncertainty and

[1] www.paraview.org



(a) (b)

Fig. 1. (a) Conductivity coefficient κ in Example 1, (b) Conductivity coefficient κ in Example 2



(a) (b)

Fig. 2. (a) Optimal state (temperature distribution) y in Example 1, (b) source function f in Example 1

input uncertainty are taken into account with PCE based approach. Stochastic parameters are approximated via Karhunen-Luéve expansion. Uncertainty quantification is performed by gPC-SG and the problem is turned into a very large linear system of equations. Gradient decent method is implemented for solving the stochastic optimization problem and as illustrated in numerical examples we obtained desired results. For future researches one can consider other types of uncertainties e.g. weakly related random variables and etc. Other types of controls such as distributed can also be taken into account. Finally transient case of optimal heating problems can be considered with stochastic parameters.

References

[1] H. M. Park and W. J. Lee, ”A New Numerical Method for the Boundary Optimal Control Problems of the Heat Conduction Equation,” *International Journal for Numerical Methods in Engineering*, vol. 53, pp. 1593-1613, 2002.

[2] J. Nulman , J. P. Krusius and A. Gat, ”Rapid thermal processing of thin gate dielectrics-oxidation of silicon,” *Electron Device Letters, IEEE.*, vol. 6, pp. 205-208, 1985.

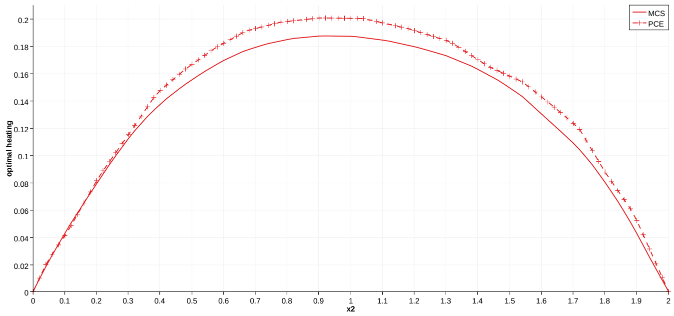
[3] H. M. Park, Y. T. Yoon and O. Y. Kim, ”Optimal control of rapid thermal processing systems by empirical reduction of modes,” *Industrial & Engineering Chemistry Research*, vol. 38, pp. 3964 -3975, 1999.

[4] D. McKinney and A. Savitsky, ”Basic optimization models for water and energy,” *Technical Report: University of Texas management, Austin*, 2003.

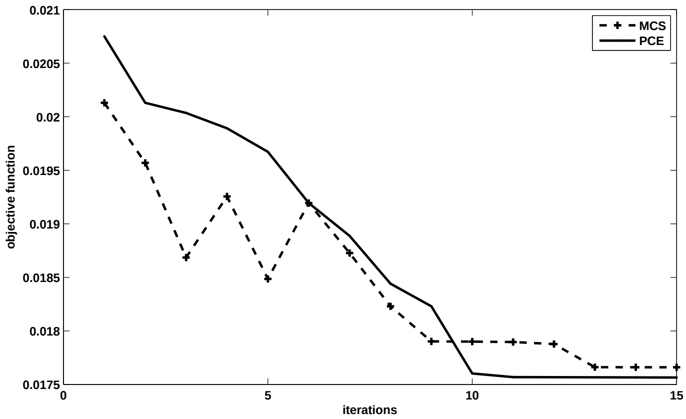
[5] K. Kunisch and B. Vexler, ”Constrained Dirichlet boundary control in L^2 for a class of evolution equations,” *SIAM Journal on Control and Optimization*, vol. 46, pp. 1726-1753, 2007.

[6] J. E. Lagnese and G. Leugering, ”Dynamic domain decomposition in approximate and exact boundary control in problems of transmission for wave equations,” *SIAM Journal on Control and Optimization*, vol. 38, pp. 503-537, 2000.

[7] Y. Zhang, ”Efficient uncertainty quantification in aerospace

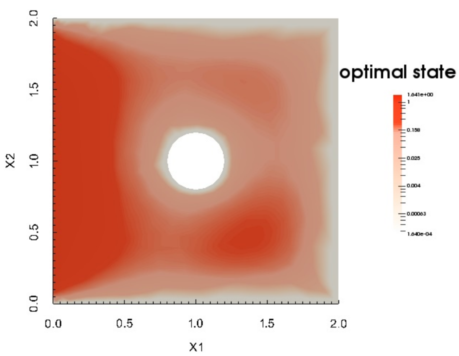


(a)

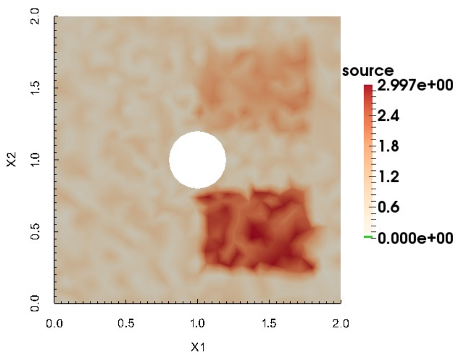


(b)

Fig. 3. (a) Optimal heating strategy u in Example 1 (gPC-SG based method (later on denoted by PCE) is presented with dash line and plus sign and MCS based methos is shown with solid line) , (b) value of objective function in Example 1 (PCE is denoted by dash line and MCS is shown with solid line)



(a)



(b)

Fig. 4. (a) optimal state y computed in Example 2, (b) source function f in Example 2

analysis and design," *Ph.D thesis, Missouri university of science and technology*, 2013.

[8] H. Tiesler, R. M. Kirby, D. Xiu and T. Preusser, "stochastic collocation for optimal control problems with stochastic pde constraints", *SIAM Journal on Control and Optimization*, Vol. 50, pp. 2659–2682, 2012.

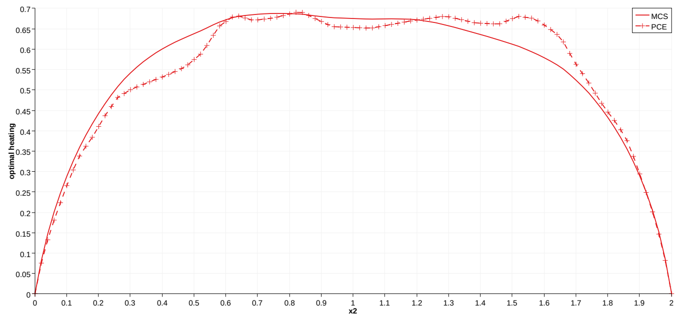
[9] O. L. William, J. C. Helton, C. A. Joslyn, S. F. Wojtkiewicz and S. Ferson, "Challenge problems: uncertainty in system response given uncertain parameters," *Reliability Engineering & System Safety*, vol. 85, pp. 11-19, 2004.

[10] L. P. Swiler, T. L. Paez, R. L. Mayes, "Epistemic uncertainty quantification tutorial," *in proceedings of the 27th International Modal Analysis Conference*, Orlando, Florida, 2009.

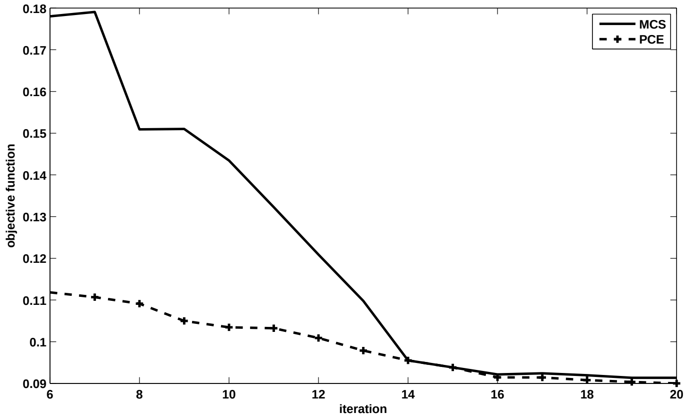
[11] K. Sepahvand, S. Marburg and H. J. Hardtke, "Uncertainty quantification in stochastic systems using polynomial chaos expansion," *International Journal of Applied Mechanics*, vol. 2, pp. 305-353, 2010.

[12] R. Ghanem, P. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Springer-Verlag, New York, 1991.

[13] M. K. Deb, I. Babuska, J. T. Oden. "Solution of stochastic partial differential equations using Galerkin nite element techniques," *Comput. Methods Appl. Mech. Eng.* vol. 190, pp.



(a)



(b)

Fig. 5. (a) Optimal heating strategy u in Example 2 (gPC-SG based method (later on denoted by PCE) is presented with dash line and plus sign and MCS based methos is shown with solid line), (b) value of objective function in Example 2 (PCE is denoted by dash line and MCS is shown with solid line), and results are presented for iterations 6 to 20

6359-6372, 2001.

[14] D. Xiu,and G. E. Karniadakis, "The Wiener-Askey polynomial chaos for stochastic differential equations," *SIAM journal on scientific computing*, vol. 24, pp. 619-44, 2002.

[15] D. Xiu and G. Karniadakis, "Modeling uncertainty in steady state diffusion problems via generalized polynomial chaos," *Computer Methods in Applied Mechanics and Engineering*, vol. 191, pp. 4927-4948, 2002.

[16] D. Xiu, andG. E. Karniadakis. "A new stochastic approach to transient heat conduction modeling with uncertainty," *International Journal of Heat and Mass Transfer*, vol. 46(24), pp. 4681-4693, 2003.

[17] J. Liu. "Uncertainty analysis of temperature prediction of biological bodies subject to randomly spatial heating," *J. Biomech.* vol. 34, pp. 1637-1642, 2001.

[18] C. Schwab and C. J. Gittelson, "Sparse tensor discretizations of high dimensional parametric and stochastic PDEs," *Acta Numerica*, vol. 20, pp. 291-467, 2011.

[19] D. Xiu, *Numerical methods for stochastic computations: a spectral method approach*, Princeton University Press, 2010.

[20] G. J Lord, C. E. Powell, T. Shardlow, *An introduction to computational stochastic PDEs*, Cambridge University Press, 2014.

[21] H. C. Lee and J. Lee, "A stochastic Galerkin method for stochastic control problems," *Communications in Computational Physics*, vol. 14(01), pp. 77-106, 2013.

[22] K. Karhunen "Uber lineare Methoden in der Wahrscheinlichkeitsrechnung," *Annales Academi Scientiarum Fennice*, vol. 37, 1947.

[23] M. Loéve. "Fonctions aléatoires de second ordre," *Revue des sciences de l'eau*, vol. 84, pp. 195-206, 1946.

[24] C. Schwab and R. A. Todor. "Karhunen-Loéve approximation of random fields by generalized fast multiple methods," *Journal of Computational Physics*, vol. 217, pp. 100-122, 2006.

[25] F. Troltzsch, *Optimal control of partial differential equations: theory, methods, and applications*, American Mathematical Society, 2010.