Structural Analysis of Complex Geometries Using the Finite Cell Method

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Abstract. The attractiveness of the Finite-Element-Method heavily depends on the ability to create a sufficient discretization of the domain of interest. Therefore, analyzing geometries of high complexity, like one obtained from medical image data, takes a large amount of time and effort, creating the need for a more efficient approach. A solution for this type of problem is provided by the novel Finite-Cell-Method. It works around the need of a demanding discretization, by resolving the geometry on an integration level. This paper will give a brief overview of the method concept in the case of linear elasticity, fields of application and current state of the development. Furthermore the method's accuracy and convergence behavior is examined with two well regarded benchmark problems. Finally to demonstrate the abilities of the novel approach, a digital model of an aluminum foam, which has been obtained with computer tomography, is analyzed in the context of structural rigidity.

Keywords: Numerical- & Solid Mechanics · Complex Geometries · FEM · FCM

1 Introduction

The Finite Element Method (FEM) has been around for decades and is one of the most essential tools in the world of modern engineering. Nevertheless in order to use the method each geometry has to be preprocessed and meshed. This task is very demanding for complex geometries. In [2] Hughes et al. have reported that this process will take up to 80% of the analyzing time. To work around this discretization process mesh free methods can be used. One of these is the so-called Finite Cell Method (FCM). It has been introduced by Parvizian et al. in [6] and a more thorough overview can be found in [8] by Schillinger and Ruess. Since its introduction in 2007 the method has been applied to multiple problems including linear elasticity [6], geometric nonlinearities [9], heat flow [12] and fluid dynamics [11]. Furthermore the method has been improved in the context of embedded interface problems [3], which contain non C^∞ continuous solutions and singularities.

In the following the main concept of the FCM will be presented. Subsequently two benchmark problems are presented to analyze the methods convergence behavior and errors. Finally it will be shown how the method can be used to analyze complex real world problems.

2 The Finite Cell Method

In this paper the well known problem of linear elasticity is considered, which is widely used in the field of engineering. The weak form can be expressed through the equations of linear and bi-linear functional, reading

$$\mathcal{B}(\boldsymbol{u},\boldsymbol{v}) = \mathcal{F}(\boldsymbol{v}) \tag{1a}$$

$$\mathcal{B}(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega} \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u})\, d\Omega \tag{1b}$$

$$\mathcal{F}(\boldsymbol{v}) = \int_{\Omega} \boldsymbol{v} \cdot \boldsymbol{b}_0\, d\Omega + \int_{\Gamma_N} \boldsymbol{v} \cdot \boldsymbol{t}_0\, d\Gamma_N . \tag{1c}$$

The primary solution of the problem is the displacement-field $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$ is the strain tensor and $\boldsymbol{C}$ the elasticity tensor, holding the material properties. $\boldsymbol{b}_0$ and $\boldsymbol{t}_0$ are the body- and external forces respectively, acting in the problem domain Ω and on parts of its boundary $\partial\Omega$. The domain boundary is divided into Γ_D, where essential boundary conditions are applied and Γ_N with natural ones. $\boldsymbol{v}$ is the test function which results from the problem transformation into the weak form.

In the classic FEM a problem like this is usually solved by dividing the problem domain into a finite number of subdomains, called elements Ω_e such that

$$\Omega \approx \bigcup_i \Omega_e^{(i)}, \tag{2}$$

where the elements are usually of polygonal (2D) shape. The conceptual approach of the FCM is displayed in figure 1. The original problem of arbitrary

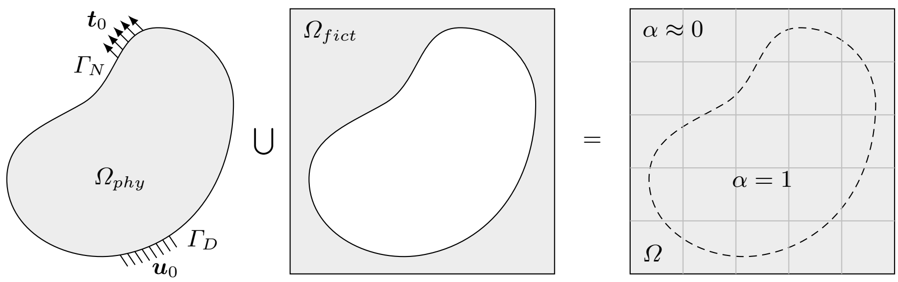


Fig. 1. Conceptual approach of the Finite Cell Method

shape, which from now on will be denoted as Ω_{phy}, is extended through a fictitious domain Ω_{fict} to trivial geometry. The mesh generation for the conjugation is now a task which can be easily performed. Due to the fact that not all elements are fully part of the actual problem they are called *cells*. To distinguish between internal and external points $\boldsymbol{x}$ with respect to the physical domain, the indicator function

$$\alpha(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \Omega_{phy} \\ 10^{-q} & \boldsymbol{x} \in \Omega_{fict} \end{cases} \tag{3}$$

is introduced. The parameter q is usually chosen to be a positive integer in the range of $q = 5 \ldots 15$, depending on the problem and solution approach. To remove the influence of the fictitious domain on the original problem, equation (1b) is modified as follows

$$\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \alpha(\boldsymbol{x}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u})\, d\Omega. \tag{4}$$

In terms of elasticity this significantly reduced the stiffness of the fictitious domain. A value of $\alpha(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \Omega_{fict}$ is not used, to avoid ill condition system matrices, which would be hard to solve using iterative solution schemes.

This new approach however also creates a few problems which need to be solved. The first one being, that with the introduction of α the problem equations become non-smooth. Thus the numeric Gauss Quadrature integration - which is usually used in the FEM - will not yield sufficiently accurate results. A well regarded and straight forward approach to overcome this is shown in figure 2. It can be seen how a finite cell is recursively divided in equally large

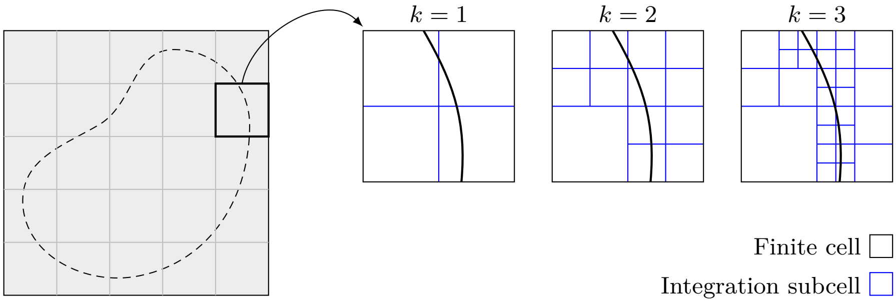


Fig. 2. Recursive cell refinement using a quad-tree scheme

sub cells, leading to a fine resolution along the physical boundary $\partial\Omega_{phy}$. After a satisfactory refinement is archived, each subcell is integrated individually.

Another issue, which has to be approached differently compared to the classic FEM, is the application of essential boundary conditions. These are usually applied at the nodes (element vertices). However, in the FCM these nodes do not generally align with the physical boundary and thus have to be applied inside the cells. In this paper the enforcement is done using the penalty method as described by Vinci in [10]. The method defines a second set of linear equations, describing the boundary conditions, which is then added to the main system of equations using the penalty method. Another method which is widely used for this purpose is NITSCHES method [7].

3 Results

For this work the previously described concepts have been implemented in the `FELiNA` framework, a small software for finite element analysis.

3.1 Validation

In order to check for the correctness of the results of the FCM and the used implementation, two commonly used benchmark problems are used. An analytical solution for these problems is known, which makes them very suitable for the validation of numeric results. The first one is shown in figure 3 and its properties and solution are given by HUGHES in [4]. The figure shows a small section of

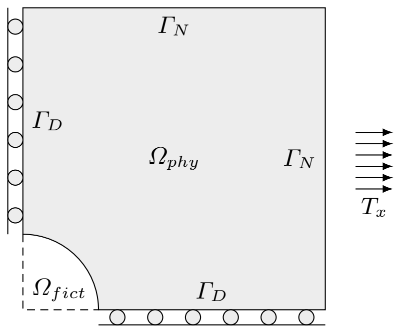


Fig. 3. Numeric benchmark setup of an infinite plate with circular hole under tension of the far field traction T_x

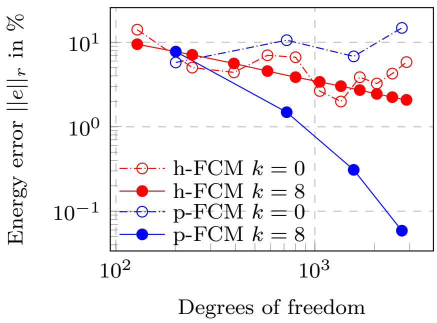


Fig. 4. Results of the FCM calculations showing how the errors decrease without integration refinement $k = 0$ and with $k = 8$

an infinite plate with a centered circular hole. Due to symmetry, only a quarter of the center has to be simulated. On the left and lower edge displacements in normal direction are locked and on the top and right boundary the exact stresses from the analytical solution are prescribed. The cutout from the hole is filled with a fictitious domain and thus the final domain is a square shape. Since the dirichlet boundaries Γ_D coincide with the mesh boundary, no weak enforcement of boundary conditions has to be used. The convergence of the model is examined by gradually refining the mesh (*h-FCM*) and increasing the polynomial degree p of the cells (*p-FCM*). The error is computed via the relative Energy norm

$$||e||_r = \left[\frac{|\Pi_{\text{exact}} - \Pi_{\text{numeric}}|}{|\Pi_{\text{exact}}|} \right]^{1/2}. \tag{5}$$

Figure 4 shows the results from these computations, with k being the subcell division depth. It is visible that the computations performed without cell refinement ($k = 0$) do not converge towards the analytical solution. However, if a refinement is used ($k = 8$) the *h-FCM* shows a linear and the *p-FCM* an exponential convergence rate.

Figure 5 shows another benchmark problem, known from [8]. It's a circular ring plate, which is fixed along its outer radius and objected to a load at its inner radius and inside the body itself. The geometry is enclosed with a fictitious domain to a square shape. The fixation along the outer edge is implemented using a weak enforcement through the penalty method. The results from the

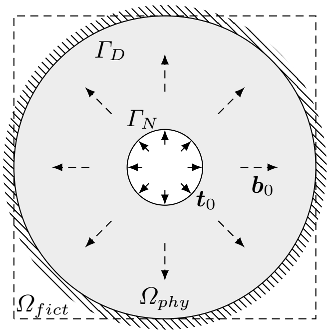


Fig. 5. Numeric benchmark setup of an annular ring plate with outer fixation implemented using the penalty-method

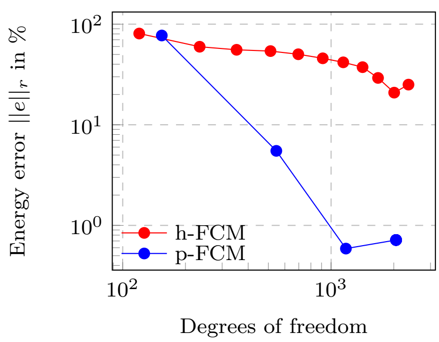
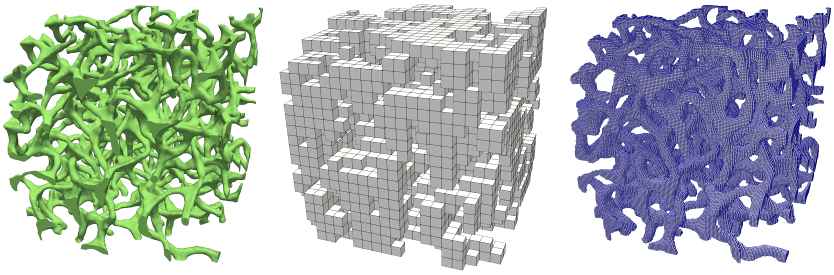


Fig. 6. Results of the FCM calculations showing how the errors decrease with increasing degrees of freedom

convergence studies are displayed in figure 6. It shows that the convergence behavior of the *h-FCM*, using cells with bi-linear basis, is significantly worse compared to the previous model and converges with a much lower rate. The *p-FCM* however falls well below an error of 1% within the first three computations, which is sufficiently accurate for most engineering purposes.

3.2 Demonstration

After it has been demonstrated that the implementation of the FCM is able to achieve accurate results, a more complex model is implemented to demonstrate real world usage. The geometry chosen for this is an aluminum foam [5], shown in figure 7 a, which is available at [1]. These materials are used e.g. aerospace industry because of their weight to mechanical properties ratio. The geometry is cut out from a larger plate, which is examined with respect to vertical compression. Thus the displacement at all faces except the top one are fixed in the normal direction. The top face is subjected to a prescribed displacement of 1% of the geometries initial height. As shown in b and c, the geometry is embedded



a) Surface model b) Unfitted mesh c) Refined subcells $k = 3$

Fig. 7. Mesh and refinement of surface model

into a cube shaped domain. All cells which are not directly connected to the geometry of interest are removed, to reduce computational effort. The cell integration is refined up to $k = 5$. To estimate the state of convergence, the mesh has been refined gradually. The energy of the corresponding model is tracked and displayed in figure 8. It is visible that the rate of change becomes small with increasing degrees of freedom, leading to the assumption that the solution is close to convergence. The stressfield of the resulting solution is shown in figure 9 and shows a physically reasonable distribution. Thinner branches of the structure are objected to higher stress than thicker ones.

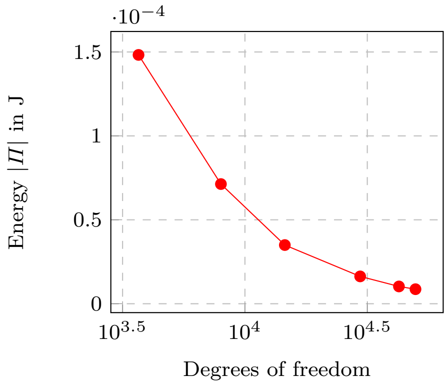


Fig. 8. Model convergence

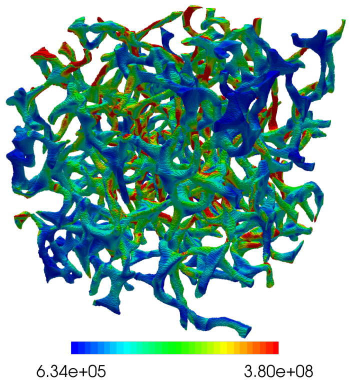


Fig. 9. Stressfield solution

4 Conclusion

This paper has shown that using the Finite Cell Method accurate results can be obtained without the need of a demanding and complex discretization of the actual geometry. The presented benchmark problems have proven that the refinement of the cell integration using a recursive subcell division is very effective, yet simple. It has been observed, that the use of weakly enforced essential boundary conditions using the penalty method will worsen the convergence behavior of cells with low polynomial degree. Nevertheless, cells of higher order will still archive sufficiently accurate results. Furthermore it has been shown that the method can handle geometries of high complexity, leading to the ability to analyze more complicated phenomena, using a real world example.

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