Exotic Tori from ATFs oder so

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1 Introduction

Definition 1.1. Let $k \in \mathbb{N}$ such that $0 < k \leq d$ and $a \in (0, \infty)$. Through nodal slides we can arrange the ATF on B_{dpq} such that the line $x_2 = a$ intersects the branch cut line between the $(k-1)$-th and k-th degenerated fibre. $T_k(a)$ is defined to be the fibre over the intersection point of these two lines.

Theorem 1.2. *Let $U \subset H^1(T_k(a), \mathbb{R}) \setminus \{\text{branch cut line}\}$. The restriction of the displacement energy germ to U is given by*

$$S^e_{T_k(a)}\Big|_U (x, y) = a + \max\{x, x(1 - kpq) - kp^2 y\}$$

so oder so ähnlich...

Let $d, p, q \in \mathbb{N}$ such that $d \geq$ and p, q coprime with $1 \leq q < p$ or $q = 0, p = 1$, and $0 < a_1 < \ldots < a_d$ real integers. Let P be the polynomial $P(z) = \prod_{i=1}^d (z^p - a_i)$. Define the manifold M_P by

$$M_P = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 + P(z_3) = 0\right\}.$$

We define the Hamiltonian system

$$\mathbf{H}(z_1, z_2, z_3) = \left(|z_3|^2, \frac{1}{2}\left(|z_1|^2 - |z_2|^2\right)\right)$$

Let μ_p be the group of p-Th roots of unity acting on M_P by

$$\mu \cdot (z_1, z_2, z_3) = \left(\mu z_1, \mu^{-1} z_2, \mu^q z_3\right), \quad \mu \in \mu_p.$$

This is a free action, so we can define the quotient $B_{dpq} = M_P / \mu_p$. The Hamiltonian system $\mathbf{H}$ is invariant under the action, so it descends to a Hamiltonian system on B_{dpq}.

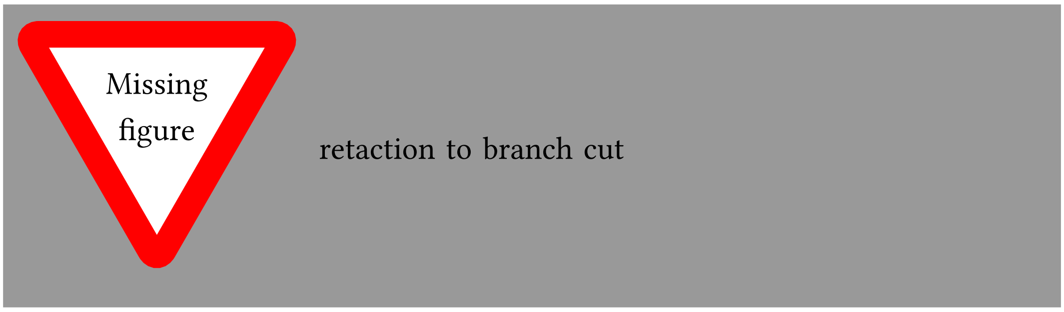

Figure 1: asdf

2 Upper bound on displacement energy: Probes

3 Short interlude: Homology of B_{dpq}

In order to calculate the lower bound for the displacement energy of a torus $L(x, y)$, definieren we will need to calculate a basis for $H_2(B_{dpq}, L(x, y))$.

B_{dpq} deformation retracts to the preimage of the branch cut line segment shown in fig. 1. This can be understood as follows: If there were no critical points on the line, this would be a solid torus $T = S^1 \times D^2$. We pick $(1, 0), (0, 1) \in H_1(\partial T)$ to be the classes generated by $S^1 \times \mathrm{pt}, \mathrm{pt} \times \partial D^2$ respectively. At each critical point we collapse a loop along homology class $(p, -q)$. Up to homotopy this is the same as attaching a disk along $(p, -q)$. Again up to homotopy we can also require that the d discs $D_1, \ldots, D_d$ are attached along ∂T. Let us call this space S.

Let us look at the long exact sequence of homology for the pair $(B_{dpq}, L(x, y))$. This pair is homotopy equivalent to $(S, \partial T)$.

$$\begin{tikzcd}
H_2(\partial T) \arrow[r, "0"] & H_2(S) \arrow[r, hook] \arrow[d, "\cong"] & H_2(S, \partial T) \arrow[r] \arrow[d, "\cong"] & H_1(\partial T) \arrow[r] \arrow[d, "\cong"] & H_1(S) \arrow[d, "\cong"] \\
& \mathbb{Z}^{d-1} & \mathbb{Z}^{d+1} & \mathbb{Z}^2 & \mathbb{Z}_p
\end{tikzcd}$$

The first horizontal map is zero since ∂T retracts to a point in S. Homology $H_2(S)$ can be seen as follows: By contracting the solid torus T in S to a circle, we see that S is homotopic to a circle with d discs glued to its boundary by a degree p map. So $H_2(S)$ is generated by spheres $\{S_2, \ldots, S_d\}$, $S_k = D_1 - D_{k+1}$. $H_2(S, \partial T)$ is generated by the discs $D_0 = \mathrm{pt} \times D^2, D_1, \ldots, D_d$. In B_{dpq}, these discs can be seen, where the disc intersecting the toric boundary collapses the $(0, 1)$ cycle in the toric fibre $L(x, y)$ and the discs intersecting the critical points collapse the $(q, -p)$ cycle (see fig. 2).

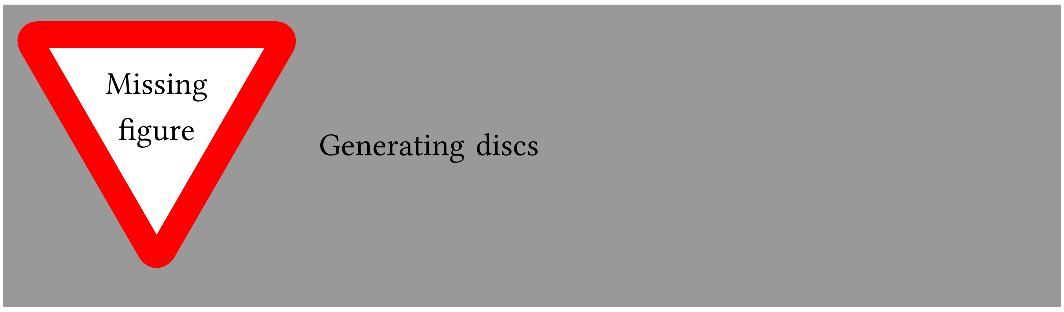

Figure 2: asdf

4 Lower bound on displacement energy: minimal J-holomorphic curves

oder nur discs?

Let $L(x, y)$ a fibre torus, where (x, y) is not over the branch cut line. In [1] it is proved that lalalala

Pick a tame almost complex structure J on B_{pdq}. Let u be a non-constant J-holomorphic sphere or disc with boundary on $L(x, y)$. Then the homology class of u can be written in terms of the generators of $H_2(B_{dpq}, L(x, y))$ described above as

Or some standard structure? Does it matter?

$$[u] = c_0 D_0 + c_1 D_1 + \sum_{k=2}^{d} c_k S_k \,.$$

The symplectic area of u is then given by

$$\int_u \omega = c_0 \int_{D_0} \omega + c_1 \int_{D_1} \omega \,,$$

as the symplectic area of the spheres S_k is zero.

By a nodal slide we can move the critical points in the moment image such that they don't occur for $\mathbf{H}_2 > 2y$. By classification of toric manifolds, $\{p \in B_{dpq} \mid \mathbf{H}_2(p) < 2y\}$ is then symplectomorphic to $B^2(2y) \times \mathbb{R} \times S^1$, where $B^2(a)$ is the 2-ball of area a. Here we can choose D_0 to be $\overline{B^2(y)} \times \{(x, \mathrm{pt})\}$, which is J-holomorphic with the standard almost complex structure. Our claim is that D_0 is the minimal J-disc.

References

[1] Yu. V. Chekanov. "Lagrangian intersections, symplectic energy, and areas of holomorphic curves". In: *Duke Mathematical Journal* 95 (1998), pp. 213–226.