Worksheet 3: Binary Classification

Name: Due October 2, 2024

Recall for model $\tilde{y} : \mathcal{X} \to \mathbb{R}$, we obtain prediction model with thresholding: for $t \in \mathbb{R}$, we define $\overline{y}_t := \mathbb{1}_{\tilde{y} \geq t}$. Typically, $\tilde{y}(\mathbb{R}) \subset [0, 1]$, because we may like to interpret $\tilde{y}(x)$ as (something like) $\mathbb{P}_{\mathcal{Y}|\mathcal{X}}(y = 1|x)$, but it isn't strictly necessary.

1. Consider joint probability space $\mathcal{X} \times \mathcal{Y} = \mathbb{R} \times \{0, 1\}$ with measure

$$\mathbb{P}_{\mathcal{X}\times\mathcal{Y}}\big((a, b) \times \{j\}\big) = \alpha_j \cdot \int_a^b f_j(x)dx,$$

where $\alpha_0 + \alpha_1 = 1$, both nonnegative, and $f_j(x) = \gamma_j e^{-\gamma_j t} \cdot \mathbb{1}_{t\geq 0}$ with $\gamma_j > 0$. Express both marginals $\mathbb{P}_{\mathcal{X}}\big((a, b)\big)$ and $\mathbb{P}_{\mathcal{Y}}(y = j)$. You may use (properties of) this density for problems 2-4 as well.

2. Show that $\mathbb{P}_{\mathcal{Y}|\mathcal{X}}(y = 1|x) = \frac{\alpha_1 f_1(x)}{\alpha_0 f_0(x)+\alpha_1 f_1(x)}$. You may wish to recall the Fundamental Theorem of Calculus, and use continuity of measure (which you may suppose without proof for both arguments of $\mathbb{P}(\cdot|\cdot)$).

3. We may treat input data $x \in \mathcal{X} = \mathbb{R}$ as a score itself. For threshold predictor $\overline{y}_t : \mathcal{X} \to \mathcal{Y}$ defined by $\overline{y}_t(x) := \mathbb{1}_{x\geq t}$, express the true positive rate, false positive rate, and precision of $\overline{y}_t$ defined as

$$\mathrm{tpr}(t) := \mathbb{P}_{\mathcal{X}|\mathcal{Y}}(\overline{y}_t = 1|y = 1),\ \mathrm{fpr}(t) := \mathbb{P}_{\mathcal{X}|\mathcal{Y}}(\overline{y}_t = 1|y = 0)\text{and } \mathrm{prec}(t) := \mathbb{P}_{\mathcal{Y}|\mathcal{X}}(y = 1|\overline{y}_t = 1).$$

4. It is typically impossible to maximally satisfy all desired objectives. For example, optimal $\mathtt{tpr} = 1$ may be realized for minimal threshold at the expense of inducing undesirable $\mathtt{fpr} = 1$ (the other extreme realizes $\mathtt{fpr} = 0$ at the expense of $\mathtt{tpr} = 0$). Suppose you are given objective function

$$f(t) := \lambda \mathtt{tpr}(t) + (1 - \lambda)\mathtt{prec}(t).$$

Explain how you would solve for $t^* = \arg\max_{t \in \mathcal{X}} f(t)$, and compute the steps–as much as you can–to do so.

5. Define equal error rate (eer) as false positive rate $eer := \mathbb{P}_{\mathcal{X}|\mathcal{Y}}(\overline{y}_{t^e} = 1 | y = 0)$ at

$$t^e := \arg\min_{t \in \tilde{y}(\mathbb{R})} |1 - \mathtt{fpr}(t) - \mathtt{tpr}(t)|.$$

Express accuracy $\mathbb{P}_{\mathcal{X} \times \mathcal{Y}}(\overline{y}_{t^e} = y)$ at eer in terms of *only* $\mathtt{tpr}(t^e)$ and $\mathtt{fpr}(t^e)$. Simplify as much as you can.*

6. For loss function $\ell_{\tilde{y}}(x, y) := -\log\left(\tilde{y}(x)^y (1 - \tilde{y}(x))^{(1-y)}\right)$, show that the optimal model $y^* : \mathcal{X} \to [0, 1]$ is calibrated (as defined in notes §4.4).

7. Suppose model score $\tilde{y} : \mathcal{X} \to [0, 1]$ has the relation $\tilde{y}(p) = \sigma(p) = \frac{1}{1 + e^{-c(2p-1)}}$ for $p(x) := \mathbb{P}_{\mathcal{Y}|\mathcal{X}}(y = 1 | x)$ and $c > 0$. Define calibrated model $\overline{y} : \mathcal{X} \to [0, 1]$ in terms of $\tilde{y}$, i.e. map $\varphi : [0, 1] \to [0, 1]$ for which $\overline{y} := \varphi \circ \tilde{y} = p$.

*Your expression should have a minimal number of terms and arithmetic operations.