Worksheet 0: Some Probability

Name: Due: August 28, 2023

For the following questions, you will be working with probability measure $\mathbb{P}_{\mathcal{X}\times\mathcal{Y}}$ on $\mathcal{X}\times\mathcal{Y} = \mathbb{R}\times\mathbb{R}$ defined by density function $f_{\mathcal{X}\times\mathcal{Y}}$:

$$f(x,y) := x \cdot y^{-1/2}\mathbb{1}_{x\geq 0, y>0, x+y\leq 1},$$

where $\mathbb{1}_{\mathrm{cond}(t)} := \begin{cases} 1 & \text{if } \mathrm{cond}(t) \\ 0 & \text{else} \end{cases}$ denotes the *indicator* function for condition $\mathrm{cond}(t)$.* Please try to fit all answers on this page.

1. Does $f_{\mathcal{X}\times\mathcal{Y}}$ define a valid pdf? If so, articulate why, else explain why not, and if not, modify f to make it a pdf. Also sketch (shading the appropriate region in $\mathbb{R}^2$) the *support* of f, i.e. the points (x,y) for which $f(x,y)\neq 0$.

2. Define the marginal density $f_{\mathcal{Y}}(y)$, express as a function of y, sketch a plot, and compute the marginal probability $\mathbb{P}_{\mathcal{Y}}(y=0)$. Also report $\lim\limits_{y_0\to 0^+} f_{\mathcal{Y}}(y_0)$ and $\lim\limits_{y_0\to 0^+} \mathbb{P}_{\mathcal{Y}}(y=y_0)$.

3. Components x, y define random variables $\mathbb{R}^2 \xrightarrow{\pi_\cdot} \mathbb{R}$ by projection, $\pi_x : (x,y)\mapsto x$ and $\pi_y(x,y)\mapsto y$. Are these random variables independent? Justify your answer.

4. Consider dataset on $\mathcal{X}\times\mathcal{Y} = \{0,1\}^2$ in the table below. The right most column denotes the output of *model* $\overline{y} : \mathcal{X}\to\mathcal{Y}$. Formally define the covariance $\mathrm{Cov}(x,y)$ as an expectation and estimate this value from the table. Also estimate accuracy $\mathbb{P}_{\mathcal{X}\times\mathcal{Y}}(\overline{y}=y)$ and true positive rate $\mathbb{P}_{\mathcal{X}|\mathcal{Y}}(\overline{y}=1|y=1)$.

x	y	$\overline{y}$
0	1	0
1	0	1
1	1	1
0	0	0
0	0	0
1	1	1
1	0	1

5. Consider, now, probability space $\mathcal{X} := \prod_{j=1}^{n}\mathcal{X}_j$ with joint measure $\mathbb{P}_{\mathcal{X}}$, and suppose that for any proper subset (projection) $\{i_1,\dots,i_k\}\subsetneq\{1,\dots,n\}$ setting $\mathcal{X}' := \prod_{j=1}^{k}\mathcal{X}_{i_j}$, we have equality measures $\mathbb{P}_{\mathcal{X}'} = \prod_{j=1}^{k}\mathbb{P}_{\mathcal{X}_{i_j}}$. Are projections $\pi_1,\dots,\pi_n$ then independent? Why or why not? (Rationale for your answer may take the form of proof, example/counterexample, "just the definition," picture, or something else.)

*While $\mathcal{X}=\mathcal{Y}=\mathbb{R}$, we distinguish components with notation to e.g. distinguish the marginals $f_{\mathcal{X}}$ and $f_{\mathcal{Y}}$.