# Exotic Tori from ATFs oder so

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## 1 Introduction

**Definition 1.1.** Let $k \in \mathbb{N}$ such that $0 < k \leq d$ and $a \in (0, \infty)$. Through nodal slides we can arrange the ATF on $B_{dpq}$ such that the line $x_2 = a$ intersects the branch cut line between the $(k-1)$-th and $k$-th degenerated fibre. $T_k(a)$ is defined to be the fibre over the intersection point of these two lines.

**Theorem 1.2.** *Let $U \subset H^1(T_k(a), \mathbb{R}) \setminus \{branch\ cut\ line\}$. The restriction of the displacement energy germ to $U$ is given by*

$$S^e_{T_k(a)}\Big|_U (x, y) = a + \max\{x, x(1 - kpq) - kp^2 y\}$$

so oder so ähnlich...

Let $d, p, q \in \mathbb{N}$ such that $d \geq$ and $p, q$ coprime with $1 \leq q < p$ or $q = 0, p = 1$, and $0 < a_1 < \ldots < a_d$ real integers. Let $P$ be the polynomial $P(z) = \prod_{i=1}^d (z^p - a_i)$. Define the manifold $M_P$ by

$$M_P = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 + P(z_3) = 0\right\}.$$

We define the Hamiltonian system

$$\mathbf{H}(z_1, z_2, z_3) = \left(|z_3|^2, \frac{1}{2}\left(|z_1|^2 - |z_2|^2\right)\right)$$

Let $\mu_p$ be the group of $p$-Th roots of unity acting on $M_P$ by

$$\mu \cdot (z_1, z_2, z_3) = \left(\mu z_1, \mu^{-1} z_2, \mu^q z_3\right), \quad \mu \in \mu_p.$$

This is a free action, so we can define the quotient $B_{dpq} = M_P / \mu_p$. The Hamiltonian system $\mathbf{H}$ is invariant under the action, so it descends to a Hamiltonian system on $B_{dpq}$.

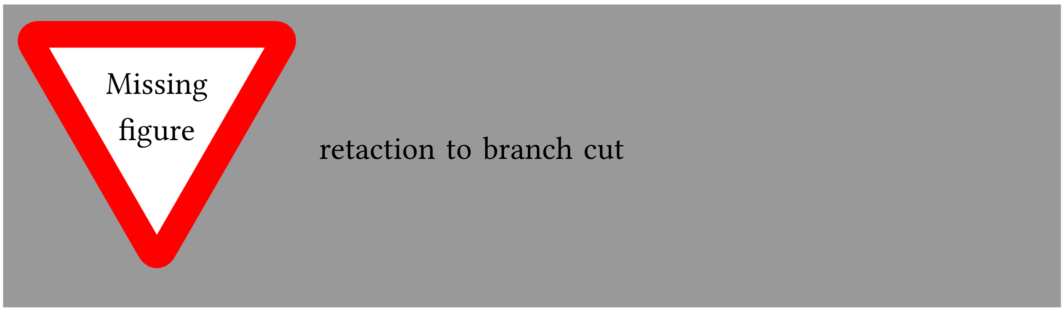


Figure 1: asdf

## 2 Upper bound on displacement energy: Probes

## 3 Short interlude: Homology of $B_{dpq}$

In order to calculate the lower bound for the displacement energy of a torus $L(x, y)$, definieren we will need to calculate a basis for $H_2(B_{dpq}, L(x, y))$.

$B_{dpq}$ deformation retracts to the preimage of the branch cut line segment shown in fig. 1. This can be understood as follows: If there were no critical points on the line, this would be a solid torus $T = S^1 \times D^2$. We pick $(1, 0), (0, 1) \in H_1(\partial T)$ to be the classes generated by $S^1 \times \mathrm{pt}, \mathrm{pt} \times \partial D^2$ respectively. At each critical point we collapse a loop along homology class $(p, -q)$. Up to homotopy this is the same as attaching a disk along $(p, -q)$. Again up to homotopy we can also require that the $d$ discs $D_1, \ldots, D_d$ are attached along $\partial T$. Let us call this space $S$.

Let us look at the long exact sequence of homology for the pair $(B_{dpq}, L(x, y))$. This pair is homotopy equivalent to $(S, \partial T)$.

$$\begin{tikzcd}
H_2(\partial T) \arrow[r, "0"] & H_2(S) \arrow[r, hook] \arrow[d, "\cong"] & H_2(S, \partial T) \arrow[r] \arrow[d, "\cong"] & H_1(\partial T) \arrow[r] \arrow[d, "\cong"] & H_1(S) \arrow[d, "\cong"] \\
& \mathbb{Z}^{d-1} & \mathbb{Z}^{d+1} & \mathbb{Z}^2 & \mathbb{Z}_p
\end{tikzcd}$$

The first horizontal map is zero since $\partial T$ retracts to a point in $S$. Homology $H_2(S)$ can be seen as follows: By contracting the solid torus $T$ in $S$ to a circle, we see that $S$ is homotopic to a circle with $d$ discs glued to its boundary by a degree $p$ map. So $H_2(S)$ is generated by spheres $\{S_2, \ldots, S_d\}$, $S_k = D_1 - D_{k+1}$. $H_2(S, \partial T)$ is generated by the discs $D_0 = \mathrm{pt} \times D^2, D_1, \ldots, D_d$. In $B_{dpq}$, these discs can be seen, where the disc intersecting the toric boundary collapses the $(0, 1)$ cycle in the toric fibre $L(x, y)$ and the discs intersecting the critical points collapse the $(q, -p)$ cycle (see fig. 2).

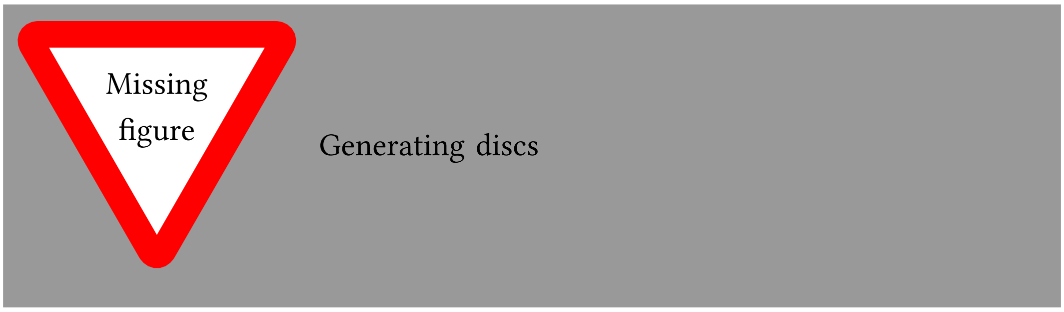


Figure 2: asdf

# 4 Lower bound on displacement energy: minimal J-holomorphic curves

oder nur discs?

Let $L(x, y)$ a fibre torus, where $(x, y)$ is not over the branch cut line. In [1] it is proved that lalalala

Pick a tame almost complex structure $J$ on $B_{pdq}$. Let $u$ be a non-constant J-holomorphic sphere or disc with boundary on $L(x, y)$. Then the homology class of $u$ can be written in terms of the generators of $H_2(B_{dpq}, L(x, y))$ described above as

Or some standard structure? Does it matter?

$$[u] = c_0 D_0 + c_1 D_1 + \sum_{k=2}^{d} c_k S_k \,.$$

The symplectic area of $u$ is then given by

$$\int_u \omega = c_0 \int_{D_0} \omega + c_1 \int_{D_1} \omega \,,$$

as the symplectic area of the spheres $S_k$ is zero.

By a nodal slide we can move the critical points in the moment image such that they don't occur for $\mathbf{H}_2 > 2y$. By classification of toric manifolds, $\{p \in B_{dpq} \mid \mathbf{H}_2(p) < 2y\}$ is then symplectomorphic to $B^2(2y) \times \mathbb{R} \times S^1$, where $B^2(a)$ is the 2-ball of area $a$. Here we can choose $D_0$ to be $B^2(y) \times \{(x, \mathrm{pt})\}$, which is J-holomorphic with the standard almost complex structure. Our claim is that $D_0$ is the minimal J-disc.

## 4.1 Minimal J-Disks don't run away

Let $(n, a)$ be two coprime integers, $\mu_n$ the group of $n$-th roots of unity. Let $\mu_n$ act on $\mathbb{C}^2$ by $\mu(z_1, z_2) = (\mu z_1, \mu^a z_2)$. Let $\mathbb{C}^2/\mu_n$ be the quotient space. This space is an orbifold, with one orbifold point at $[0, 0]$.

We define the Hamiltonian system on $\mathbb{C}^2/\mu_n$ by

$$\mathrm{G}(z_1, z_2) = \frac{1}{2}\left(\frac{1}{n}\left(|z_1|^2 + a|z_2|^2\right), |z_2|^2\right) .$$

**Lemma 4.1.** *Let $r_+ > r_- \geq 0$. Let $u : \Sigma \to (B_{r_+} \setminus B_{r_-})/\mu_n$ be a $\mathcal{J}$-holomorphic curve such that the closure of $u(\Sigma)$ intersects $\partial B_{r_-}$. Then the symplectic area of $u$ is at least $\pi(r_+^2 - r_-^2$.*

*Proof.* Let $r \in (r_-, r_+)$ such that the intersection $u(\Sigma) \cap \partial B_r$ is transversal. Then this intersection is an immersed 1-dimensional manifold, so it is a collection of immersed circles. Let $\gamma$ be a parametrization of one of these circles. We choose a local holomorphic reparametrization of $u$ as follows:

$$\begin{aligned} \tilde{u} &: S^1 x I \to \mathbb{C}^n \\ u(t,0) &= \gamma(t) \end{aligned}$$

ja, was jetzt?

Then

$$\begin{aligned} F'(r) &= \int_{u(\Sigma)\cap\partial B_r} u^*\omega \\ &\geq \int_{S^1} \omega(\frac{\partial u \circ s}{\partial t}, \frac{\partial u \circ s}{\partial r})\mathrm{d}t \\ &= \int_{S^1} \omega(\frac{\partial u}{\partial t} + \frac{\partial s}{\partial t}\frac{\partial u}{\partial s}, \frac{\partial s}{\partial r}\frac{\partial u}{\partial s})\mathrm{d}t \\ &= \int_{S^1} \frac{\partial s}{\partial r}\omega(\frac{\partial u}{\partial t}, i\frac{\partial u}{\partial t})\mathrm{d}t \\ &= \int_{S^1} \frac{\partial s}{\partial r}|\dot{\gamma}(t)|^2\mathrm{d}t \\ &\geq l^2(\gamma)\int_{S^1} \frac{\partial r}{\partial s}\mathrm{d}t \end{aligned}$$

Also

$$l^2(\gamma) \geq \int_{S^1} \gamma^*\alpha_n = \int_{S^1} \langle\dot{\gamma}, \xi\rangle\mathrm{d}t$$

□

# References


[1] Yu. V. Chekanov. “Lagrangian intersections, symplectic energy, and areas of holomorphic curves”. In: *Duke Mathematical Journal* 95 (1998), pp. 213–226.