Effect of an additional death on period e0

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Theory: What happens to period e0 if there is one more death at age z?

We calculate period life expectancy using age-specific mortality rates that are estimated from observed deaths and exposure ($\approx$ mid-year population) at each age.

Period mortality rates are

$$\mu_a = \frac{D_a}{N_a}$$

for ages $a = 0, 1, 2, ...$ The cumulative mortality risk up to any specific age x is

$$H_x = \int_0^x \mu_a \, da$$

and the probability that a newborn experiencing subject to current age-specific rates would survive to age x is

$$\ell_x = \exp(-H_x)$$

Period life expectancy is

$$e_0 = \int_0^\infty \ell_x \, dx \; = \int_0^\infty \exp(-H_x) \, dx$$

and if some quantity θ were to change, its effect on period life expectancy would be

$$\frac{\partial e_0}{\partial \theta} \; = \; -\int_0^\infty \frac{\partial H_x}{\partial \theta} \exp(-H_x) \, dx \; = \; -\int_0^\infty \frac{\partial H_x}{\partial \theta} \, \ell_x \, dx$$

Rewriting H_x in its integral form and changing the order of integration produces the following

$$\begin{aligned}
\frac{\partial e_0}{\partial \theta} &= -\int_0^\infty \frac{\partial}{\partial \theta} \Big[\int_0^x \mu_a \, da \Big] \, \ell_x \, dx \\
&= -\int_0^\infty \Big[\int_0^x \frac{\partial \mu_a}{\partial \theta} \, da \Big] \, \ell_x \, dx \\
&= -\int_0^\infty \Big[\int_a^\infty \ell_x \; dx \Big] \frac{\partial \mu_a}{\partial \theta} da \\
&= -\int_0^\infty T_a \frac{\partial \mu_a}{\partial \theta} da
\end{aligned}$$

where T_a is the demographic abbreviation for expected person-years lived, per newborn, after age a.

If the θ that changes is the number of deaths at one specific age z, then

$$\frac{\partial e_0}{\partial D_z} \; = \; -T_z \frac{\partial \mu_z}{\partial D_z} \; = \; -\frac{T_z}{N_z}$$

Notice that this is always negative, which makes sense: **any increase in period deaths can only lower period life expectancy.**

Example: England & Wales, 2018

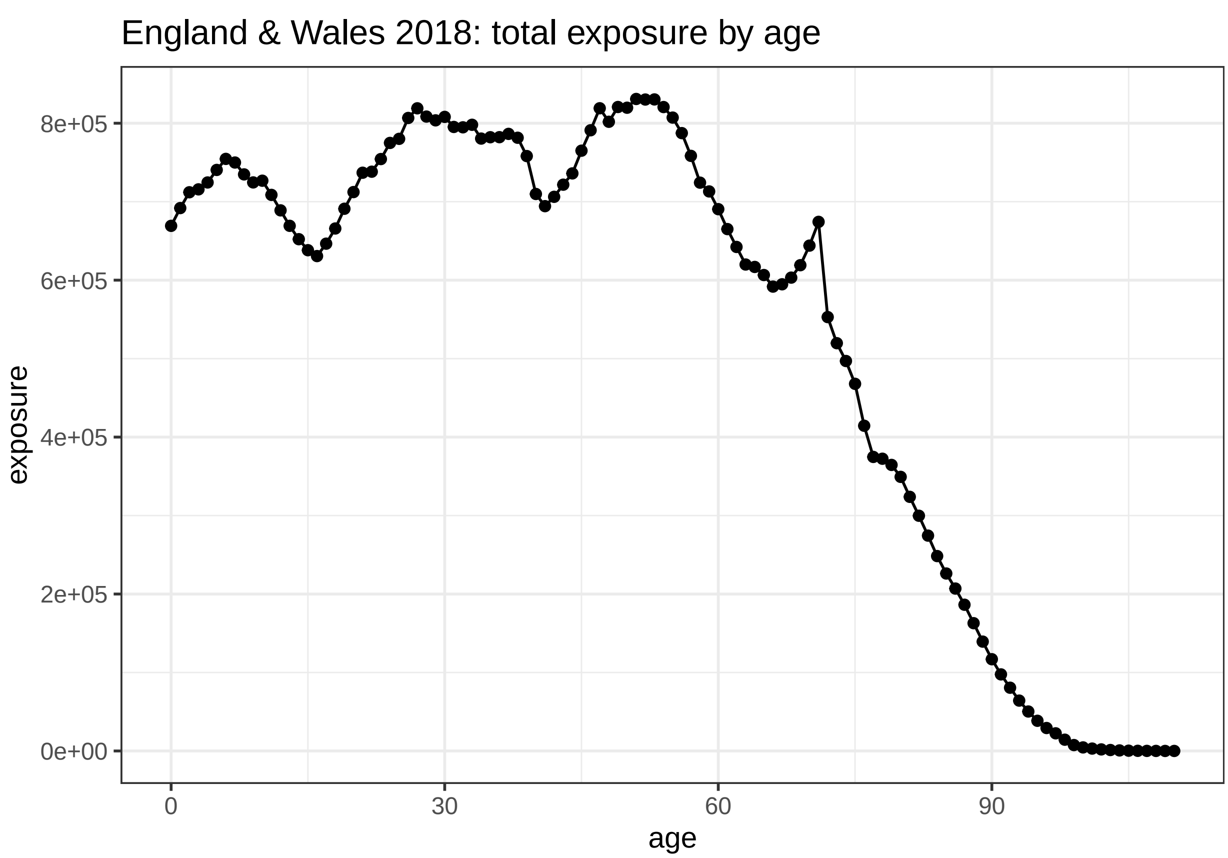
Start by grabbing deaths and exposure data for England & Wales, 2018, from the Human Mortality Database.

```
library(tidyverse)
```

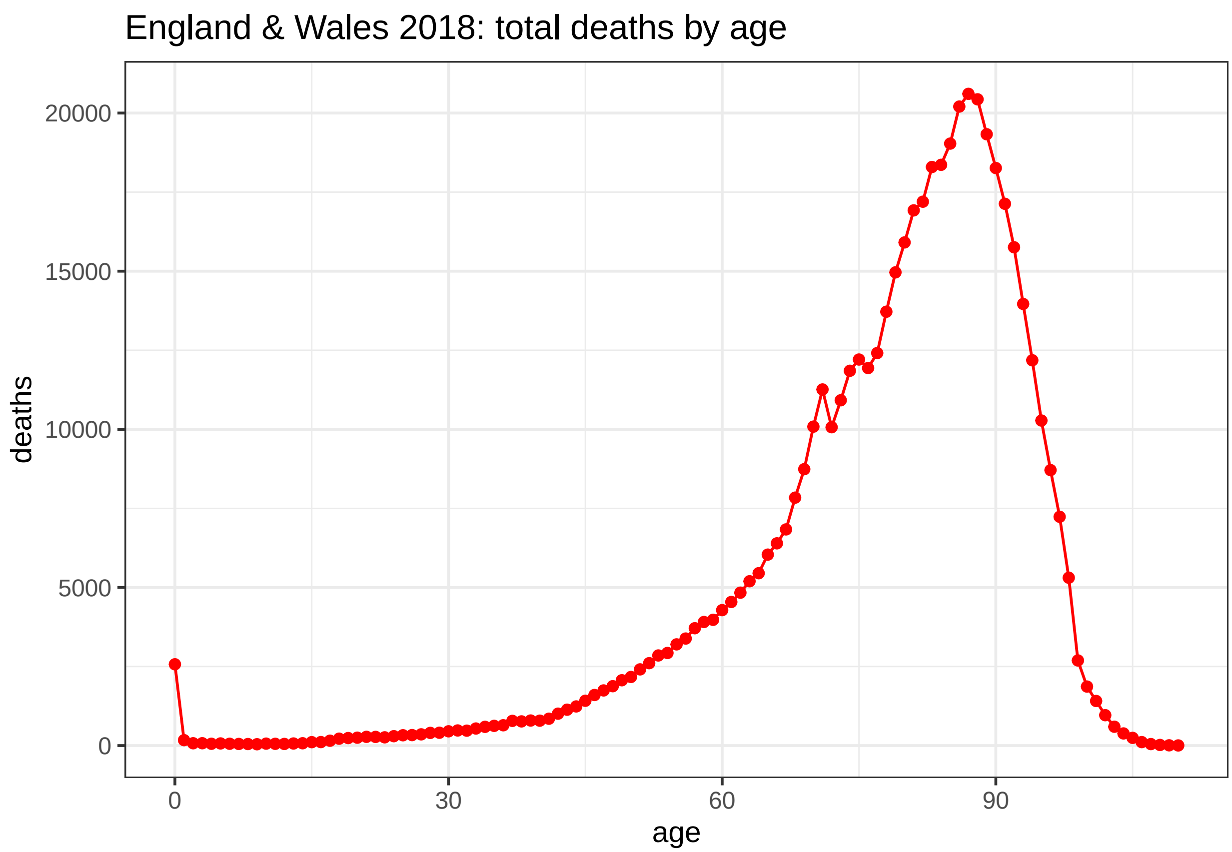
```
## -- Attaching packages ----------- tidyverse 1.3.0 --
## v ggplot2 3.2.1     v purrr   0.3.3
## v tibble  2.1.3     v dplyr   1.0.0
## v tidyr   1.0.2     v stringr 1.4.0
## v readr   1.3.1     v forcats 0.4.0
## -- Conflicts -------------- tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

```
library(HMDHFDplus)

age = 0:110

# --- running the code below, with your HMD username/password
#     will produce the data used next
#
# deaths = readHMDweb(CNTRY='GBRTENW', item='Deaths_1x1',
#                     username=myUN, password = myPW) %>%
#          filter(Year == 2018) %>%
#          pull(Total)
#
# exposure = readHMDweb(CNTRY='GBRTENW', item='Exposures_1x1',
#                     username=myUN, password = myPW) %>%
#          filter(Year == 2018) %>%
#          pull(Total)

deaths = c(2571, 172, 74, 75, 58, 68, 58, 52, 49, 43,
           62, 56, 52, 66, 74, 108, 111, 155, 219, 238,
           250, 277, 274, 261, 297, 327, 332, 355, 402,
           407, 451, 478, 472, 540, 593, 624, 643, 782,
           765, 793, 788, 850, 1009, 1135, 1238, 1418,
           1599, 1744, 1878, 2066, 2170, 2410, 2606,
           2850, 2928, 3200, 3385, 3711, 3908, 3976, 4283,
           4541, 4835, 5196, 5450, 6038, 6395, 6835, 7840,
           8740, 10085, 11260, 10066, 10918, 11851, 12207,
           11937, 12411, 13718, 14963, 15909, 16923, 17197,
           18294, 18363, 19034, 20205, 20607, 20434, 19330,18261,
           17131, 15757, 13964, 12183, 10278, 8710, 7235, 5308,
           2692, 1866, 1410, 961, 596, 384, 244.08, 109.98,
           47.56, 20.09, 8.55, 4.75)

exposure = c(669210.73, 691861.02, 712004.53, 715739.78,
             724495.76, 740452.11, 754531.92, 749897.73,
             734785.26, 724573.81, 726744.81, 708641.67,
             688934.33, 669226.8, 652195.19, 638155.24, 630686.9,
             646508.56, 665833.27, 690973.51, 712259.35,
             736907.72, 738219.28, 754322.4, 774900.77,
             780075.31, 806703.85, 818993, 808465.65, 803667.66,
```

```
             808063.81, 795247.17, 794788.94, 798010.28,
             780420.81, 782175.75, 782237.78, 786424.28,
             781489.57, 758138.79, 709801.34, 694317.72,
             706222.42, 721685.61, 736031.27, 764986.11,
             791020.76, 819059.8, 801836.14, 820679.92,
             819766.86, 830890.06, 830177.27, 830330.93,
             820611.52, 807171.72, 787401.85, 758391.75,
             724262.96, 713065.81, 690461.42, 665083.58,
             642302.3, 619905.53, 616813.7, 606548.56, 591775.06,
             594740.49, 603223.79, 619066.1, 644018.51,
             674298.27, 552941.48, 519673.65, 496971.56,
             467863.69, 414458.55, 374726.01, 372418.07,
             364512.81, 349289.51, 323837.66, 299739.11,
             274434.48, 248359.86, 226145.16, 206983.97,
             186341.08, 162814.27, 139347.85, 116899.8, 97594.19,
             80572.22, 64176.24, 50281.62, 38456.32, 29221.5,
             22483.32, 14335.43, 7534.56, 4527.78, 3024.29,
             1969.86, 1190.85, 687.88, 355.12, 171.99, 78.79,
             34.08, 14.56, 8.49)

ggplot() +
  geom_point(aes(x=age, y=exposure)) +
  geom_line(aes(x=age, y=exposure)) +
  labs(title='England & Wales 2018: total exposure by age') +
  theme_bw()
```



```r
ggplot() +
  geom_point(aes(x=age, y=deaths), color='red') +
  geom_line(aes(x=age, y=deaths), color='red') +
  labs(title='England & Wales 2018: total deaths by age') +
  theme_bw()
```



Now calculate the period life expectancy:

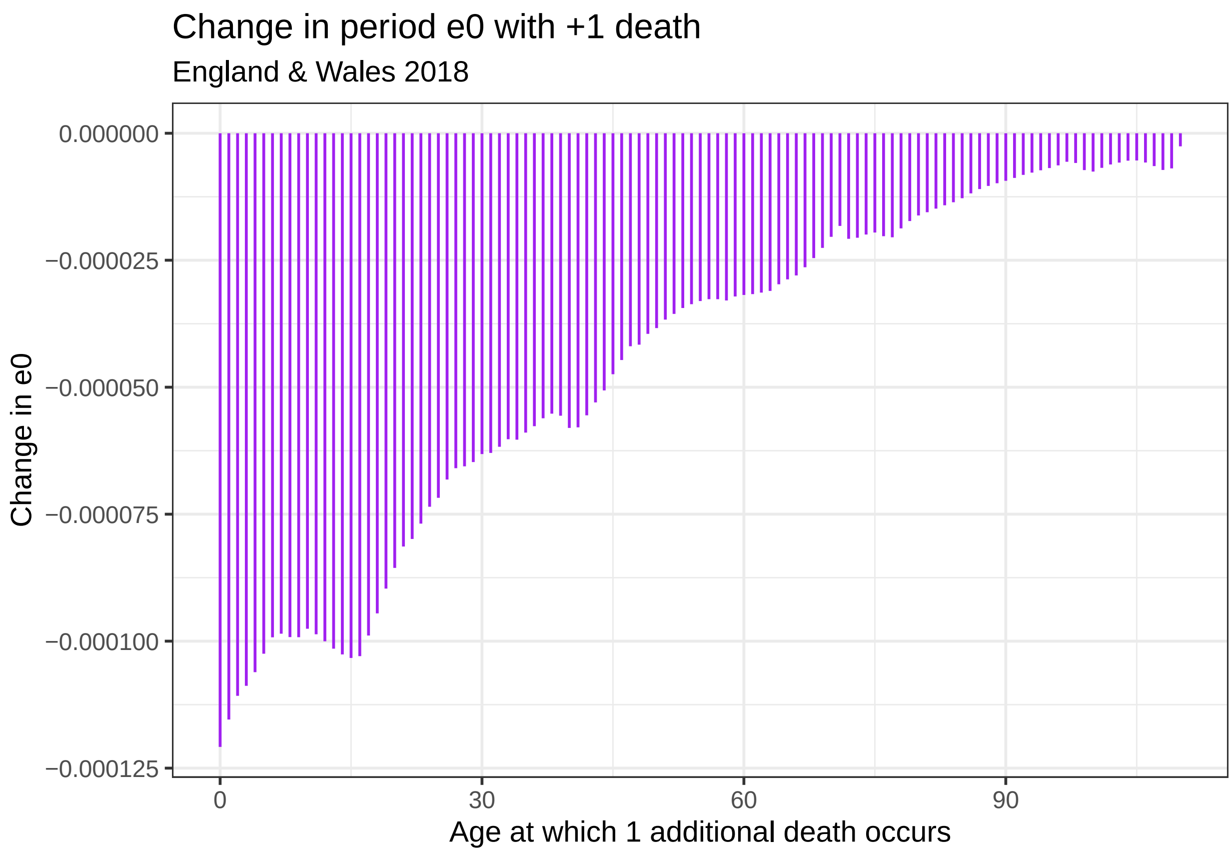
```r
e0 = function(D,N) {
  mu = D/N
  H  = cumsum(c(0,mu))
  lx = exp(-H)
  sum( (tail(lx,-1) + head(lx,-1))/2)
}

e0_actual = e0(deaths,exposure)

round(e0_actual, 2)
```

```
## [1] 81.36
```

Calculate the changes in period life expectancy if there were +1 death at ages 0 or 1 or 2 or .. 110. Plot the results.

```r
delta_e0 = sapply(0:110,
                  function(x) {
                    new_deaths = deaths + 1*(age==x)
                    new_e0 = e0(new_deaths,exposure)
```

```
                    return(new_e0 - e0_actual)
                  })

G = ggplot() +
  geom_segment(aes(x=age,y=0,xend=age,yend=delta_e0),color='purple') +
  labs(title='Change in period e0 with +1 death',
       subtitle='England & Wales 2018',
       x = 'Age at which 1 additional death occurs',
       y = 'Change in e0') +
  theme_bw()

print(G)
```



Overlay the theoretical values (some small differences due to the integer nature of the observed deaths)

```
Tx = function(D,N,x) {
  mu = D/N
  H  = cumsum(c(0,mu))
  lx = exp(-H)
  lx = lx[0:110 >= x]
  sum( (tail(lx,-1) + head(lx,-1))/2)
}

# theoretical effect is -Tx/Nx for one additional
# death at age x

Tvals = sapply(0:110, function(x) Tx(deaths,exposure,x))
```

```r
G + geom_line(aes(x=age,y=-Tvals/exposure),
              size=1.5,color='navy', alpha=.50) +
    geom_text(aes(x=70,y=-8e-5), label='+theoretical calculation',
              color='navy')
```