Math2411 - Calculus II - Spring 2022
Final Exam Formula Sheet

University of Colorado Denver / College of Liberal Arts and Sciences

Department of Mathematics

Convergence/Divergence Tests

Divergence Test: If $\sum a_k$ converges then $\lim\limits_{k\to\infty} a_k = 0$. Equivalently, if $\lim\limits_{k\to\infty} a_k \neq 0$ then $\sum a_k$ diverges.

Geometric Series Test: Let r and a be real numbers. If $|r| < 1$, then $\displaystyle\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$, then $\displaystyle\sum_{k=0}^{\infty} ar^k$ diverges.

Integral Test: Suppose f is a continuous, positive, and decreasing function for all $x \geq 1$ and suppose $a_k = f(k)$ for $k = 1, 2, 3, \ldots$. Then the following series and improper integral

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

Ratio Test: Let $\sum a_k$ be an infinite series and let $r = \lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$. Then we have the following.

1. If $0 \leq r < 1$ the series converges.
2. If $r > 1$ the series diverges.
3. If $r = 1$ the test is inconclusive.

Root Test: Let $\sum a_k$ be an infinite series with positive terms and let $\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$. Then we have the following.

1. If $0 \le \rho < 1$ the series converges.
2. if $\rho > 1$ the series diverges.
3. If $\rho = 1$ the test is inconclusive.

Direct Comparison Test: Let $\sum a_k$ and $\sum b_k$ be infinite series with positive terms.

1. $0 \le a_k \le b_k$ and $\sum b_k$ converges, then $\sum a_k$ also converges.
2. $0 \le b_k \le a_k$ and $\sum b_k$ diverges, then $\sum a_k$ also diverges.

Limit Comparison Test: Let $\sum a_k$ and $\sum b_k$ be infinite series with positive terms and

$$L = \lim_{k\to\infty} \frac{a_k}{b_k}.$$

1. If $0 < L < \infty$ (that is L is a positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
2. If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ also converges.
3. If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ also diverges.

Alternating Series Test: The alternating series $\sum (-1)^k a_k$ converges provided

1. the terms of the sequence are eventually decreasing in magnitude; i.e., $0 \le a_{k+1} < a_k$ for big enough k.
2. $\lim_{k\to\infty} a_k = 0$.

Alternating Series Remainder: Let $R_n = |S - S_n|$ be the remainder in approximating the value of a convergent alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$. Then $R_n \le a_{n+1}$.

Taylor Polynomial for the function f centered at a

$$p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Important Taylor Series

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \cdots, \quad \text{for } -1 < x < 1 \\
\ln(1-x) &= -\sum_{k=1}^{\infty} \frac{x^k}{k} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \quad \text{for } -1 \le x < 1 \\
\ln(x) &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots, \quad \text{for } 0 < x \le 2 \\
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \quad \text{for all } x \in \mathbb{R} \\
\sin(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots, \quad \text{for all } x \in \mathbb{R} \\
\cos(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots, \quad \text{for all } x \in \mathbb{R} \\
\tan^{-1}(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots, \quad \text{for } -1 \le x \le 1
\end{aligned}$$

Integral Application Formulas

Area of a Region Between Two Curves: Suppose that f and g are continuous and $f(x) \ge g(x)$ whenever $a \le x \le b$. Then the area between the curves $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ equals

$$\text{Area} = \int_a^b f(x) - g(x)\, dx$$

Disc Method for Solid of Revolution About the $\mathbf{x}$***-Axis:*** Suppose that f is continuous and $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the curve $y = f(x)$ and the x-axis is revolved about the x-axis, the formula for the volume of the solid of revolution is given by

$$\text{Volume} = \pi \int_a^b \left[f(x)\right]^2 dx.$$

Surface Area for Solid of Revolution About the $\mathbf{x}$***-Axis:*** Suppose that f is continuous and $f(x) \geq 0$ on the interval $[a, b]$ and rotated about the x-axis. Then the formula for the surface area of the surface of revolution is given as

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

Arclength: Let f have continuous derivative on the interval $[a, b]$. Then the arclength on the curve $y = f(x)$ on the interval is given as

$$s = \int_{x=a}^{x=b} \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

Work Formula for Pumping Water: Suppose we need to pump water out of a tank of height h with water depth d. If the surface area of an arbitrary cross section of water is given as $A(y)$ then the work required to pump the water out of the tank is

$$\text{Work} = \rho g \int_{y=0}^{y=d} A(y)(h - y)\, dy.$$

Trigonometric Substitution Formulas

1. ***Integral contains*** $\sqrt{a^2 - x^2}$: Let $x = a \sin\theta$.
2. ***Integral contains*** $\sqrt{x^2 + a^2}$: Let $x = a \tan\theta$.
3. ***Integral contains*** $\sqrt{x^2 - a^2}$: Let $x = a \sec\theta$.

Some Useful Trigonometric Identities

1. $\sin^2(x) + \cos^2(x) = 1$
2. $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$
3. $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$
4. $\sin(2x) = 2\sin(x)\cos(x)$
5. $\sec^2(x) = \tan^2(x) + 1$

Numerical Integral Approximation Formulas

1. Midpoint Rule: $\displaystyle\int_a^b f(x)\,dx \approx \Big(f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)\Big)\triangle x$, where x_i^* is the midpoint of the i^{th} interval.
2. Trapezoid Rule: $\displaystyle\int_a^b f(x)\,dx \approx \frac{1}{2}\Big(f(x_0)+2f(x_1+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\Big)\triangle x.$
3. Simpson's Rule: $\displaystyle\int_a^b f(x)\,dx \approx \Big(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots 4f(x_{n-1}) + f(x_n)\Big)\frac{\triangle x}{3}.$

An Important Definition

We define the ***exponential*** as $e^a = \displaystyle\lim_{k\to\infty}\left(1 + \frac{a}{k}\right)^k.$