Exotic Tori from ATFs oder so

JoJoJo

February 17, 2023

Todo list

1 Introduction

Definition 1.1. Let $k \in \mathbb{N}$ such that $0 < k \leq d$ and $a \in (0, \infty)$. Through nodal slides we can arrange the ATF on B_{dpq} such that the line $x_2 = a$ intersects the branch cut line between the $(k-1)$-th and k-th degenerated fibre. $T_k(a)$ is defined to be the fibre over the intersection point of these two lines.

Theorem 1.2. *Let $U \subset H^1(T_k(a), \mathbb{R}) \setminus \{\text{branch cut line}\}$. The restriction of the displacement energy germ to U is given by*

$$S^e_{T_k(a)}\Big|_U (x, y) = a + \max\{x, x(1 - kpq) - kp^2 y\}$$

so oder so ähnlich...

Let $d, p, q \in \mathbb{N}$ such that $d \geq$ and p, q coprime with $1 \leq q < p$ or $q = 0, p = 1$, and $0 < a_1 < \ldots < a_d$ real integers. Let P be the polynomial $P(z) = \prod_{i=1}^{d}(z^p - a_i)$. Define the manifold M_P by

$$M_P = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 + P(z_3) = 0\right\}.$$

We define the Hamiltonian system

$$\mathbf{H}(z_1, z_2, z_3) = \left(|z_3|^2, \frac{1}{2}\left(|z_1|^2 - |z_2|^2\right)\right)$$

Let μ_p be the group of p-Th roots of unity acting on M_P by

$$\mu \cdot (z_1, z_2, z_3) = \left(\mu z_1, \mu^{-1} z_2, \mu^q z_3\right), \quad \mu \in \mu_p .$$

This is a free action, so we can define the quotient $B_{dpq} = M_P/\mu_p$. The Hamiltonian system $\mathbf{H}$ is invariant under the action, so it descends to a Hamiltonian system on B_{dpq}.

2 Upper bound on displacement energy: Probes

3 Short interlude: Homology of B_{dpq}

In order to calculate the lower bound for the displacement energy of a torus $L(x, y)$, we will need to calculate a basis for $H_2(B_{dpq}, L(x, y))$.

definieren

B_{dpq} deformation retracts to the preimage of the branch cut line segment shown in figure 1. This can be understood as follows: If there were no critical points on the line, this would be a solid torus $T = S^1 \times D^2$. We pick $(1, 0), (0, 1) \in H_1(\partial T)$ to be the classes generated by $S^1 \times \mathrm{pt}, \mathrm{pt} \times \partial D^2$ respectively. At each critical point we collapse a loop along homology class $(p, -q)$. Up to homotopy this is the same as attaching a disk along $(p, -q)$. Again up to homotopy we can also require that the d discs $D_1, \ldots, D_d$ are attached along ∂T. Let us call this space S.

Let us look at the long exact sequence of homology for the pair $(B_{dpq}, L(x, y))$. This pair is homotopy equivalent to $(S, \partial T)$.

$$\begin{tikzcd}
H_2(\partial T) \arrow[r, "0"] & H_2(S) \arrow[r, hook] \arrow[d, "\cong"] & H_2(S, \partial T) \arrow[r] \arrow[d, "\cong"] & H_1(\partial T) \arrow[r] \arrow[d, "\cong"] & H_1(S) \arrow[d, "\cong"] \\
& \mathbb{Z}^{d-1} & \mathbb{Z}^{d+1} & \mathbb{Z}^2 & \mathbb{Z}_p
\end{tikzcd}$$

The first horizontal map is zero since ∂T retracts to a point in S. Homology $H_2(S)$ can be seen as follows: By contracting the solid torus T in S to a circle, we see that

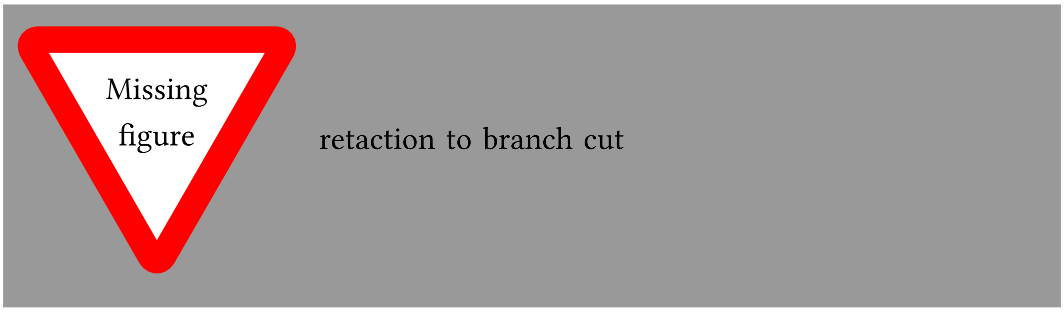

Figure 1: asdf

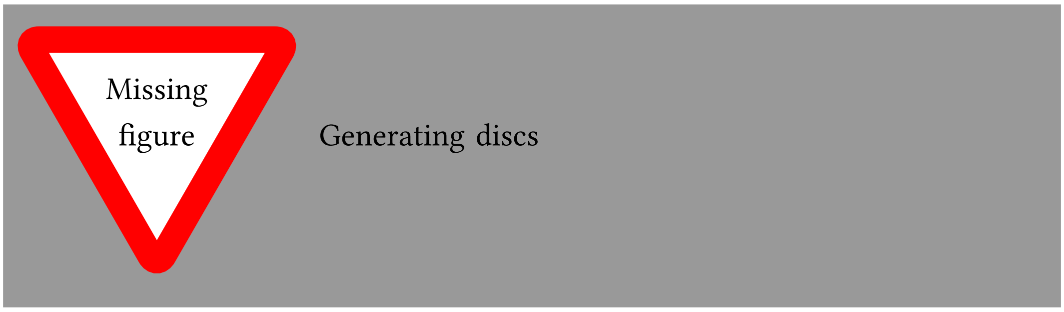

Figure 2: asdf

S is homotopic to a circle with d discs glued to its boundary by a degree p map. So $H_2(S)$ is generated by spheres $\{S_2, ..., S_d\}$, $S_k = D_1 - D_{k+1}$. $H_2(S, \partial T)$ is generated by the discs $D_0 = \mathrm{pt} \times D^2, D_1, ..., D_d$. In B_{dpq}, these discs can be seen, where the disc intersecting the toric boundary collapses the $(0, 1)$ cycle in the toric fibre $L(x, y)$ and the discs intersecting the critical points collapse the $(q, -p)$ cycle (see figure 2).

4 Lower Bound on Displacement Energy: Minimal J-holomorphic Curves

Let $L(x, y)$ a fibre torus, where (x, y) is not over the branch cut line. In [1] the following is proven:

Theorem 4.1. *Let $L \subset X$ be a Lagrangian submanifold. Then the displacement energy satisfies*

$$e(L) \geq \min \sigma_D(X, J), \sigma_S(X, J)$$

4.1 D_0 is a Minimal J-Disk in B_{dpq}

By a nodal slide we can move the critical points in the moment image such that they don't occur for $\mathbf{H}_2 > 3y$. By classification of toric manifolds, $U_{D_0} = \mathbf{H}^{-1}((x -$

$\varepsilon, x+\varepsilon) \times [0, y+\varepsilon))$ is then symplectomorphic to $B^2(y+\varepsilon) \times (x-\varepsilon, x+\varepsilon) \times S^1$, where $B^2(a)$ is the 2-ball of area a. Here we can choose D_0 to be $\overline{B^2(y)} \times \{(x, \mathrm{pt})\}$, which is J-holomorphic with the standard almost complex structure J_0. Our claim is that D_0 is a minimal J-curve in $H_2(B_{dpq}, L(x,y))$.

what is the standard acs?

Let J be some tameextension of $J_0|_{U_{D_0}}$.

Was sind die bedingungen an die J's nochmal genau?

Let $u : \Sigma \to B_{dpq}$ be a non-constant J-holomorphic curve having possibly a boundary on $L(x,y)$. Then the homology class of u can be written in terms of the generators of $H_2(B_{dpq}, L(x,y))$ described in section 3:

$$[u] = c_0 D_0 + c_1 D_1 + \sum_{k=2}^{d} c_k S_k \,.$$

The symplectic area of u is then given by

$$\int_u \omega = c_0 \int_{D_0} \omega + c_1 \int_{D_1} \omega \,,$$

as the symplectic area of the spheres S_k is zero.

Assume that $c_1 \neq 0$. Let $U_0 = \mathbf{H}^{-1}(\mathbb{R} \times (y, 3y))$. Then $u(\Sigma) \cap U_0 \neq \emptyset$, and for every $\varepsilon > 0$ we find a ball embedding $B^4(y-\varepsilon) \to U_0$ such that the origin of the embedded ball lies on $u(\Sigma)$. By the monotonicity theorem, we have that $\int_u \omega \geq y$, so $\int_u \omega \geq \int_{D_0} \omega$, which was what we wanted to show.

ist das der richtige name? referenz?

4.2 D_0 stays a Minimal J-Disks for Suitable Embeddings

Let (n, a) be two coprime integers, μ_n the group of n-th roots of unity. Let μ_n act on $\mathbb{C}^2$ by $\mu(z_1, z_2) = (\mu z_1, \mu^a z_2)$. Let $A(n,a) = \mathbb{C}^2/\mu_n$ be the quotient space. This space is an orbifold, with one orbifold point at $[(0,0)]$.

We define the Hamiltonian system on $A(n,a)$ by

$$\mathrm{G}(z_1, z_2) = \frac{1}{2}\left(|z_2|^2, \frac{1}{n}\left(|z_1|^2 + a|z_2|^2\right)\right) \,. \tag{1}$$

With this Hamiltonian system the moment polytope is a wedge with edges pointing along vectors $(1,0), (n,a)$, as seen in figure 3. $A(n,a)$ has a almost complex structure J coming from the canonical complex structure on $\mathbb{C}^2$.

Let $B(a) \in \mathbb{C}^n$ be the open ball in $\mathbb{C}^n$ of radius $\sqrt{\frac{a}{\pi}}$. In [2, Appendix A] the following lemma is proven:

Lemma 4.2. *Let $a_+ > a_- \geq 0$. Let $u : \Sigma \to B(a_+) \setminus \overline{B(a_-)}$ be a J-holomorphic curve such that the closure of $u(\Sigma)$ in $\mathbb{C}^n$ intersects $\partial B(a_-)$. Then $\int_u \omega \geq a_+ - a_-$.*

We give the slight generalization:

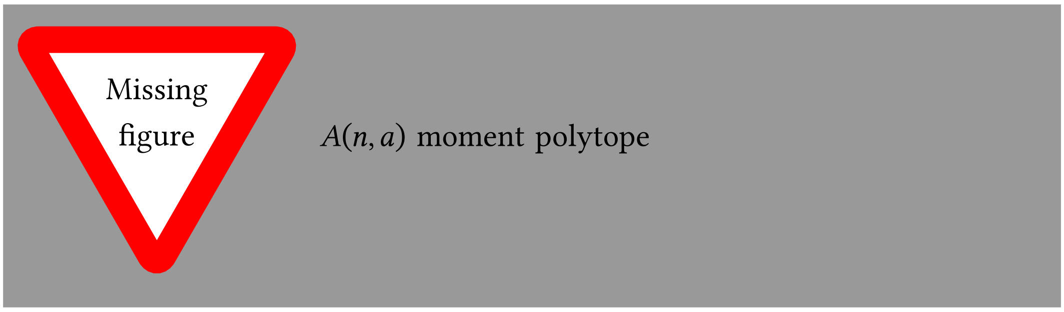

Figure 3: Moment polytope of $A(n, a)$ with given by Hamiltonian system G

Lemma 4.3. *Let* $a_+ > a_- > 0$, *and*

$$X = \mathbf{G}^{-1}(\{c_1(0,1) + c_2(n,a) \mid a_- < c_1 + c_2 < a_+\}) \subset A(n,a)\ ,$$

equipped with the almost complex structure of $A(n,a)$.

Let $u : \Sigma \to X$ *be a J-holomorphic curve whose closure intersects*

$$\mathbf{H}^{-1}(\{c_1(0,1) + c_2(n,a) \mid a_- = c_1 + c_2\})\ .$$

Then $\int_u \omega \geq a_+ - a_-$.

Remark 4.4. Suppose we have a moment polytope Δ of a (almost) toric symplectic manifold or orbifold $\mathbf{H} : M \to \Delta$ with two non-parallel edges given by the two primitive vectors u_1, u_2, as in figure 4. Suppose without loss of generality that the edges intersect in the origin. Then the subset

$$X = \mathbf{H}^{-1}(\{c_1 u_1 + c_2 u_2 \mid a_- < c_1 + c_2 < a_+\})$$

with $a_\pm$ such that $a_\pm u_1, a_\pm u_2 \in \Delta$, can be transformed by a $T \in \mathrm{GL}(\mathbb{Z}^2)$, such that $Tu_1 = (0,1), Tu_2 = (n,a)$, for some coprime integers n, a.

With this transformation we can view X as a subset of $A(n,a)$. Equipping M with an extension of the almost complex structure coming from $A(n,a)$, we get that J-curves in M intersecting

$$\mathbf{H}^{-1}(\{c_1 u_1 + c_2 u_2 \mid a_- = c_1 + c_2\})$$

must have at least area $a_+ - a_-$.

Proof. Since the action of μ_n is free in $(\mathbb{C}^*)^2$, the projection map $\pi : (\mathbb{C}^*)^2 \to A(n,a) \setminus \{[(0,0)]\}$ is an n-fold covering map.

As shown below, the preimage $\pi^{-1}(X)$ is $B(na_+) \setminus \overline{B(na_-)}$, and u lifts to a J-curve $\tilde{u}$, i.e. a curve making the diagram

$$\begin{tikzcd} \Sigma' \arrow[r, "\tilde{u}"] \arrow[d, "\tilde{\pi}"] & \mathbb{C}^2 \arrow[d, "\pi"] \\ \Sigma \arrow[r, "u"] & A(n,a) \end{tikzcd}$$

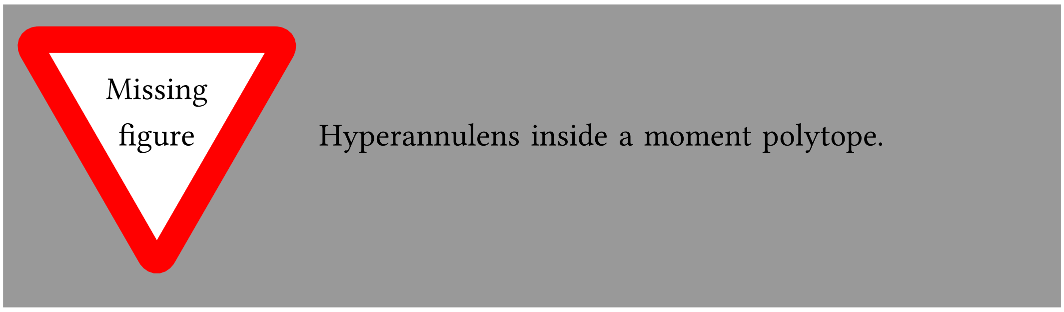

Figure 4: Hyperannulens inside a moment polytope.

NOOOOOOOOOOOO!

commute, where $\tilde{\pi} : \Sigma' \to \Sigma$ is some n-fold covering of Σ.

Using lemma 4.2, we get that the symplectic area of $\tilde{u}$ is at least $n(a_+ - a_-)$, and since $\tilde{u}$ is an n-fold covering of u, u has at least symplectic area $a_+ - a_-$, as desired.

To show that $\pi^{-1}(X) = B(na_+) \setminus \overline{B(na_-)}$, note that the Hamiltonian system $\mathbf{G}$ from equation (1), is given by a linear transformation of the standard system on $\mathbb{C}^2$

$$\mathbf{H}(z_1, z_2) = \frac{1}{2}\left(|z_1|^2, |z_2|^2\right),$$

given by the matrix

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{n} & \frac{a}{n} \end{pmatrix},$$

Das ist jetzt nicht wie bei Evans linkswirkend, sondern normal...

whose inverse maps $\mathbf{G}(X)$ to $\mathbf{H}(B(na_+) \setminus \overline{B(na_-)})$, and since π maps fibres to fibres, the claim follows. □

Define $B_{dpq}(a) := \mathbf{G}^{-1}\{c_1(0,1) + c_2(dp^2, dpq - 1) \mid c_1 + c_2 < a\}$, and let (X, ω, J) be a symplectic manifold with tame almost complex structure.

was ist G genau?

gibts davon noch eine schönere definition?

Suppose we have an embedding $B_{dqp}(2a) \to X$, and that all J-spheres in X have at least area a. Using remark 4.4 and lemma 4.3, we get that the minimal J-curve with boundary on the torus $T_k(a)$ has area a, and since there are no smaller J-spheres by assumption, using theorem 4.1, we get that the displacement energy of $T_k(a)$ is at least a.

Brauchen wir hier noch ein ε platz?

References

[1] Yu. V. Chekanov. “Lagrangian intersections, symplectic energy, and areas of holomorphic curves”. In: *Duke Mathematical Journal* 95 (1998), pp. 213–226.

[2] Yu. V. Chekanov and Felix Schlenk. “Lagrangian product tori in tame symplectic manifolds”. In: *arXiv: Symplectic Geometry* (2015).