

Exotic Tori from ATFs oder so

JoJoJo

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1 Introduction

Definition 1.1. Let $k \in \mathbb{N}$ such that $0 < k \leq d$ and $a \in (0, \infty)$. Through nodal slides we can arrange the ATF on $B_{d,pq}$ such that the line $x_2 = a$ intersects the branch cut line between the $(k - 1)$ -th and k -th degenerated fibre. $T_k(a)$ is defined to be the fibre over the intersection point of these two lines.

Theorem 1.2. Let $U \subset H^1(T_k(a), \mathbb{R}) \setminus \{\text{branch cut line}\}$. The restriction of the displacement energy germ to U is given by

$$S_{T_k(a)}^e \Big|_U (x, y) = a + \max\{x, x(1 - kpq) - kp^2y\}$$

so oder so ähnlich...

Let $d, p, q \in \mathbb{N}$ such that $d \geq$ and p, q coprime with $1 \leq q < p$ or $q = 0, p = 1$, and $0 < a_1 < \dots < a_d$ real integers. Let P be the polynomial $P(z) = \prod_{i=1}^d (z^p - a_i)$. Define the manifold M_p by

$$M_p = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 + P(z_3) = 0\}.$$

We define the Hamiltonian system

$$H(z_1, z_2, z_3) = \left(|z_3|^2, \frac{1}{2}(|z_1|^2 - |z_2|^2) \right)$$

Let μ_p be the group of p -Th roots of unity acting on M_p by

$$\mu \cdot (z_1, z_2, z_3) = (\mu z_1, \mu^{-1} z_2, \mu^q z_3), \quad \mu \in \mu_p.$$

This is a free action, so we can define the quotient $B_{dpq} = M_p / \mu_p$. The Hamiltonian system H is invariant under the action, so it descends to a Hamiltonian system on B_{dpq} .

2 Upper bound on displacement energy: Probes

3 Short interlude: Homology of B_{dpq}

In order to calculate the lower bound for the displacement energy of a torus $L(x, y)$, we will need to calculate a basis for $H_2(B_{dpq}, L(x, y))$.

definieren

B_{dpq} deformation retracts to the preimage of the branch cut line segment shown in figure 1. This can be understood as follows: If there were no critical points on the line, this would be a solid torus $T = S^1 \times D^2$. We pick $(1, 0), (0, 1) \in H_1(\partial T)$ to be the classes generated by $S^1 \times \text{pt}, \text{pt} \times \partial D^2$ respectively. At each critical point we collapse a loop along homology class $(p, -q)$. Up to homotopy this is the same as attaching a disk along $(p, -q)$. Again up to homotopy we can also require that the d discs D_1, \dots, D_d are attached along ∂T . Let us call this space S .

Let us look at the long exact sequence of homology for the pair $(B_{dpq}, L(x, y))$. This pair is homotopy equivalent to $(S, \partial T)$.

$$\begin{array}{ccccccc} H_2(\partial T) & \xrightarrow{0} & H_2(S) & \hookrightarrow & H_2(S, \partial T) & \longrightarrow & H_1(\partial T) \longrightarrow H_1(S) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ & & \mathbb{Z}^{d-1} & & \mathbb{Z}^{d+1} & & \mathbb{Z}^2 \\ & & & & & & \downarrow \cong \\ & & & & & & \mathbb{Z}_p \end{array}$$

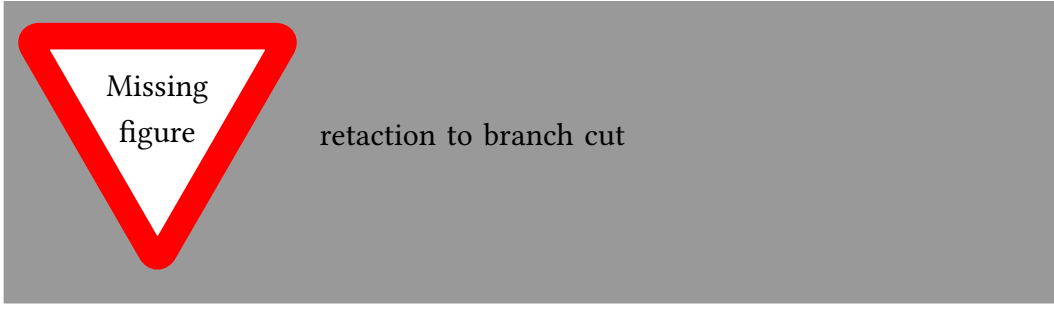


Figure 1: asdf

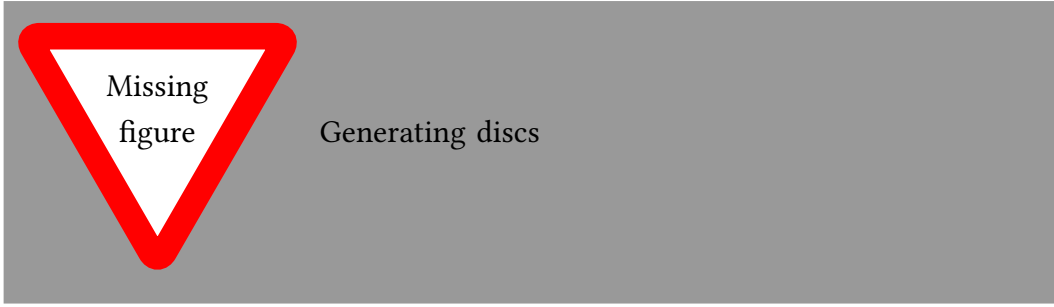


Figure 2: asdf

The first horizontal map is zero since ∂T retracts to a point in S . Homology $H_2(S)$ can be seen as follows: By contracting the solid torus T in S to a circle, we see that S is homotopic to a circle with d discs glued to its boundary by a degree p map. So $H_2(S)$ is generated by spheres $\{S_2, \dots, S_d\}$, $S_k = D_1 - D_{k+1}$. $H_2(S, \partial T)$ is generated by the discs $D_0 = \text{pt} \times D^2, D_1, \dots, D_d$. In B_{dpq} , these discs can be seen, where the disc intersecting the toric boundary collapses the $(0, 1)$ cycle in the toric fibre $L(x, y)$ and the discs intersecting the critical points collapse the $(q, -p)$ cycle (see figure 2).

4 Lower Bound on Displacement Energy: Minimal J-holomorphic Curves

Let $L(x, y)$ a fibre torus, where (x, y) is not over the branch cut line. In [1] the following is proven:

Theorem 4.1. *Let $L \subset X$ be a Lagrangian submanifold. Then the displacement energy satisfies*

$$e(L) \geq \min \sigma_D(X, J), \sigma_S(X, J)$$

4.1 D_0 is a Minimal J-Disk in B_{dpq}

By a nodal slide we can move the critical points in the moment image such that they don't occur for $H_2 > 3y$. By classification of toric manifolds, $U_{D_0} = \mathbf{H}^{-1}((x - \varepsilon, x + \varepsilon) \times [0, y + \varepsilon))$ is then symplectomorphic to $B^2(y + \varepsilon) \times (x - \varepsilon, x + \varepsilon) \times S^1$, where $B^2(a)$ is the 2-ball of area a . Here we can choose D_0 to be $\overline{B^2(y)} \times \{(x, \text{pt})\}$, which is J-holomorphic with the standard almost complex structure J_0 . Our claim is that D_0 is a minimal J-curve in $H_2(B_{dpq}, L(x, y))$.

what is the standard acs?

Let J be some tame extension of $J_0|_{U_{D_0}}$.

Was sind die bedingungen an die J's nochmal genau?

Let $u : \Sigma \rightarrow B_{dpq}$ be a non-constant J-holomorphic curve having possibly a boundary on $L(x, y)$. Then the homology class of u can be written in terms of the generators of $H_2(B_{dpq}, L(x, y))$ described in section 3:

$$[u] = c_0 D_0 + c_1 D_1 + \sum_{k=2}^d c_k S_k .$$

The symplectic area of u is then given by

$$\int_u \omega = c_0 \int_{D_0} \omega + c_1 \int_{D_1} \omega ,$$

as the symplectic area of the spheres S_k is zero.

Assume that $c_1 \neq 0$. Let $U_0 = \mathbf{H}^{-1}(\mathbb{R} \times (y, 3y))$. Then $u(\Sigma) \cap U_0 \neq \emptyset$, and for every $\varepsilon > 0$ we find a ball embedding $B^4(y - \varepsilon) \rightarrow U_0$ such that the origin of the embedded ball lies on $u(\Sigma)$. By the monotonicity theorem, we have that $\int_u \omega \geq y$, so $\int_u \omega \geq \int_{D_0} \omega$, which was what we wanted to show.

ist das der richtige name? referenz?

Hier wäre wahrscheinlich irgend so ein monotonicity argument schöner (muss man vielleicht auch nicht so vorsichtig mit den verschiedenen J's sein...

4.2 D_0 stays a Minimal J-Disks for Suitable Embeddings

Let (n, a) be two coprime integers, μ_n the group of n -th roots of unity. Let μ_n act on \mathbb{C}^2 by $\mu(z_1, z_2) = (\mu z_1, \mu^a z_2)$. Let $A(n, a) = \mathbb{C}^2 / \mu_n$ be the quotient space. This space is an orbifold, with one orbifold point at $[(0, 0)]$.

We define the Hamiltonian system on $A(n, a)$ by

$$G(z_1, z_2) = \frac{1}{2} \left(|z_2|^2, \frac{1}{n} (|z_1|^2 + a|z_2|^2) \right) . \quad (1)$$

With this Hamiltonian system the moment polytope is a wedge with edges pointing along vectors $(1, 0)$, (n, a) , as seen in figure 3. $A(n, a)$ has a almost complex structure J coming from the canonical complex structure on \mathbb{C}^2 .

Let $B(a) \in \mathbb{C}^n$ be the open ball in \mathbb{C}^n of radius $\sqrt{\frac{a}{\pi}}$. In [2, Appendix A] the following lemma is proven:

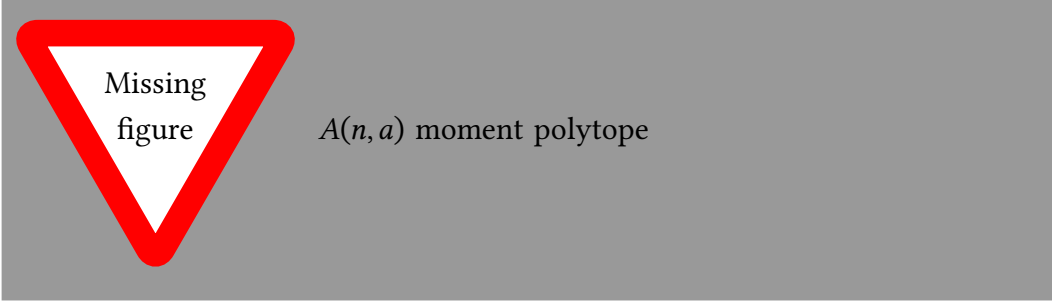


Figure 3: Moment polytope of $A(n, a)$ with given by Hamiltonian system G

Lemma 4.2. *Let $a_+ > a_- \geq 0$. Let $u : \Sigma \rightarrow B(a_+) \setminus \overline{B(a_-)}$ be a \mathcal{J} -holomorphic curve such that the closure of $u(\Sigma)$ in \mathbb{C}^n intersects $\partial B(a_-)$. Then $\int_u \omega \geq a_+ - a_-$.*

We give the slight generalization:

Lemma 4.3. *Let $a_+ > a_- > 0$, and*

$$X = \mathbf{G}^{-1}(\{c_1(0, 1) + c_2(n, a) \mid a_- < c_1 + c_2 < a_+\}) \subset A(n, a),$$

equipped with the almost complex structure of $A(n, a)$.

Let $u : \Sigma \rightarrow X$ be a \mathcal{J} -holomorphic curve whose closure intersects

$$\mathbf{H}^{-1}(\{c_1(0, 1) + c_2(n, a) \mid a_- = c_1 + c_2\}).$$

Then $\int_u \omega \geq a_+ - a_-$.

Remark 4.4. Suppose we have a moment polytope Δ of a (almost) toric symplectic manifold or orbifold $\mathbf{H} : M \rightarrow \Delta$ with two non-parallel edges given by the two primitive vectors u_1, u_2 , as in figure 4. Suppose without loss of generality that the edges intersect in the origin. Then the subset

$$X = \mathbf{H}^{-1}(\{c_1 u_1 + c_2 u_2 \mid a_- < c_1 + c_2 < a_+\})$$

with a_{\pm} such that $a_{\pm} u_1, a_{\pm} u_2 \in \Delta$, can be transformed by a $T \in \text{GL}(\mathbb{Z}^2)$, such that $Tu_1 = (0, 1), Tu_2 = (n, a)$, for some coprime integers n, a .

With this transformation we can view X as a subset of $A(n, a)$. Equipping M with an extension of the almost complex structure coming from $A(n, a)$, we get that \mathcal{J} -curves in M intersecting

$$\mathbf{H}^{-1}(\{c_1 u_1 + c_2 u_2 \mid a_- = c_1 + c_2\})$$

must have at least area $a_+ - a_-$.

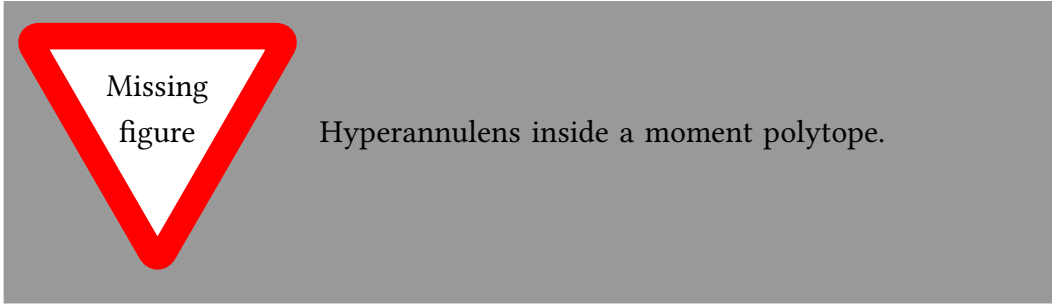


Figure 4: Hyperannulens inside a moment polytope.

NOOOOOOOOOOOOO!

Proof. Since the action of μ_n is free in $(\mathbb{C}^*)^2$, the projection map $\pi : (\mathbb{C}^*)^2 \rightarrow A(n, a) \setminus \{(0, 0)\}$ is an n -fold covering map.

As shown below, the preimage $\pi^{-1}(X)$ is $B(na_+) \setminus \overline{B(na_-)}$, and u lifts to a J-curve \tilde{u} , i.e. a curve making the diagram

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\tilde{u}} & \mathbb{C}^2 \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ \Sigma & \xrightarrow{u} & A(n, a) \end{array}$$

commute, where $\tilde{\pi} : \Sigma' \rightarrow \Sigma$ is some n -fold covering of Σ .

Using lemma 4.2, we get that the symplectic area of \tilde{u} is at least $n(a_+ - a_-)$, and since \tilde{u} is an n -fold covering of u , u has at least symplectic area $a_+ - a_-$, as desired.

To show that $\pi^{-1}(X) = B(na_+) \setminus \overline{B(na_-)}$, note that the Hamiltonian system G from equation (1), is given by a linear transformation of the standard system on \mathbb{C}^2

$$H(z_1, z_2) = \frac{1}{2}(|z_1|^2, |z_2|^2),$$

given by the matrix

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{n} & \frac{a}{n} \end{pmatrix},$$

whose inverse maps $G(X)$ to $H(B(na_+) \setminus \overline{B(na_-)})$, and since π maps fibres to fibres, the claim follows. \square

Define $B_{dpq}(a) := G^{-1}\{c_1(0, 1) + c_2(dp^2, dpq - 1) \mid c_1 + c_2 < a\}$, and let (X, ω, J) be a symplectic manifold with tame almost complex structure.

Suppose we have an embedding $B_{dpq}(2a) \rightarrow X$, and that all J-spheres in X have at least area a . Using remark 4.4 and lemma 4.3, we get that the minimal J-curve with boundary on the torus $T_k(a)$ has area a , and since there are no smaller J-spheres by assumption, using theorem 4.1, we get that the displacement energy of $T_k(a)$ is at least a .

Das ist jetzt nicht wie bei Evans linkswirkend, sondern normal...

was ist G genau?

gibts davon noch eine schönere definition?

Brauchen wir hier noch ein ε platz?

References

- [1] Yu. V. Chekanov. “Lagrangian intersections, symplectic energy, and areas of holomorphic curves”. In: *Duke Mathematical Journal* 95 (1998), pp. 213–226.
- [2] Yu. V. Chekanov and Felix Schlenk. “Lagrangian product tori in tame symplectic manifolds”. In: *arXiv: Symplectic Geometry* (2015).