Exotic Tori from ATFs oder so

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1 Introduction

Definition 1.1. Let $k \in \mathbb{N}$ such that $0 < k \leq d$ and $a \in (0, \infty)$. Through nodal slides we can arrange the ATF on B_{dpq} such that the line $x_2 = a$ intersects the branch cut line between the $(k-1)$-th and k-th degenerated fibre. $T_k(a)$ is defined to be the fibre over the intersection point of these two lines.

Theorem 1.2. *Let $U \subset H^1(T_k(a), \mathbb{R}) \setminus \{\text{branch cut line}\}$. The restriction of the displacement energy germ to U is given by*

$$S^e_{T_k(a)}\Big|_U (x, y) = a + \max\{x, x(1 - kpq) - kp^2 y\}$$

so oder so ähnlich...

Let $d, p, q \in \mathbb{N}$ such that $d \geq$ and p, q coprime with $1 \leq q < p$ or $q = 0, p = 1$, and $0 < a_1 < \dots < a_d$ real integers. Let P be the polynomial $P(z) = \prod_{i=1}^{d}(z^p - a_i)$. Define the manifold M_P by

$$M_P = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 + P(z_3) = 0\right\}.$$

We define the Hamiltonian system

$$\mathbf{H}(z_1, z_2, z_3) = \left(|z_3|^2, \frac{1}{2}\left(|z_1|^2 - |z_2|^2\right)\right)$$

Let ρ_p be the group of p-th roots of unity acting on M_P by

$$\rho \cdot (z_1, z_2, z_3) = \left(\rho z_1, \rho^{-1} z_2, \rho^q z_3\right), \quad \rho \in \rho_p.$$

This is a free action, so we can define the quotient $B_{dpq} = M_P/\rho_p$. The Hamiltonian system $\mathbf{H}$ is invariant under the action, so it descends to a Hamiltonian system on B_{dpq}. As in [3, Chapter 6], we can remove a ray going through the critical values in the moment image, and use the flux map to obtain a moment map μ generating a Hamiltonian torus action everywhere except on the critical points, and having moment image $\Delta_{B_{dpq}}$ as in figure 1.

Kann man besser formulieren.

The position of the nodes can be varied along this ray by nodal slides.

...which only affect some ε-neighboorhood.

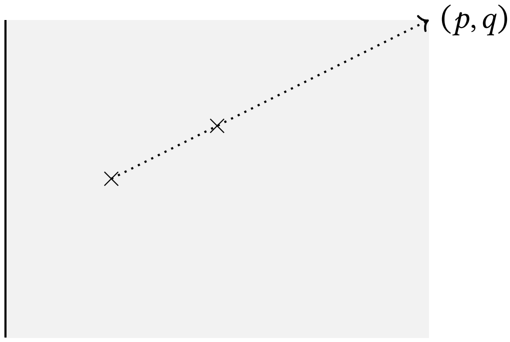
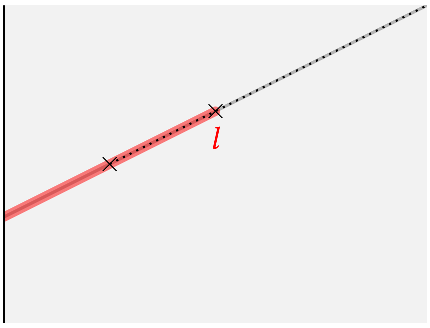
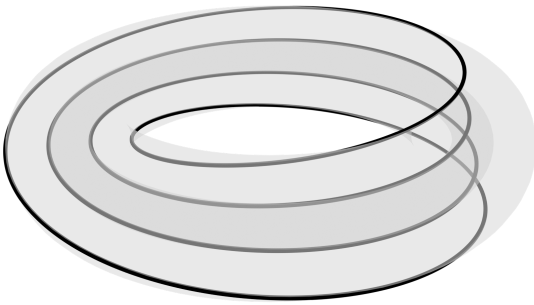


Figure 1: Moment image of B_{dpq} under μ



(a) Retraction to branch cut line



(b) The cycles marked in black (here $p = 2, q = 1$) are collapsed to a point.

Figure 2: Calculation of the homology of B_{dpq}

2 Homology of B_{dpq}

In order to calculate the lower bound for the displacement energy of a Lagrangian fibre torus $T(x, y) = \mu^{-1}(\{(x, y)\})$, we will need to calculate a basis for $H_2\big(B_{dpq}, T(x, y)\big)$.

B_{dpq} deformation retracts to the preimage of the branch cut line segment l shown in red in figure 2a, by first vertically shrinking the space onto the ray in direction (p, q), and then compressing the part of the ray that is to the right of all the critical points.

Das ist etwas dumm formuliert, lohnt es sich das besser zu formulieren?

The preimage $\mu^{-1}(l)$ can be understood as follows: If there were no critical points on the line, this would be a solid torus $T = S^1 \times D^2$. We pick $(1, 0), (0, 1) \in H_1(\partial T)$ to be the classes generated by $S^1 \times \mathrm{pt}, \mathrm{pt} \times \partial D^2$ respectively. For each critical fibre $k \in \{1, \ldots, d\}$ we collapse a loop along the homology class $(-q, p)$, as in figure 2b. Up to homotopy this is the same as attaching a disk D_k along $(-q, p)$. Again up to homotopy we can also require that the d discs $D_1, \ldots, D_d$ are attached along ∂T. Let us call this space S.

Let us look at the long exact sequence of homology for the pair $\big(B_{dpq}, T(x, y)\big)$.

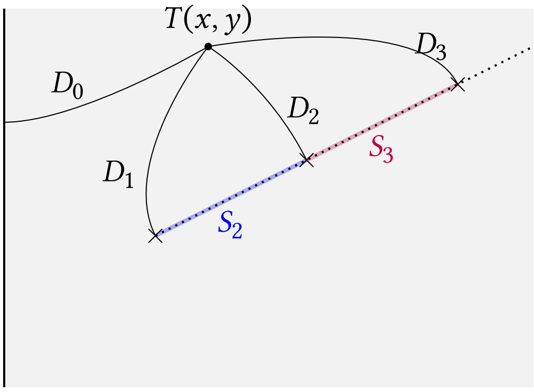


Figure 3: The disks $D_0, \ldots, D_d$ generating the homology $H_2\big(B_{dpq}, T(x,y)\big)$

This pair is homotopy equivalent to $(S, \partial T)$.

$$\begin{tikzcd}
H_2(\partial T) \arrow[r, "0"] & H_2(S) \arrow[r, hook] \arrow[d, "\cong"] & H_2(S, \partial T) \arrow[r] \arrow[d, "\cong"] & H_1(\partial T) \arrow[r] \arrow[d, "\cong"] & H_1(S) \arrow[d, "\cong"] \\
 & \mathbb{Z}^{d-1} & \mathbb{Z}^{d+1} & \mathbb{Z}^2 & \mathbb{Z}_p
\end{tikzcd}$$

The first horizontal map is zero since ∂T retracts to a circle in S. The homology $H_2(S)$ can be seen as follows: By contracting the solid torus T in S to a circle, we see that S is homotopic to a circle with d discs glued to its boundary by a degree p map. So $H_2(S)$ is generated by spheres $\{S_2, \ldots, S_d\}$, $S_k = D_{k-1} - D_k$. $H_2(S, \partial T)$ is generated by the discs $D_0 = \mathrm{pt} \times D^2, D_1, \ldots, D_d$. In B_{dpq}, these discs can be seen, where the disc intersecting the toric boundary collapses the $(0,1)$ cycle in the toric fibre $T(x,y)$ and the discs intersecting the critical points collapse the $(-q,p)$ cycle (see figure 3). The elements $S_2, \ldots, S_d \in H_2(B_{dpq})$ can be realized by embedded Lagragian spheres fibering over the segments between the nodes in the ATF – these are so-called *visible Lagrangians*, see [3, section 7.4]. The boundary map $\partial\colon H_2(S, \partial T) \to H_1(\partial T)$ is given by $\partial D_0 = (0,1)$, $\partial D_i = (-q, p)$, meaning that the last horizontal map $H_1(\partial T) \to H_1(S)$ maps $(0,1)$ to the generator of $\mathbb{Z}_p$.

3 Lower Bound on Displacement Energy: Minimal J-holomorphic Curves

Let $T(x,y)$ a fibre torus, where (x,y) is not over the branch cut line. In [1] the following is proven:

Theorem 3.1. *Let (X, ω, J) be a symplectic manifold with ω-tame almost complex structure J. Let $L \subset X$ be a compact Lagrangian submanifold. Then the displacement*

tame/geometrically bounded

energy satisfies

$$e(L) \geq \min\{\sigma_D(X, L, J), \sigma_S(X, J)\}$$

Fibres $T(\lambda p, \lambda q)$ for $\lambda > 0$ are monotone whenever they are not on a node. We compute displacement energy of the non-monotone fibres. Note that $T(x_0, y_0)$ yields a well-defined torus up to Hamiltonian isotopy, i.e. it is independent of the nodal slides, see .

das müsste man auch noch zeigen oder verrefferenzen, ist aber irgendwie irrelevant...

referenz, kein guter platz hierfür

We want to use theorem 3.1 to find a lower bound, so first we need to choose a suitable almost complex structure on B_{dpq}

3.1 Buffer Zone Lemma

Let (n, a) be two coprime integers, ρ_n the group of n-th roots of unity. Let ρ_n act on $\mathbb{C}^2$ by $\rho(z_1, z_2) = (\rho z_1, \rho^a z_2)$. Let $A(n, a) = \mathbb{C}^2/\rho_n$ be the quotient space. This space is an orbifold, with one orbifold point at $[(0, 0)]$. The space S^3/ρ_n is the lens space $L(n, a)$, so $A(n, a)$ is the cone over $L(n, a)$.

We define the Hamiltonian system on $A(n, a)$ by

$$\mathbf{G}(z_1, z_2) = \frac{1}{2}\left(|z_2|^2, \frac{1}{n}\left(|z_1|^2 + a|z_2|^2\right)\right). \tag{1}$$

With this Hamiltonian system the moment polytope is a wedge with edges pointing along vectors $(1, 0), (n, a)$, as seen in figure 4. $A(n, a)$ has a almost complex structure J descending from the canonical complex structure on $\mathbb{C}^2$.

We have the commutative diagram

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "\pi"] \arrow[d, "\mathbf{H}"] & A(n,a) \arrow[d, "\mathbf{G}"] \\ \Delta_{\mathbb{C}^2} \arrow[r, "L_{A(n,a)}"] & \Delta_{A(n,a)} \end{tikzcd} \tag{2}$$

where the left vertical map $\mathbf{H}$ is given by $(z_1, z_2) \mapsto \frac{1}{2}\left(|z_1|^2, |z_2|^2\right)$, and the bottom map is a linear transformation given by the matrix

$$L_{A(n,a)} = \begin{pmatrix} 0 & 1 \\ \frac{1}{n} & \frac{a}{n} \end{pmatrix}.$$

Let $B(a) \in \mathbb{C}^n$ be the open ball in $\mathbb{C}^n$ of radius $\sqrt{\frac{a}{\pi}}$. In [2, appendix A] the following lemma is proven:

Lemma 3.2. *Let $a_+ > a_- \geq 0$. Let $u : \Sigma \to B(a_+) \setminus \overline{B(a_-)}$ be a J-holomorphic curve such that the closure of $u(\Sigma)$ in $\mathbb{C}^n$ intersects $\partial B(a_-)$. Then $\int_u \omega \geq a_+ - a_-$.*

We give the slight generalization:

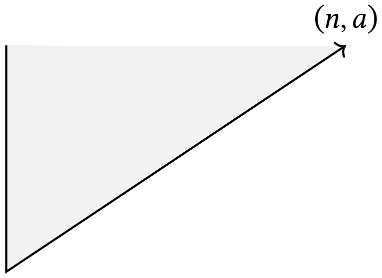


Figure 4: Moment image of $A(n,a)$ with given by Hamiltonian system G

Lemma 3.3. *Let $a_+ > a_- > 0$, and*

$$X = \mathbf{G}^{-1}(\{c_1(0,1) + c_2(n,a) \mid a_- < c_1 + c_2 < a_+\}) \subset A(n,a) \ ,$$

equipped with the almost complex structure of $A(n,a)$.

Let $u : \Sigma \to X$ be a J-holomorphic curve whose closure intersects

$$\mathbf{G}^{-1}(\{c_1(0,1) + c_2(n,a) \mid a_- = c_1 + c_2\}) \ .$$

Then $\int_u \omega \geq a_+ - a_-$.

Remark 3.4. Suppose we have a moment polytope Δ of a (almost) toric symplectic manifold or orbifold $\mathbf{H} : M \to \Delta$ with two non-parallel edges given by the two primitive vectors u_1, u_2, as in figure 5. Suppose without loss of generality that the edges intersect in the origin. Then the subset

$$X = \mathbf{H}^{-1}(\{c_1 u_1 + c_2 u_2 \mid a_- < c_1 + c_2 < a_+\})$$

with $a_\pm$ such that $a_\pm u_1, a_\pm u_2 \in \Delta$, can be transformed by a $T \in \mathrm{GL}(\mathbb{Z}^2)$, such that $Tu_1 = (0,1), Tu_2 = (n,a)$, for some coprime integers n, a.

With this transformation we can view X as a subset of $A(n,a)$. Equipping M with an extension of the almost complex structure coming from $A(n,a)$, we get that J-curves in M intersecting

$$\mathbf{H}^{-1}(\{c_1 u_1 + c_2 u_2 \mid a_- = c_1 + c_2\})$$

must have at least area $a_+ - a_-$.

Proof. Since the action of ρ_n is free in $(\mathbb{C}^*)^2$, the projection map $\pi : (\mathbb{C}^*)^2 \to A(n,a) \setminus \{[(0,0)]\}$ is an n-fold covering map.

The using the commutative diagram 2, we can compute the preimage

$$\pi^{-1}(X) = (\mathbf{H} \circ L_{A(n,a)})^{-1}(\mathbf{G}(X)) = B(na_+) \setminus B(na_-) \ .$$

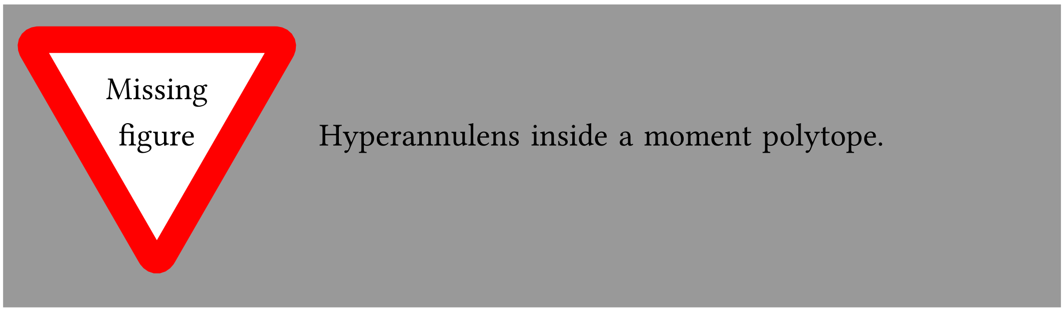


Figure 5: Hyperannulens inside a moment polytope.

NOOOOOOOOOOOO!

A J-curve $u: \Sigma \to A(n, a)$ lifts to a J-curve $\tilde{u}$, i.e. a curve making the diagram

$$\begin{tikzcd} \Sigma' \arrow[r, "\tilde{u}"] \arrow[d, "\tilde{\pi}"] & \mathbb{C}^2 \arrow[d, "\pi"] \\ \Sigma \arrow[r, "u"] & A(n, a) \end{tikzcd}$$

commute, where $\tilde{\pi}: \Sigma' \to \Sigma$ is some n-fold covering of Σ.

Using lemma 3.2, we get that the symplectic area of $\tilde{u}$ is at least $n(a_+ - a_-)$, and since $\tilde{u}$ is an n-fold covering of u, u has at least symplectic area $a_+ - a_-$, as desired. □

3.2 Almost complex structure for B_{dpq}

Lemma 3.5. *There is an almost complex structure on B_{dpq}*

$$\min\{\sigma_D(B_{dpq}, T(x_0, y_0), J), \sigma_S(B_{dpq}, J)\} = x_0.$$

Proof. Independently of the choice of J, there are no J-holomorphic spheres in B_{dpq}, since $H_2(B_{dpq})$ admits a basis the elements of which can be realized by Lagrangian spheres as discussed in section 2.

After a mutation on B_{dpq}, we get a modified moment map $\tilde{\mu}$, with moment image $\tilde{\Delta}_{B_{dpq}}$ as in figure 6, satisfying the commutative diagram

$$\begin{tikzcd} & B_{dpq} \arrow[dl, "\mu"'] \arrow[dr, "\tilde{\mu}"] & \\ \Delta_{B_{dpq}} \arrow[rr, "M"] & & \tilde{\Delta}_{B_{dpq}} \end{tikzcd} \tag{3}$$

where M is the mutation map given by the matrix

$$\begin{pmatrix} dqp + 1 & -dp^2 \\ dq^2 & -dpq + 1 \end{pmatrix}$$

Figure 6: Moment image of B_{dpq} after mutation

for points below the branch cut line, and the identity for points above.

Let $\varepsilon > 0$. Using a nodal slide we can assume that the nodes are all located in $U_l = \{c_1(0,1) + c_2(dp^2, dpq-1) \mid c_1 + c_2 < \varepsilon\}$, marked in blue in figure 6. Removing U_l, the moment image $\Delta_{B_{dpq}}$ is the same as that of $A(dp^2, dpq-1)$ with the corresponding corner U_l^A removed. Since the moment images are the same, by Delzant we get a equivariant symplectomorphism

$$B_{dpq} \setminus \tilde{\mu}^{-1}(U_l) \to A(dp^2, dpq-1) \setminus \mathbf{G}^{-1}(U_l^A) ,$$

and we can equip B_{dpq} with a almost complex structure J obtained by extending the almost complex structure of $A(dp^2, dpq-1)$ to all of B_{dpq} such that J tames ω.

Chasing a point (x, y) through the maps

$$\Delta_{B_{d,p,q}} \setminus U_l \xrightarrow{M} \tilde{\Delta}_{A(n,a)} \setminus U_l^A \xrightarrow{L^{-1}_{A(n,a)}} \Delta_{\mathbb{C}} \setminus \pi(B(\varepsilon))$$

we obtain the point $(x + dp(py - qx), x)$ or $(x, x - dp(py - qx))$ depending on whether (x, y) lies above or below the branch cut line.

Suppose $u\colon (\Sigma, \partial\Sigma) \to (B_{dpq}, T(x_0, y_0))$ is any J-holomorphic disk, and $D_0, D_1, S_2, \ldots, S_d$ representatives of the homology classes generating $H_2(B_{dpq}, T(x_0, y_0))$ as described in section 2. If we have $[u] = [D_0]$ for the homology classes, the areas are also equal, so $\int_u \omega = \int_{D_0} \omega = x_0$. Suppose $[u]$ is not some multiple of $[D_0]$. As discussed in section 2, this means that $[u]$ has some non-zero component of $[D_1], [S_2], \ldots, [S_d]$. This means that $u(\Sigma) \cap U_l \neq \emptyset$ (with U_l defined as above), and in particular $u(\Sigma) \cap \partial U_l \neq \emptyset$. So we can use lemma 3.3 on the region

$$\tilde{\mu}^{-1}\{c_1(0,1) + c_2(dp^2, dpq-1) \mid \varepsilon < c_1 + c_2 < 2x_0 + dp|qx - py|\} ,$$

where does this weird thing come from

which gives us

$$\int_u \omega \geq 2x_0 + dp|qx - py| - \varepsilon \geq x_0 ,$$

if we assume that we have chosen $\varepsilon \leq x_0 + dp|qx - py|$. $\square$

Proposition 3.6. *Let $T(x_0, y_0)$ be a non-monotone ATF-fibre of B_{dpq}. Then it has displacement energy*

$$e(B_{dpq}, T(x_0, y_0)) \geq x_0.$$

Proof. We use theorem 3.1 together with lemma 3.5. □

Add proof of theorem 1.2.

3.3 D_0 stays a Minimal J-Disks for Suitable Embeddings

Define $B_{dpq}(a) := \tilde{\mu}^{-1}\{c_1(0,1) + c_2(dp^2, dpq - 1) \mid c_1 + c_2 < a)\}$, and let (X, ω, J) be a symplectic manifold with tame almost complex structure.

gibts davon noch eine schönere definition?

Suppose we have an embedding $B_{dqp}(2a) \to X$, and that all J-spheres in X have at least area a. Using remark 3.4 and lemma 3.3, we get that the minimal J-curve with boundary on the torus $T_k(a)$ has area a, and since there are no smaller J-spheres by assumption, using theorem 3.1, we get that the displacement energy of $T_k(a)$ is at least a.

Brauchen wir hier noch ein ε platz?

4 Upper bound on displacement energy: Probes

References

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