Neutrino-Nucleus Interactions

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OUTLINE

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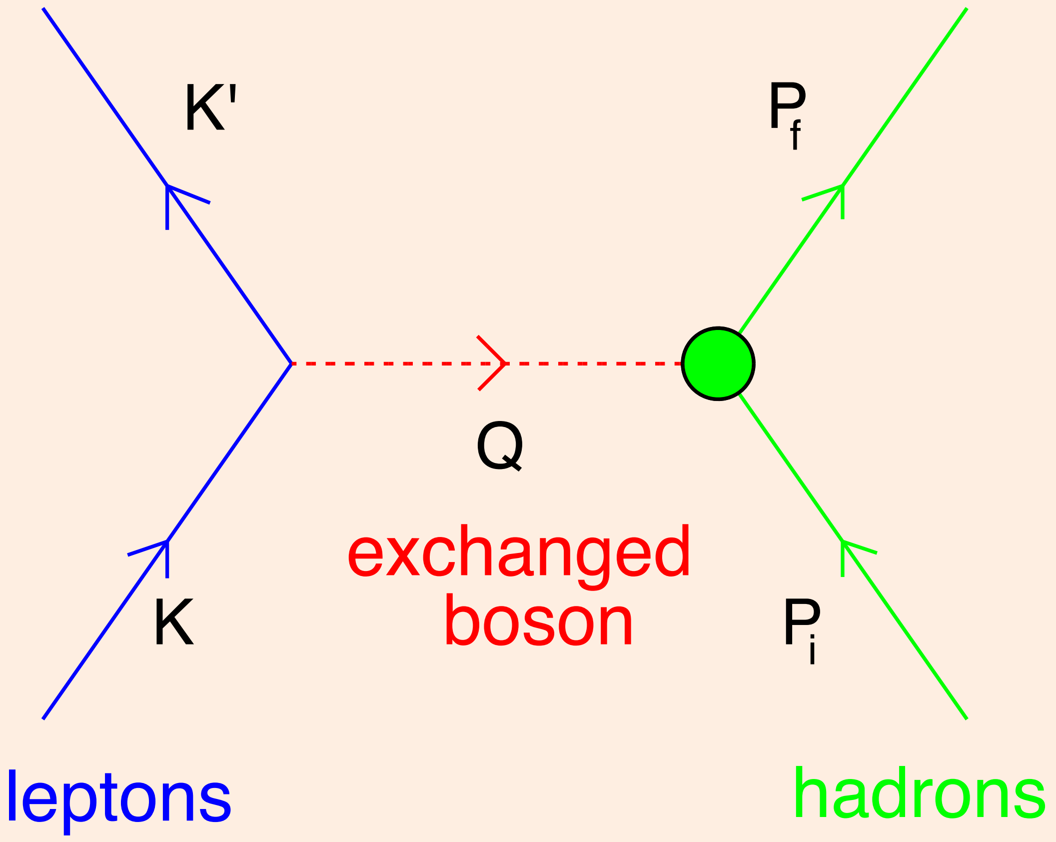
1. Inclusive electron scattering: general form of cross section

Inclusive: only K' measured



All semi-leptonic electroweak reactions are closely related. In the one-boson-exchange approximation (*i.e.*, one γ, $W^{\pm}$ or Z exchange) we have the general form

$$\sigma \sim \eta_{\mu\nu} W^{\mu\nu} = \eta^{s}_{\mu\nu} W^{\mu\nu}_{s} + \eta^{a}_{\mu\nu} W^{\mu\nu}_{a},$$

where s and a stand for symmetric and anti-symmetric under exchange of μ and ν.

Specifically, consider first inclusive electron scattering $\leftrightarrow$ (e, e') where only the scattered electron is presumed to be detected. Furthermore, assume that no polarizations are involved (for the general situation of parity-conserving inclusive electron scattering with polarizations see TWD, *Prog. Part. Nucl. Phys.* **13** (1985) 183 and TWD and A. S. Raskin, *Ann. Phys.* **169** (1986) 247), in which case only the symmetric terms above enter. Consider the hadronic side of the diagram above: we have two independent 4-vectors to work with, namely, Q^μ and P^μ_i, since $P^\mu_f = P^\mu_i + Q^\mu$ can be eliminated. Or, instead of P^μ_i, we can use

$$V^\mu_i \equiv \frac{1}{M_i} \left(P^\mu_i - \left(\frac{Q \cdot P_i}{Q^2} \right) Q^\mu \right)$$

which has the advantage that $Q \cdot V_i = 0$. The conventions here are that 4-vectors are written as capital letters

$$A^\mu = \left(A^0, A^1, A^2, A^3 \right) = \left(A^0, \mathbf{a} \right),$$

where 3-vectors are written in lower case. We have, for instance, that $A \cdot B = A^0 B^0 - \mathbf{a} \cdot \mathbf{b}$, with $a = |\mathbf{a}|$, *etc.*

We proceed to prove a familiar theorem which will be generalized below. The most general symmetric hadronic tensor in this case may be written

$$\begin{aligned} [W_s^{\mu\nu}]_{incl}^{unpol} &= X_{incl}^{(1)} g^{\mu\nu} + X_{incl}^{(2)} Q^\mu Q^\nu + X_{incl}^{(3)} V_i^\mu V_i^\nu \\ &\quad + X_{incl}^{(4)} \left(Q^\mu V_i^\mu + V_i^\mu Q^\nu \right), \end{aligned}$$

where the $X_{incl}^{(i)}$ are invariant functions of the available scalars. The latter are $Q^2 = \omega^2 - q^2$, $Q \cdot P_i$ and $P_i^2 = M_i^2$, where the only dynamical variables are Q^2 and $Q \cdot P_i$ (or q and ω, or Q^2 and $\nu = \omega$, or Q^2 and $x\ldots$ it does not matter which set of two variables, since they are all simply related). Thus,

$$X_{incl}^{(i)} = X_{incl}^{(i)} \left(Q^2, Q \cdot P_i \right).$$

Now specializing to PC electron scattering where

$$W^{\mu\nu} \sim J^{\mu *} J^\nu$$

with J^μ involving matrix elements of the EM current. Since this is a conserved current we have $Q \cdot J = 0$, and hence that $Q_\mu W_s^{\mu\nu} = 0$. This immediately implies that

$$\left(X_{incl}^{(1)} + X_{incl}^{(2)} Q^2 \right) Q^\nu + \left(X_{incl}^{(4)} Q^2 \right) V_i^\nu = 0$$

and therefore, by the theorem of linear algebra, that

$$X_{incl}^{(1)} + X_{incl}^{(2)} Q^2 = X_{incl}^{(4)} = 0.$$

Written more conventionally we then have proven that

$$[W_s^{\mu\nu}]_{incl}^{unpol} = W_{incl}^{(1)} \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) + W_{incl}^{(2)} V_i^\mu V_i^\nu$$

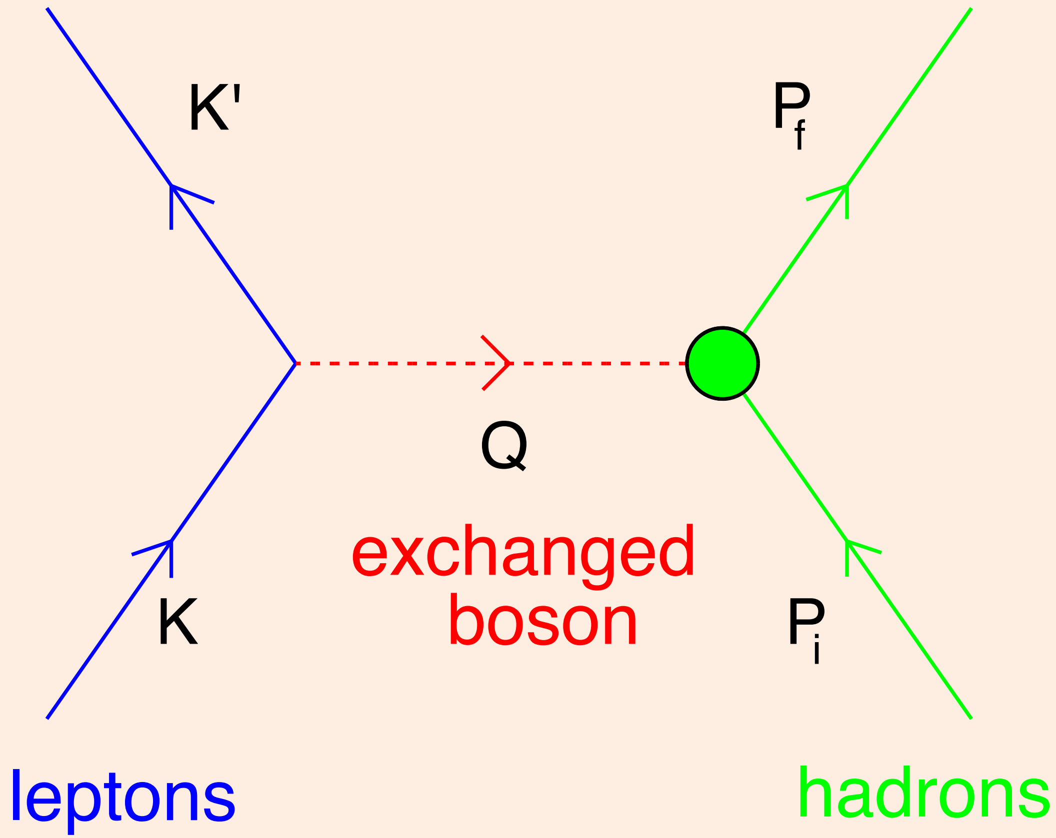
with

$$W_{incl}^{(i)} = W_{incl}^{(i)} \left(Q^2, Q \cdot P_i \right),$$

namely the standard result where one has two independent response functions, each a function of two independent scalars.

2. Inclusive charge-changing neutrino reactions: general σ

Inclusive: only K' measured



Next consider inclusive charge-changing neutrino and anti-neutrino reactions, (ν_ℓ, ℓ^-) and $(\overline{\nu}_\ell, \ell^+)$, where $\ell = e$, μ or τ. In this case we have both vector (V) and axial-vector (A) currents and so two differences from the above derivation occur. First, one can show that the symmetric terms involve only products of two polar-vectors (VV) or two axial-vectors (AA), and since the axial-vector current is not conserved, we cannot reduce the number of responses as we did above using the continuity equation, leaving us with four rather than two symmetric responses. Second, it is now possible to have anti-symmetric terms,

$$[W_a^{\mu\nu}]_{incl}^{unpol} = X_{incl}^{(5)} \epsilon^{\mu\nu\alpha\beta} Q_\alpha V_{i\beta} + X_{incl}^{(6)} \left(Q^\mu V_i^\mu - V_i^\mu Q^\nu \right).$$

Note that, although we have used the same symbols $X_{incl}^{(i)}$ for the invariant response functions, these are in general not the same as those that entered for inclusive electron scattering. Since the anti-symmetric contributions can all be shown to involve VA interferences, and since the vector current is assumed to be conserved in weak interactions processes (CVC hypothesis), one has that $X_{incl}^{(6)} = 0$ in the same way that $X_{incl}^{(4)}$ was shown to vanish above, leaving us with five independent response functions $X_{incl}^{(i)}$, $i = 1 \ldots 5$ each a function of two scalars as before.

Assuming CPT invariance, and moreover T and hence CP invariance, using C to go from neutrinos to anti-neutrinos must be compensated using P; the consequence of this is that the anti-symmetric VA terms reverse sign when the flavor is reversed.

Let us write the CCν result in a different form to make contact with other treatments of the problem (see, for instance, O. Moreno, TWD, J. W. Van Orden and W. P Ford, *Phys. Rev.* **D90** (2014) 013014 [MDVF]):

For inclusive scattering one has

$$\begin{aligned}\eta^s_{\mu\nu}W^{\mu\nu}_s &\sim \widehat{V}_{CC}W^{CC}_{incl} + \widehat{V}_{CL}W^{CL}_{incl} + \widehat{V}_{LL}W^{LL}_{incl} + \widehat{V}_{T}W^{T}_{incl} \\ \eta^a_{\mu\nu}W^{\mu\nu}_a &\sim \widehat{V}_{T'}W^{T'}_{incl},\end{aligned}$$

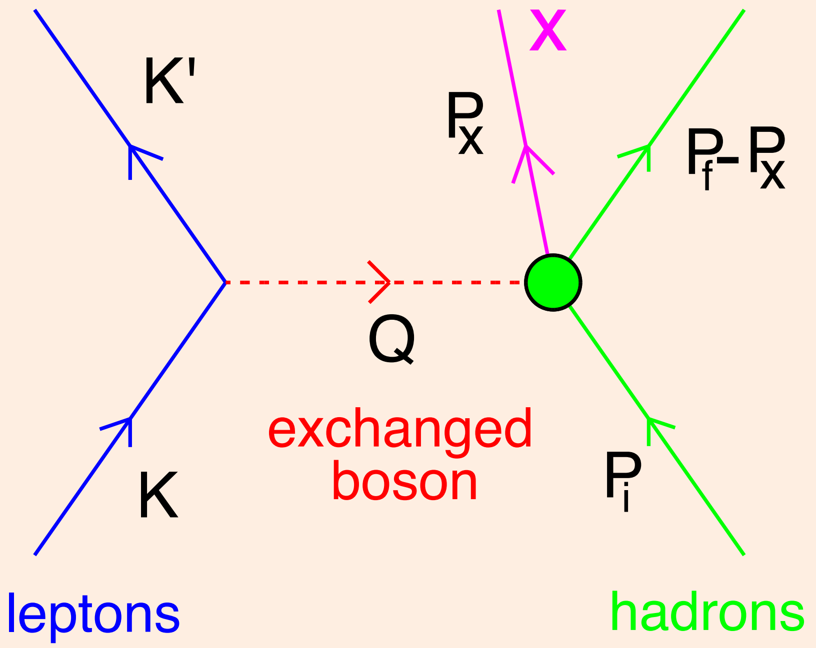
where each of the 5 responses is a function of the neutrino momentum k and 2 other variables, for instance (k', θ), the muon momentum and the lepton scattering angle, or (q, ω), the 3-momentum transfer and energy transfer:

$$\begin{aligned}W^K_{incl} &= W^K_{incl}(k; k', \theta) \\ &= W^K_{incl}(k; q, \omega),\end{aligned}$$

where $K = CC, CL, LL, T$ and T'. The factors V_K are the leptonic kinematic factors ("Rosenbluth factors") which can be found, for instance, in MDVF.

3. Semi-inclusive electron scattering: general extensions

Semi-inclusive: K' and P_x measured



Next let us sketch the similar developments for semi-inclusive electron scattering $\leftrightarrow (e, e'x)$, where one presumes that, in addition to the scattered electron, some particle x is detected in coincidence. This means that, in addition to the two 4-vectors Q^μ and P_i^μ or V_i^μ used above, we now also have P_x^μ or

$$V_x^\mu \equiv P_x^\mu - \left(\frac{Q \cdot P_x}{Q^2} \right) Q^\mu$$

with $Q \cdot V_x = 0$.

The general forms for the hadronic tensors in this case (no polarizations are assumed; for the more general semi-inclusive case with polarizations see A. S. Raskin and TWD, *Ann. Phys.* **191** (1989) 78) we have the following:

$$\begin{aligned}
[W_s^{\mu\nu}]_{semi}^{unpol} &= X_{semi}^{(1)} g^{\mu\nu} + X_{semi}^{(2)} Q^\mu Q^\nu + X_{semi}^{(3)} V_i^\mu V_i^\nu \\
&\quad + X_{semi}^{(4)} \left(Q^\mu V_i^\mu + V_i^\mu Q^\nu \right) + X_{semi}^{(5)} V_x^\mu V_x^\nu \\
&\quad + X_{semi}^{(6)} \left(Q^\mu V_x^\mu + V_x^\mu Q^\nu \right) + X_{semi}^{(7)} \left(V_i^\mu V_x^v + V_x^\mu V_i^\nu \right) \\
[W_a^{\mu\nu}]_{semi}^{unpol} &= + X_{semi}^{(8)} \left(Q^\mu V_i^\mu - V_i^\mu Q^\nu \right) + X_{semi}^{(9)} \left(Q^\mu V_x^\mu - V_x^\mu Q^\nu \right) \\
&\quad + X_{semi}^{(10)} \left(V_i^\mu V_x^v - V_x^\mu V_i^\nu \right);
\end{aligned}$$

note that no ϵ-terms can occur here, since the tensors must behave as the products two polar-vectors.

Upon employing the continuity equation again (these steps can be seen in TWD, *Prog. Part. Nucl. Phys.* **13** (1985) 183 or in Chapter 16 of *Foundations of Nuclear and Particle Physics*, TWD, J. A. Formaggio, B. R. Holstein, R. G. Milner and B. Surrow, Cambridge University Press (2017)), one finds that

$$\begin{aligned}\left(X_{semi}^{(1)} + X_{semi}^{(2)} Q^2\right) Q^\nu + \left(X_{semi}^{(4)} Q^2\right) V_i^\nu + \left(X_{semi}^{(6)} Q^2\right) V_x^\nu &= 0 \\ \left(X_{semi}^{(8)} Q^2\right) V_i^\nu + \left(X_{semi}^{(9)} Q^2\right) V_x^\nu &= 0\end{aligned}$$

and hence that

$$X_{semi}^{(1)} + X_{semi}^{(2)} Q^2 = X_{semi}^{(4)} = X_{semi}^{(6)} = X_{semi}^{(8)} = X_{semi}^{(9)} = 0$$

leaving 4 symmetric contributions together with a new 5th anti-symmetric contribution.

For unpolarized semi-inclusive parity-conserving electron scattering the leptonic tensor is symmetric and thus the 5th response does not enter; however, with polarized electrons it does enter, even if parity-violating effects are neglected.

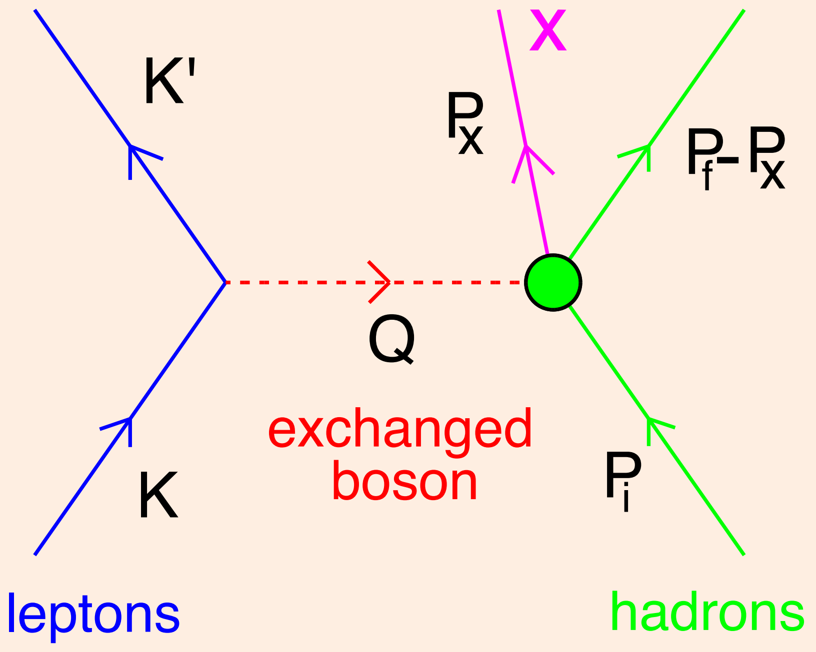
Each of the five invariant functions that remain is a function of four dynamical scalars, for instance, Q^2, $Q \cdot P_i$, $Q \cdot P_x$ and $P_i \cdot P_x$.

Also, if one integrates over all allowed kinematics for the outgoing particle x, does this for all open channels, and then adds the results one recovers the inclusive cross section.

The procedures here may be extended to more exclusive reactions of the type $(e, e' x_1 \cdots x_n)$ with $n \geq 2$ as discussed in TWD, *Prog. Part. Nucl. Phys.* **13** (1985) 183.

4. Semi-inclusive CCν reactions: general extensions

Semi-inclusive: K' and P_x measured



The semi-inclusive CCν formalism for reactions of the type $(\nu_\ell, \ell^- N)$ and $(\overline{\nu}_\ell, \ell^+ N)$, where $N = p$ or n, is presented in MDVF and follows directly from the ideas just sketched (generalizations to other particles $x \neq N$ are trivial). The differences from the electron scattering case are similar to what happened for inclusive scattering: since now the axial-vector current is present and since it is not conserved, one has 4 symmetric and 1 anti-symmetric contributions of VV type, 7 symmetric and 3 anti-symmetric contributions of AA type, and 3 contributions of VA type.

Let us next consider some issues that have to do with how CCν are studied experimentally versus what is more natural theoretically; here we make contact with MDVF including some of the nomenclature used there.

… thus, to begin, we assume that the muon's kinematics are known (k',θ) for given neutrino momentum (k), and that a nucleon is detected having $(p_N,\theta_N^L,\phi_N^L)$ in the laboratory system

Next, we change variables to the q-system where coordinates are specified with respect to the direction of the 3-momentum transfer: (p_N,θ_N,ϕ_N)

Note that as the neutrino momentum k changes the direction of q also changes, and thus (θ_N,ϕ_N) change

The general form of the cross section involves the contraction of the leptonic and hadronic tensors:

$$\sigma \sim \eta_{\mu\nu} W^{\mu\nu} = \eta^s_{\mu\nu} W^{\mu\nu}_s + \chi\, \eta^a_{\mu\nu} W^{\mu\nu}_a,$$

where $\chi = 1$ for incident neutrinos and $\chi = -1$ for antineutrinos.

In contrast, for semi-inclusive reactions one has more terms. Specifically, for CCν reactions (see MDVF), one has the following completely general structure:

$$\begin{aligned} \eta^s_{\mu\nu} W^{\mu\nu}_s \quad &\sim \quad \widehat{V}_{CC} W^{CC}_{semi} + \widehat{V}_{CL} W^{CL}_{semi} + \widehat{V}_{LL} W^{LL}_{semi} \\ &\quad + \widehat{V}_T W^{T}_{semi} + \widehat{V}_{TT} W^{TT}_{semi} + \widehat{V}_{TC} W^{TC}_{semi} + \widehat{V}_{TL} W^{TL}_{semi} \end{aligned}$$

$$\eta^a_{\mu\nu} W^{\mu\nu}_a \sim \widehat{V}_{T'} W^{T'}_{semi} + \widehat{V}_{TC'} W^{TC'}_{semi} + \widehat{V}_{TL'} W^{TL'}_{semi}$$

$$\begin{aligned} W^K_{semi} \quad &= \quad W^K_{semi}(k; k', \theta; p_N, \theta^L_N, \phi^L_N) \\ &= \quad W^K_{semi}(k; q, \omega; p, \mathcal{E}, \phi_N). \end{aligned}$$

Total: 10 response functions, each a function of 6 variables. (Actually, in general there are 16 classes of response for electroweak reactions of all types; for CCν reactions 6 cases do not have the corresponding leptonic factors, labeled $\underline{TT}, \underline{TC}.\underline{TL}, \underline{CL}', \underline{TC}', \underline{TL}'$.)

The ϕ_N dependence can be made explicit, leaving 6 individual responses, each a function of 5 variables, say $(k; q, \omega; p, \mathcal{E})$:

$$
\begin{aligned}
W^{CC}_{semi} &= \frac{1}{\rho^2}\left\{\rho^2 X_1 + \rho\nu^2 X_2 + X_3 + 2\sqrt{\rho}\nu X_4 \right. \\
&\quad \left. + H^2 X_5 + 2\sqrt{\rho}\nu H X_6 + 2H X_7\right\} \\
W^{CL}_{semi} &= \frac{\nu}{\rho^2}\left\{\rho X_2 + X_3 + \sqrt{\rho}(\frac{1}{\nu}+\nu) X_4 \right. \\
&\quad \left. + H^2 X_5 + \sqrt{\rho}(\frac{1}{\nu}+\nu) H X_6 + 2H X_7\right\} \\
W^{LL}_{semi} &= \frac{1}{\rho^2}\left\{-\rho^2 X_1 + \rho X_2 + \nu^2 X_3 + 2\sqrt{\rho}\nu X_4 \right. \\
&\quad \left. + \nu^2 H^2 X_5 + 2\sqrt{\rho}\nu H X_6 + 2\nu^2 H X_7\right\} \\
W^{T}_{semi} &= -2X_1 + X_5 \eta_T^2 \\
W^{TT}_{semi} &= -X_5 \eta_T^2 \cos 2\phi_N \\
W^{TC}_{semi} &= \frac{2\sqrt{2}}{\rho}\eta_T \left\{H X_5 + \sqrt{\rho}\nu X_6 + X_7\right\} \cos\phi_N \\
W^{TL}_{semi} &= \frac{2\sqrt{2}}{\rho}\eta_T \left\{\nu H X_5 + \sqrt{\rho} X_6 + \nu X_7\right\} \cos\phi_N \\
W^{T'}_{semi} &= \frac{1}{\sqrt{\rho}}\left\{Z_1 + H Z_2\right\} \\
W^{TC'}_{semi} &= \frac{2\sqrt{2}}{\rho}\eta_T \left\{-\left(\sqrt{\rho}\nu Y_2 + Y_3\right)\sin\phi_N + \left(\sqrt{\rho} Z_2 + \nu Z_3\right)\cos\phi_N\right\} \\
W^{TL'}_{semi} &= \frac{2\sqrt{2}}{\rho}\eta_T \left\{-\left(\sqrt{\rho} Y_2 + \nu Y_3\right)\sin\phi_N + \left(\sqrt{\rho}\nu Z_2 + Z_3\right)\cos\phi_N\right\},
\end{aligned}
$$

where

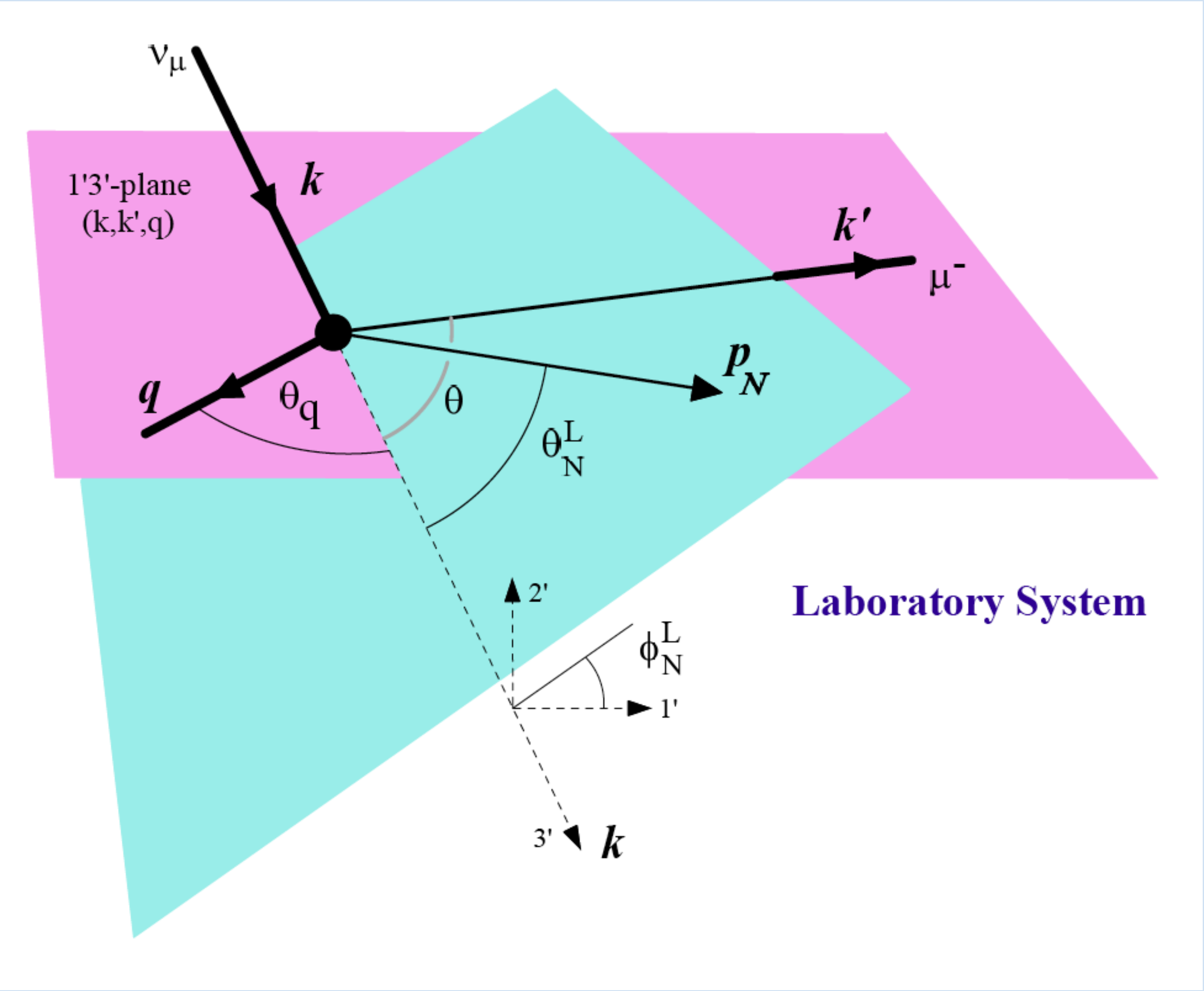
$$
\begin{aligned}
\eta_T &\equiv \frac{p_N}{m_N}\sin\theta_N \\
H &\equiv \frac{1}{m_N}\left[E_N - \nu p_N \cos\theta_N\right],
\end{aligned}
$$

and where the $X\,s$, $Y\,s$ and $Z\,s$ are functions of $(k; q, \omega; p, \mathcal{E})$, that is, are responses in 5-dimensional space.

5. Semi-inclusive CCν reactions: experiment versus theory

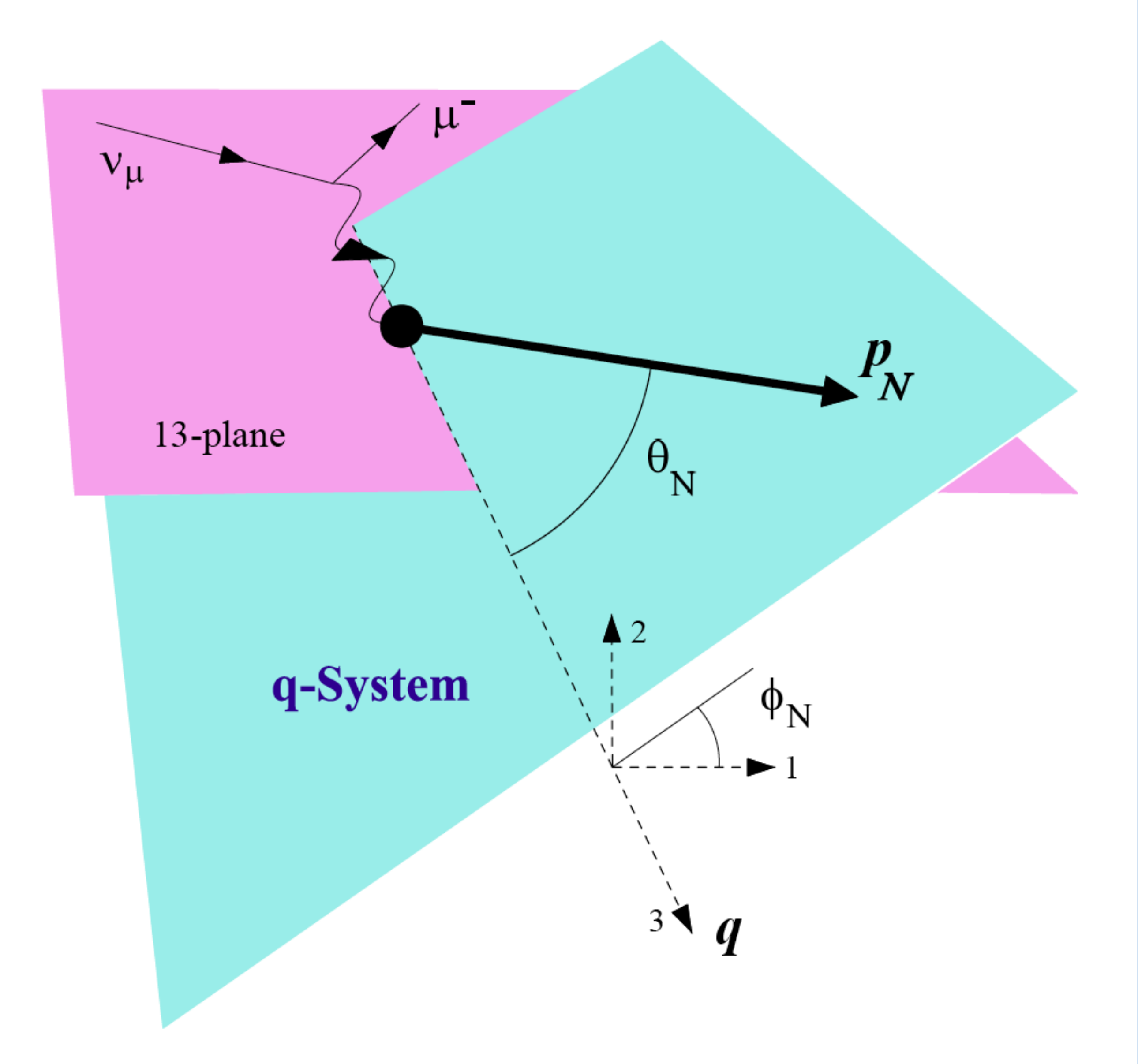
… thus, to begin, we assume that the muon's kinematics are known (k',θ) for given neutrino momentum (k), and that a nucleon is detected having $(p_N,\theta_N{}^L,\phi_N{}^L)$ in the laboratory system

5. Semi-inclusive CCν reactions: experiment versus theory



… thus, to begin, we assume that the muon's kinematics are known (k',θ) for given neutrino momentum (k), and that a nucleon is detected having $(p_N,\theta_N{}^L,\phi_N{}^L)$ in the laboratory system

Next, we change variables to the q-system where coordinates are specified with respect to the direction of the 3-momentum transfer: (p_N,θ_N,ϕ_N)



ν_μ
μ^-
p_N
13-plane
θ_N
2
q-System
ϕ_N
1
3
q

… thus, to begin, we assume that the muon's kinematics are known (k', θ) for given neutrino momentum (k), and that a nucleon is detected having $(p_N, \theta_N^L, \phi_N^L)$ in the laboratory system

Next, we change variables to the q-system where coordinates are specified with respect to the direction of the 3-momentum transfer: (p_N, θ_N, ϕ_N)

Note that as the neutrino momentum k changes the direction of q also changes, and thus (θ_N, ϕ_N) change

… however, these variables are better suited to treating the general form of the semi-inclusive cross section

6. Semi-inclusive CCν reactions: "trajectories"

All of this is fine; however, it does not capture where the nuclear response is large or small. To do this, it is better to change variables yet again to the missing energy and missing momentum, E_m and p_m, as is well known from studies of (e,e'p) reactions.

Actually I will use $\mathcal{E}$ and $\mathbf{p} = -\mathbf{p}_m$, as is traditional in scaling analyses

($\mathcal{E}$ is approximately $E_m - E_s$)

See the Appendix for a summary of all of these coordinate transformations and definitions of all variables

Example: $^{16}O(\nu_\mu,\mu^- p)$ with

$k' = 1$ GeV/c for the muon

$\theta = 10$ deg.

$p_N = 50$ MeV/c for the proton

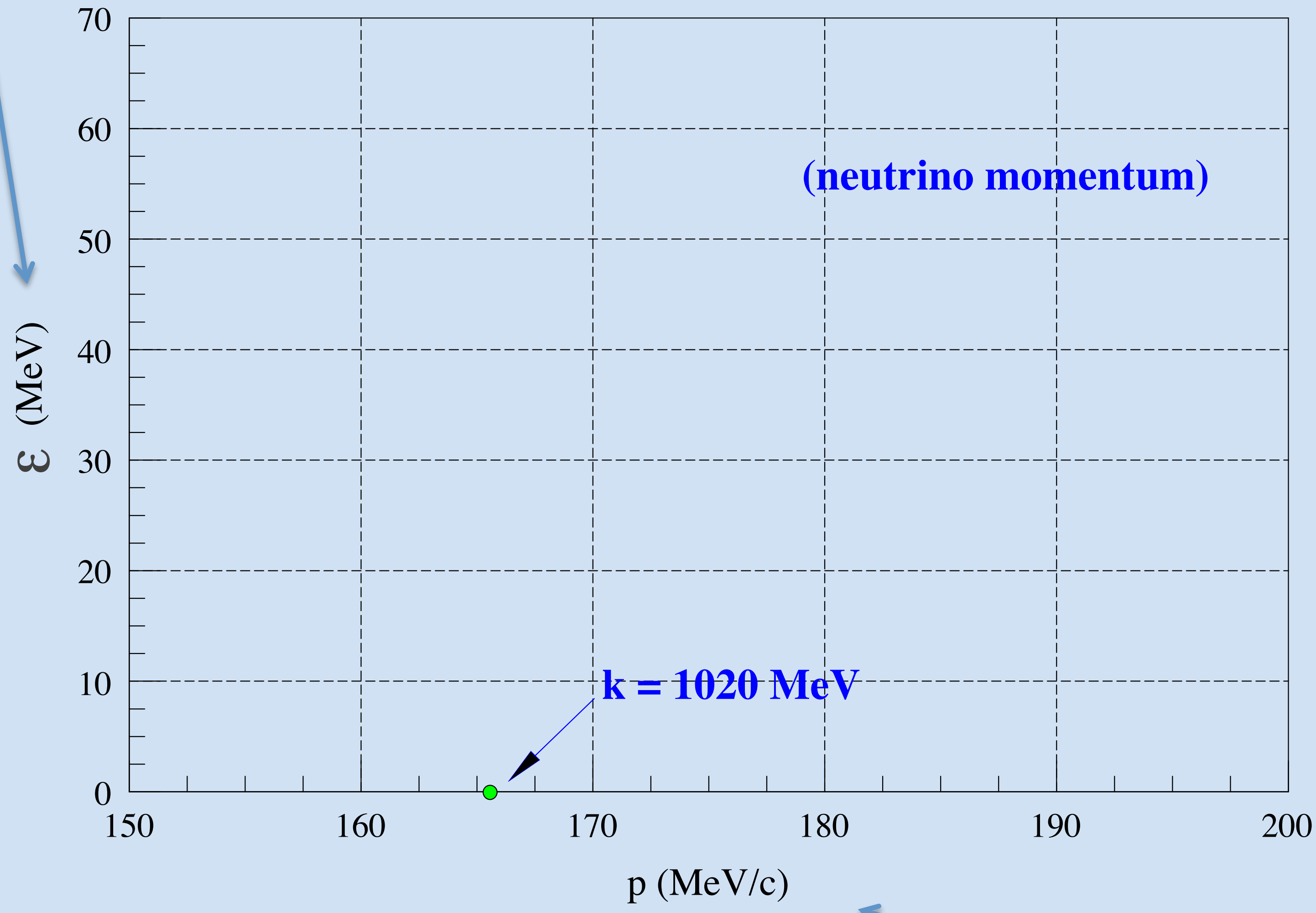
$\theta_N^L = 10$ deg.

$\phi_N^L = 180$ deg.

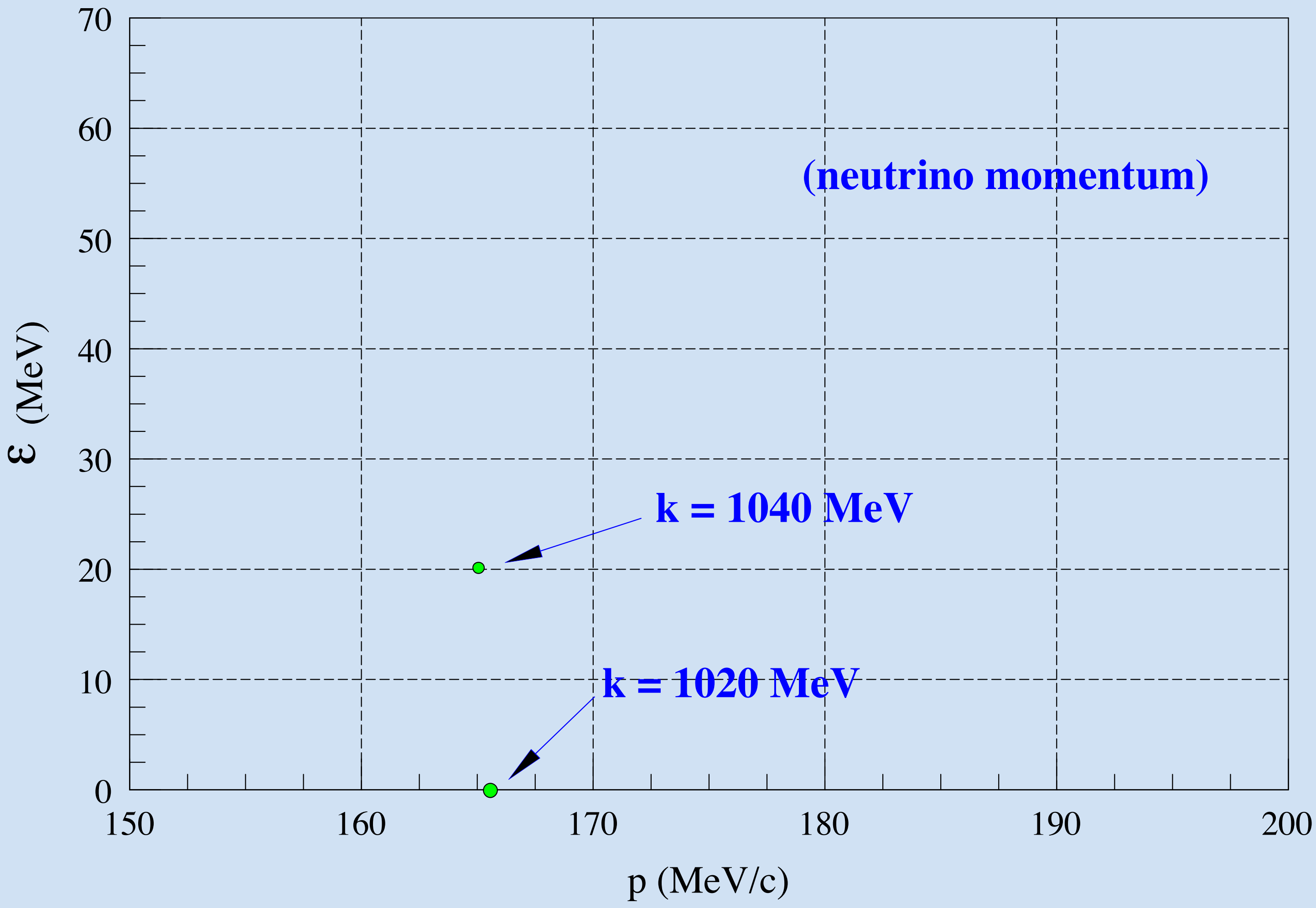
all of which will be kept fixed for this example

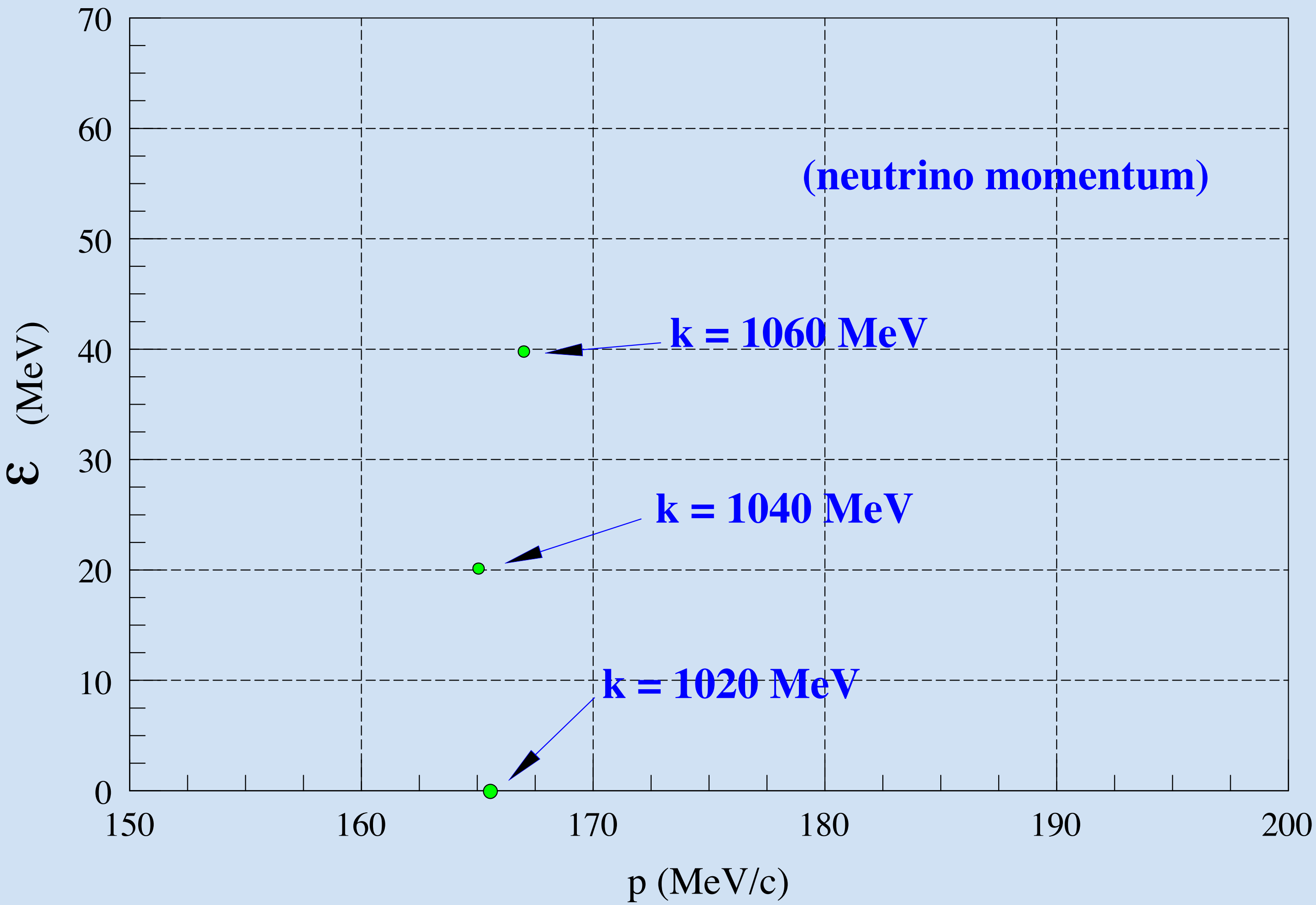
… starting with neutrino momentum k = 1020 MeV/c

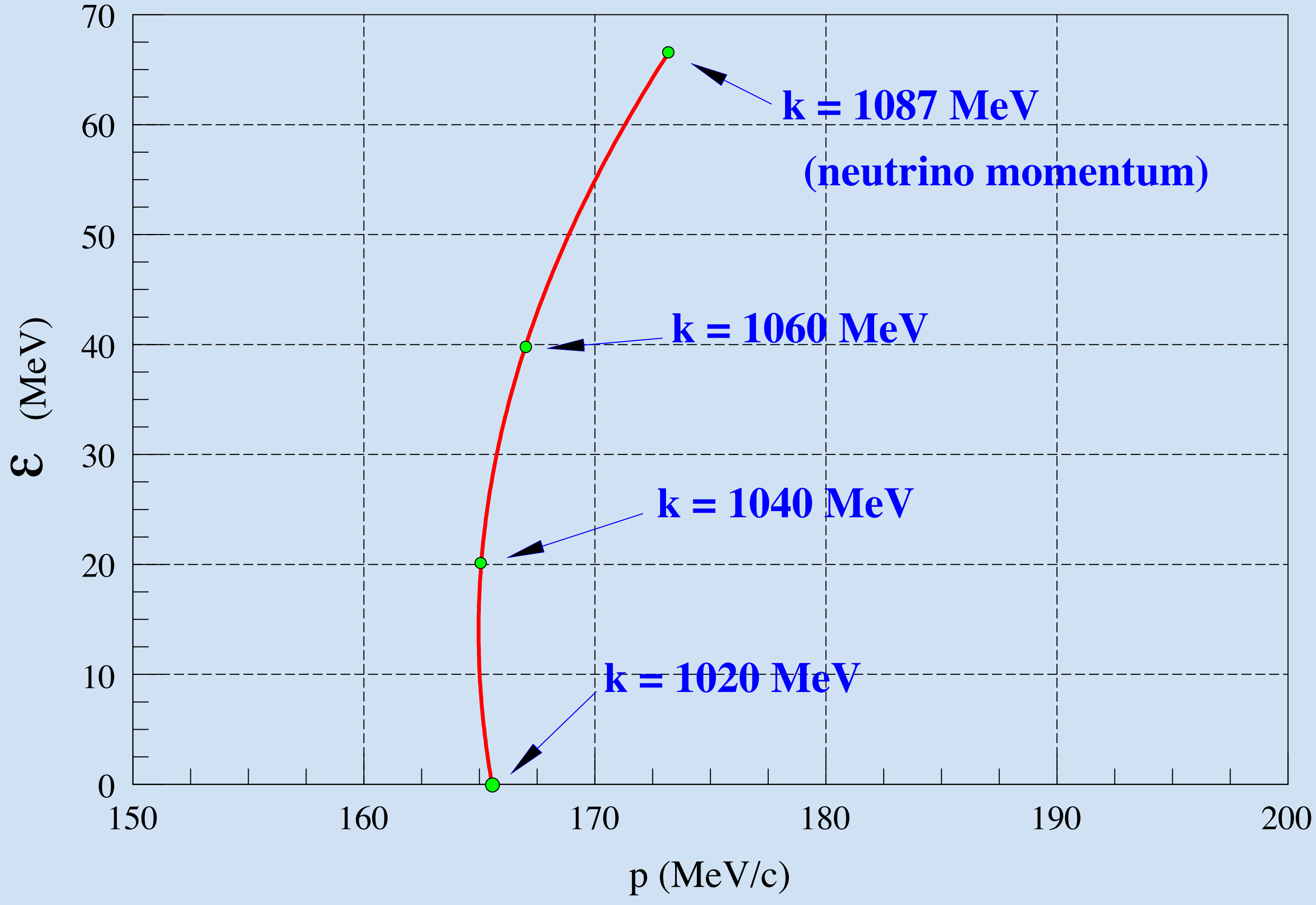
Missing energy – Separation energy



Missing momentum


(neutrino momentum)
k = 1040 MeV
k = 1020 MeV
ε (MeV)
p (MeV/c)
0
10
20
30
40
50
60
70
150
160
170
180
190
200

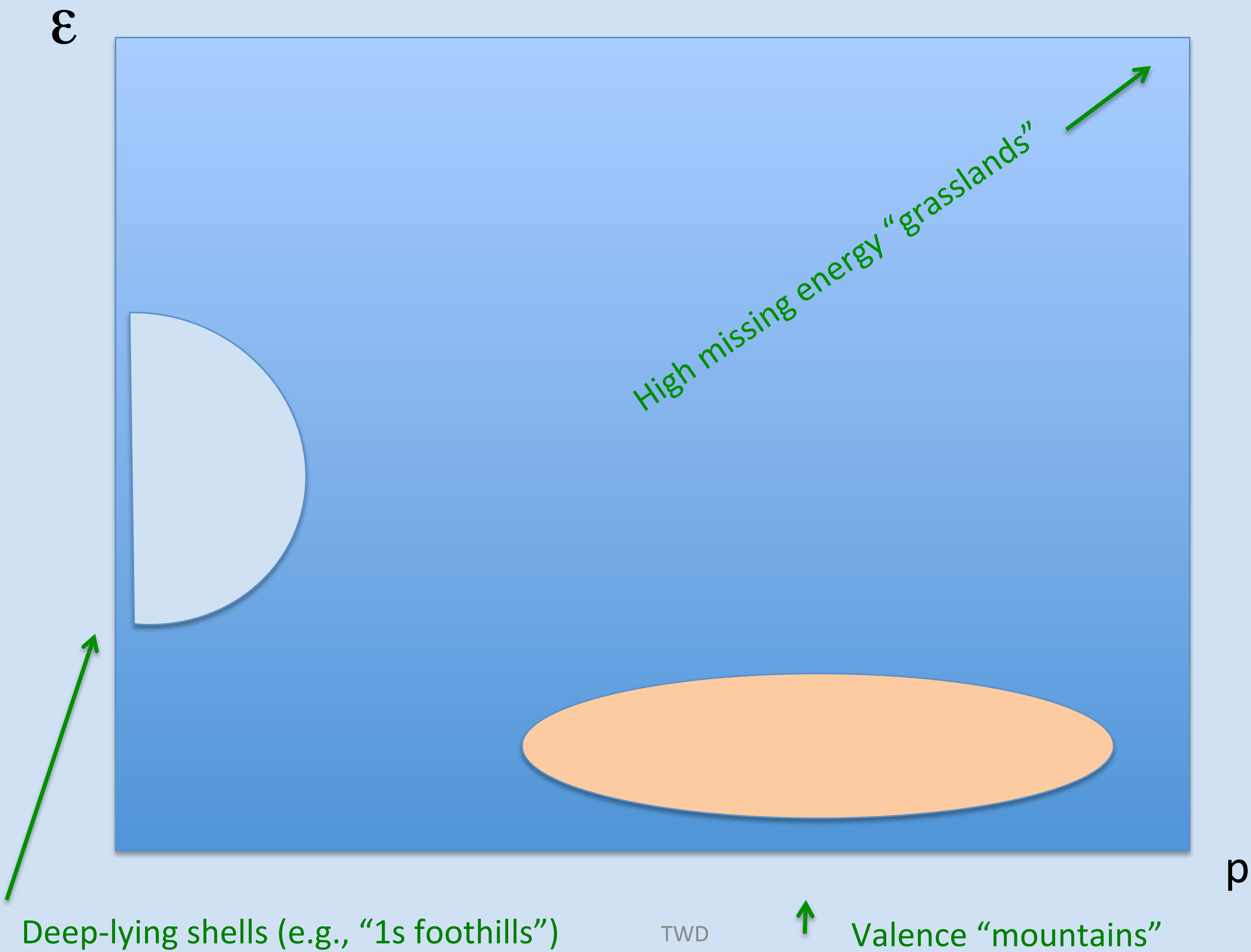

70
60
50
40
30
20
10
0
150
160
170
180
190
200
ε (MeV)
p (MeV/c)
(neutrino momentum)
k = 1060 MeV
k = 1040 MeV
k = 1020 MeV



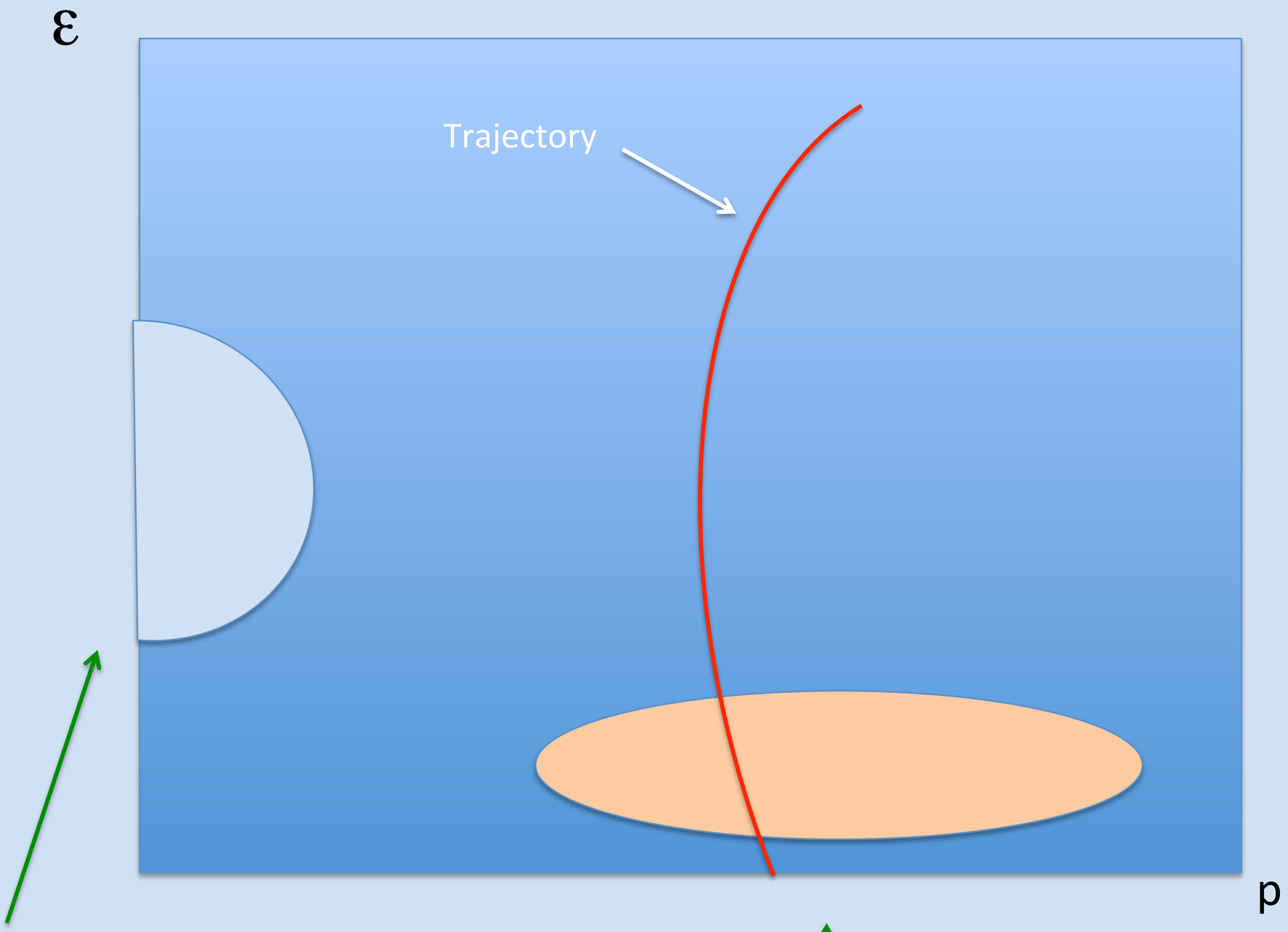
k = 1087 MeV
(neutrino momentum)
k = 1060 MeV
k = 1040 MeV
k = 1020 MeV
ε (MeV)
p (MeV/c)
0
10
20
30
40
50
60
70
150
160
170
180
190
200

Discussion of the **landscape** and **trajectories** in the missing energy – missing momentum plane:
this is where modeling comes into play

Generic landscape



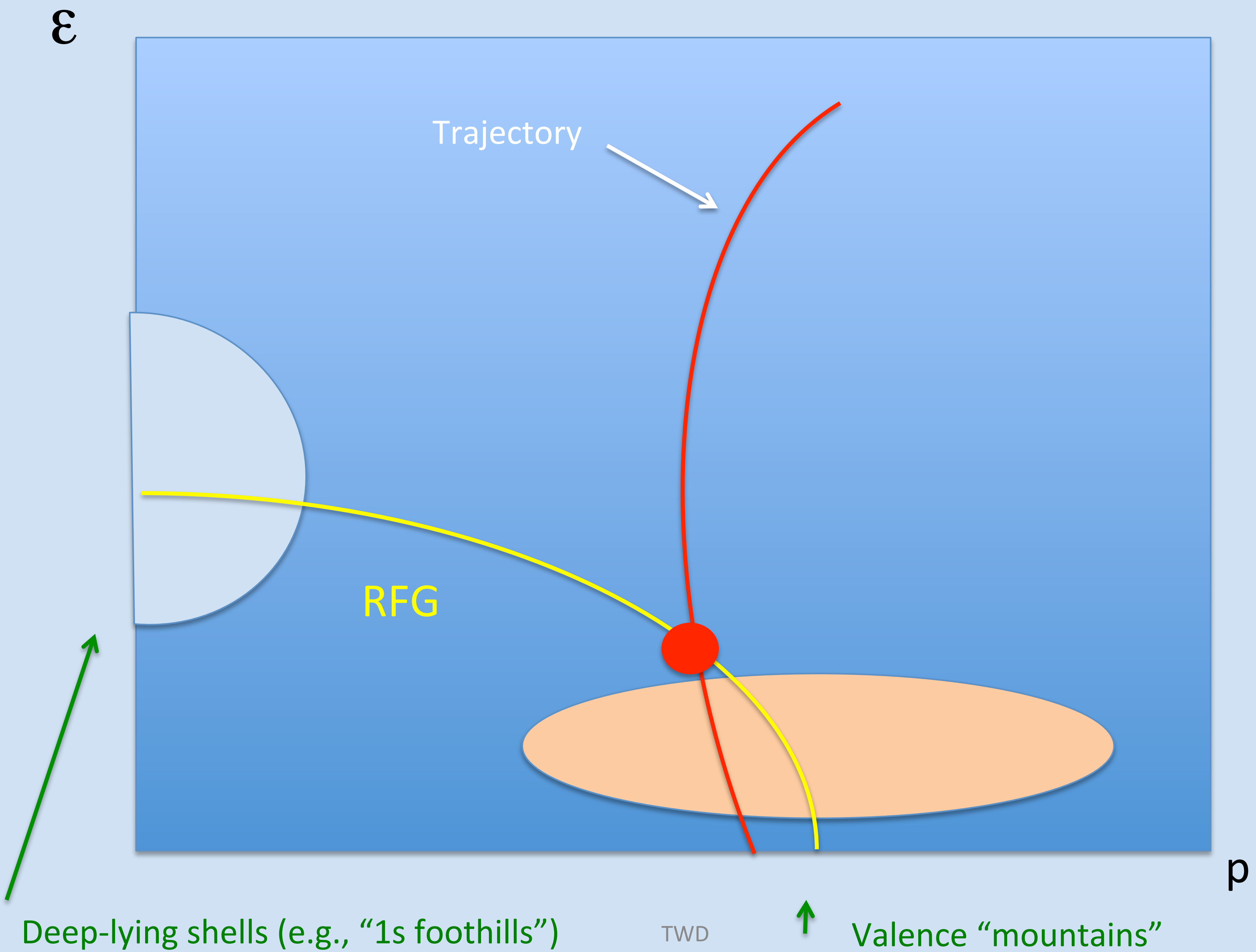
Generic landscape



Deep-lying shells (e.g., “1s foothills”)

Valence “mountains”

Generic landscape



7. Summary

- For inclusive scattering integrals must be done over the “landscape” whose boundaries are determined by the lepton kinematics

- Most modeling is done for inclusive reactions and there one finds that, as long as the models used are relativistic, the results are not dramatically different (see our recent analysis of T2K oxygen results)

- However, inevitably experimental studies must rely on semi-inclusive simulations, or, indeed, may want the extra hadronic information as in measurements using TPCs

- Despite the fact that inclusive relativistic modeling is reasonably robust, semi-inclusive modeling using naïve models such as the RFG is clearly suspect

- One special case exists: the deuteron. There a delta-function line is all that is found in the “landscape” and so detecting (say) a muon and a proton yields a unique value for the neutrino energy (see a recent study by J. W. Van Orden and TWD of CCν reactions on heavy water)


Thank you...

8. Appendix on semi-inclusive trajectories

The following is a summary of the material in O. Moreno, TWD, J. W. Van Orden and W. P. Ford, *Phys. Rev.* **D90** (2014) 013014 [MDVF]. For these developments we begin in the "laboratory system" defined by the incident neutrino beam and the final-state charged lepton momentum. We assume a given neutrino momentum k and hence energy $\epsilon = \sqrt{k^2 + m^2}$, where m is the neutrino mass (usually taken to be zero). In the laboratory system we take the $3'$ axis to be along the incident neutrino momentum, the final-state charged lepton to lie in the $1' - 3'$ plane, and the normal to that plane to be in the $2'$ direction (see Fig. 1). The final-state charged lepton then has 3-momentum

$$\mathbf{k}' = k'(\sin\theta \mathbf{u}_{1'} + \cos\theta \mathbf{u}_{3'}), \tag{1}$$

where θ is the neutrino-charged lepton scattering angle. Defining m' to be the final-state charged lepton's mass we have for its energy $\epsilon' = \sqrt{k'^2 + m'^2}$. Then, with $M^2 \equiv (m^2 + m'^2)/2$, we have for the energy transfer, the 4-momentum transfer squared, and the 3-momentum transfer, respectively, the following:

$$\begin{aligned} \omega &= \epsilon - \epsilon' && (2) \\ -Q^2 &= q^2 - \omega^2 = 2(\epsilon\epsilon' - kk'\cos\theta - M^2) && (3) \\ q &= \sqrt{-Q^2 + \omega^2}. && (4) \end{aligned}$$

Of course, one must have

$$-Q^2 > 0 \text{ (spacelike)}. \tag{5}$$

If we define

$$E_f \equiv M_A^0 + \omega \tag{6}$$

the final hadronic energy, where M_A^0 is the target rest mass, and note that the final total hadronic 3-momentum is

$$\mathbf{p}_f = \mathbf{q} \tag{7}$$

in the laboratory system where the target is at rest, then the invariant mass of the final hadronic state is given by

$$W_A = \sqrt{E_f^2 - p_f^2} = \sqrt{(M_A^0 + \omega)^2 - q^2}. \tag{8}$$

Since this must at least be greater than or equal to the breakup threshold, we have

$$W_A \geq W_{A-1}^0 + m_N = M_A^0 + E_s, \tag{9}$$

where W_{A-1}^0 is the ground state invariant mass of the A-1 system, m_N is the nucleon mass and E_s is the separation energy. This leads to the constraint

$$\omega \geq \omega_T \equiv E_s + \frac{(-Q^2) + E_s^2}{2M_A^0}. \tag{10}$$

Next let us rotate to the coordinate system oriented along the direction of the momentum transfer; this system is denoted the "q-system". The q-axis is oriented in the $1' - 3'$ plane at an angle θ_q, where

$$\begin{aligned} \cos\theta_q &= \frac{1}{q}\left(k - k'\cos\theta\right) && (11) \\ \sin\theta_q &= \frac{1}{q}k'\sin\theta. && (12) \end{aligned}$$

Letting the rotated system have axes 3 (along $\mathbf{q}$), 2 (normal to the lepton scattering plane and equal to the $2'$ direction), together with 1 (forming a right-handed coordinate system 123; see Fig. 2), we have the following relating the unit vectors:

$$\begin{aligned} \mathbf{u}_{1'} &= \cos\theta_q \mathbf{u}_1 - \sin\theta_q \mathbf{u}_3 && (13) \\ \mathbf{u}_{2'} &= \mathbf{u}_2 && (14) \\ \mathbf{u}_{3'} &= \sin\theta_q \mathbf{u}_1 + \cos\theta_q \mathbf{u}_3. && (15) \end{aligned}$$

We now assume that a nucleon is detected in the laboratory system with 4-momentum $p_N^\mu = (E_N, p_N)$, where

$$E_N = \sqrt{p_N^2 + m_N^2}; \tag{16}$$

that is, in the laboratory system

$$\mathbf{p}_N = p_N(\sin\theta_N^L \cos\phi_N^L \mathbf{u}_{1'} + \sin\theta_N^L \sin\phi_N^L \mathbf{u}_{2'} + \cos\theta_N^L \mathbf{u}_{3'}), \tag{17}$$

where the magnitude of the nucleon's 3-momentum is denoted $p_N \equiv |\mathbf{p}_N|$ and the direction in the laboratory system (the neutrino-charged lepton system) is specified by the angles (θ_N^L, ϕ_N^L). Of course the 3-momentum may also be written in terms of the q-system unit vectors

$$\mathbf{p}_N = p_N(\sin\theta_N \cos\phi_N \mathbf{u}_1 + \sin\theta_N \sin\phi_N \mathbf{u}_2 + \cos\theta_N \mathbf{u}_3), \tag{18}$$

where now the angles (θ_N, ϕ_N) are employed (see Fig. 2). Upon equating these two expressions and using Eqs. (13-15) that relate the unit vectors we have the following relationships:

$$\sin\theta_N \cos\phi_N = \sin\theta_N^L \cos\phi_N^L \cos\theta_q + \cos\theta_N^L \sin\theta_q \tag{19}$$

$$\sin\theta_N \sin\phi_N = \sin\theta_N^L \sin\phi_N^L \tag{20}$$

$$\cos\theta_N = -\sin\theta_N \cos\phi_N \sin\theta_q + \cos\theta_N^L \cos\theta_q. \tag{21}$$

Since the polar angle θ_N is assumed to be in the range $\left[0^0, 180^0\right]$, the sine is non-negative and so

$$\sin\theta_N = \sqrt{1 - \cos^2\theta_N}, \tag{22}$$

where $\cos\theta_N$ is given by Eq. (21), and hence the azimuthal angle in the q-systen is determined by

$$\cos\phi_N = \frac{1}{\sin\theta_N}\left(\sin\theta_N^L \cos\phi_N^L \cos\theta_q + \cos\theta_N^L \sin\theta_q\right) \tag{23}$$

$$\sin\phi_N = \frac{1}{\sin\theta_N}\sin\theta_N^L \sin\phi_N^L \tag{24}$$

using Eqs. (19,20,22). Knowing when the sines and cosines here are positive or negative we can determine the quadrant in which the angle ϕ_N lies. This fixes all of the variables in the q-system in terms of the laboratory variables.

Next it is useful to transform to variables that are well-suited to modeling and where the nuclear dynamics come into play. One can define the missing 3-momentum by

$$\mathbf{p}_m \equiv -\mathbf{p} = \mathbf{q} - \mathbf{p}_N \tag{25}$$

and immediately have

$$p = \sqrt{p_N^2 + q^2 - 2p_N q \cos\theta_N}, \tag{26}$$

namely, given in terms of the variables above. The energy of the recoiling (in general excited) A-1 system is given by

$$E_{A-1} = E_f - E_N \equiv \sqrt{W_{A-1}^2 + p^2}, \tag{27}$$

that is, in terms of the results given in Eqs. (6,16). Here W_{A-1} is the invariant mass of the A-1 system, again, in general in an excited state. It is convenient, as usual, to define the following effective excitation energy (the "daughter energy difference")

$$\mathcal{E} \equiv E_{A-1} - E_{A-1}^0 \geq 0, \tag{28}$$

where

$$E_{A-1}^0 = \sqrt{\left(W_{A-1}^0\right)^2 + p^2}. \tag{29}$$

This definition is such that $\mathcal{E}$ =0 when the final hadronic state is the ground state of the A-1 system. Typically one has the approximation

$$\mathcal{E} = (E_m - E_s)\left[1 - \frac{p^2}{2W_{A-1}W_{A-1}^0} + \cdots\right] \simeq E_m - E_s, \tag{30}$$

where E_m is the traditional definition of the missing energy, $E_m \equiv W_{A-1} + m_N - M_A^0$.

Finally, we have the variables that characterize the allowed region in the $(\mathcal{E}, p)$-plane:

$$y = \frac{1}{W_A^2}[Z_1 - Z_2] \quad (31)$$

$$Y = \frac{1}{W_A^2}[Z_1 + Z_2], \quad (32)$$

where W_A is given in Eq. (8) and

$$Z_1 = (M_A^0 + \omega) \times \sqrt{\Lambda - W_{A-1}^0(W_{A-1}^0 + m_N)}\sqrt{\Lambda - W_{A-1}^0(W_{A-1}^0 - m_N)} \quad (33)$$

$$Z_2 = q\Lambda \quad (34)$$

with

$$\Lambda = \frac{1}{2}\left[W_A^2 + \left(W_{A-1}^0\right)^2 - m_N^2\right]. \quad (35)$$

Two curves in the $(\mathcal{E}, p)$-plane may be defined:

$$\begin{aligned}\mathcal{E}^{\pm}(q,y;p) \quad &\equiv \quad \sqrt{m_N^2+(q+y)^2}-\sqrt{m_N^2+(q\pm p)^2} \\ &\quad + \sqrt{\left(W_{A-1}^0\right)^2+y^2}-\sqrt{\left(W_{A-1}^0\right)^2+p^2} \qquad (36)\end{aligned}$$

and, since

$$\mathcal{E}^-(q,y;p)-\mathcal{E}^+(q,y;p)=\frac{4qp}{\sqrt{m_N^2+(q+p)^2}+\sqrt{m_N^2+(q-p)^2}}\geq 0, \qquad (37)$$

one has $\mathcal{E}^-(q,y;p) \geq \mathcal{E}^+(q,y;p)$ (and equal to zero only when $p=0$). From the original derivation of these results we have that the following constraints must be satisfied, defining the physical region:

$$\max(0, \mathcal{E}^+(q,y;p)) \leq \mathcal{E} \leq \mathcal{E}^-(q,y;p). \qquad (38)$$

Thus, as long as the constraints in Eqs. (5,10,38) are all satisfied, one has a physical point in the $(\mathcal{E}, p)$-plane for fixed final-state charged lepton kinematics together with fixed laboratory system nucleon kinematics, all for a given neutrino energy. As the neutrino energy changes (for fixed final-state charged lepton kinematics and fixed laboratory system nucleon kinematics) the other variables defined above may also change (some must) and accordingly a different point will be found in the $(\mathcal{E}, p)$-plane. When the entire range of neutrino energies is spanned from threshold to very large values, a trajectory will the emerge as the line of solutions for given charged lepton and nucleon kinematics.