

Provably Efficient and Agile Randomized Q-Learning

He Wang*
CMU

Xingyu Xu*
CMU

Yuejie Chi†
Yale

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Abstract

While Bayesian-based exploration often demonstrates superior empirical performance compared to bonus-based methods in model-based reinforcement learning (RL), its theoretical understanding remains limited for model-free settings. Existing provable algorithms either suffer from computational intractability or rely on stage-wise policy updates which reduce responsiveness and slow down the learning process. In this paper, we propose a novel variant of Q-learning algorithm, refereed to as RANDOMIZEDQ, which integrates *sampling-based exploration with agile, step-wise, policy updates*, for episodic tabular RL. We establish a sublinear regret bound $\tilde{O}(\sqrt{H^5SAT})$, where S is the number of states, A is the number of actions, H is the episode length, and T is the total number of episodes. In addition, we present a logarithmic regret bound $O\left(\frac{H^6SA}{\Delta_{\min}} \log^5(SAHT)\right)$ when the optimal Q-function has a positive suboptimality Δ_{\min} . Empirically, RANDOMIZEDQ exhibits outstanding performance compared to existing Q-learning variants with both bonus-based and Bayesian-based exploration on standard benchmarks.

Keywords: Q-learning, learning rate randomization, Bayesian exploration

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*Department of Electrical and Computer Engineering, Carnegie Mellon University. Email: {hew2,xingyuxu}@andrew.cmu.edu.

†Department of Statistics and Data Science, Yale University. Email: yuejie.chi@yale.edu.

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1 Introduction

In reinforcement learning (RL) (Sutton, 1988), an agent aims to learn an optimal policy that maximizes its cumulative rewards through interactions with an unknown environment. Broadly speaking, RL algorithms can be categorized into two main approaches—model-based and model-free methods—depending on whether they first learn a model of the environment and plan within it, or directly learn the optimal policy from experience. While model-based approaches offer advantages in sample efficiency, model-free algorithms tend to be more computationally efficient and take less space complexity, making them more attractive for deployment in many real-world applications, such as games (Mnih et al., 2015) and language model training (Hong et al., 2025).

As the one of fundamental challenges in RL, the *exploitation-exploration dilemma* remains particularly difficult to address in the model-free paradigm, i.e., the learned policy needs to carefully balance between exploiting current observations and exploring unseen state-action pairs to maximize total rewards in the long term. To manage the trade-off, most provably efficient model-free algorithms adopt the principle of *optimism in the face of uncertainty*, incentivizing exploration by assigning bonuses to uncertain outcomes, constructed from their upper confidence bound (UCB) (Lai and Robbins, 1985). In particular, prior works (Jin et al., 2018; Zhang et al., 2020; Li et al., 2021) showed that Q-learning augmented with tailored bonus functions achieve comparable sample complexity to their model-based counterparts.

In contrast to bonus-based exploration methods aforementioned, Bayesian-based approaches have gained increasing attention for their superior empirical performance (Osband et al., 2016a; Fortunato et al., 2018). These approaches enhance efficient exploration by leveraging the inherent randomness in sampling from posteriors that are updated based on prior observations. However, theoretical understandings have been limited, where the majority of prior work has focused on model-based RL (Osband et al., 2013; Agrawal and Jia, 2017). When it comes to model-free RL, research is even more limited in several aspects. Dann et al. (2021) proposed a *sample-efficient* algorithm that draws Q-functions directly from the posterior distribution. However, this approach suffers from *computational inefficiency*. More recently, Tiapkin et al. (2024) introduced posterior sampling via randomized learning rates, but unfortunately they only provided theoretical guarantees¹ for *stage-wise policy updates*, which are known to be inefficient in practice as this staging approach does not allow agents to respond agilely to the environment. To this end, it is natural to ask:

*Is it possible to design a model-free RL algorithm with Bayesian-based exploration, achieving **sample efficiency**, **computational efficiency**, and **agile policy updates**?*

1.1 Main contribution

To answer this question, we focus on learning a near-optimal policy through Bayesian-based Q-learning, in a provably sample- and computation-efficient manner. As in Jin et al. (2018); Dann et al. (2021); Tiapkin et al. (2024), throughout this paper, we consider tabular, finite-horizon Markov Decision Processes (MDPs) in the online setting. Below we summarize the highlights of this work:

¹A careful examination of their proof reveals a critical technical gap in their analysis. We provide a novel fix with substantial new analyses, which fortunately preserves their claimed theoretical guarantee. We discuss this in more detail in Section 4.1.

Key Property	Conditional-PS (Dann et al., 2021)	Staged-RandQL (Tiapkin et al., 2024)	RandQL (Tiapkin et al., 2024)	RANDOMIZEDQ (This Work)
Computational tractability	\times	✓	✓	✓
Agile policy update	✓	\times	✓	✓
Gap-independent regret guarantee	✓	✓	\times	✓
Gap-dependent regret guarantee	\times	\times	\times	✓

Table 1: Comparison with the most relevant model-free RL methods with bayesian-based exploration in tabular settings. A ✓ indicates the method possesses the corresponding property, while a \times denotes its absence. We identify and fix a technical gap in (Tiapkin et al., 2024), which preserves the gap-independent regret guarantee of Staged-RandQL. Notably, our method uniquely achieves *computational tractability*, *agile policy updates*, and *provable regret guarantees*, distinguishing it from prior work.

- We propose RANDOMIZEDQ, a sampling-based of Q-learning algorithm which leverages tailored randomized learning rates to enable both efficient exploration and agile policy updates.
- We establish a gap-independent regret bound on the order of $\tilde{O}(\sqrt{H^5 SAT})$, where S is the number of states, A is the number of actions, H is the episode length, and T is the number of episodes.
- Under a strictly positive sub-optimality gap Δ_{\min} of the optimal Q-function, we further prove a logarithmic regret bound of $O(H^6 SA \log^5(SAHT)/\Delta_{\min})$. To the best of our knowledge, this is the first result showing model-free algorithms can achieve logarithmic regret via sampling-based exploration.
- Empirically, RANDOMIZEDQ consistently outperforms existing bonus-based and Bayesian-based model-free algorithms on standard exploration benchmarks, validating its efficacy.

A detailed comparison with pertinent works is provided in the Table 1.

1.2 Related works

In this section, we discuss closely-related prior works on optimistic Q-learning and online RL with Bayesian-based exploration, focusing on the tabular setting.

Q-learning with bonus-based exploration. Q-learning and its variants (Watkins, 1989; Mnih et al., 2013; Strehl et al., 2006) are among the most widely studied model-free RL algorithms. To understand its theoretical guarantees, several works have equipped Q-learning with UCB bonuses derived from the principle of optimism in the face of uncertainty (Jin et al., 2018; Zhang et al., 2020; Li et al., 2021; Yang et al., 2021; Zheng et al., 2025). Notably, Jin et al. (2018) first introduced UCB-Q, which augments Q-learning with Hoeffding-type or Bernstein-type bonuses and established a nearly optimal regret bound. Building upon this, Zhang et al. (2020) proposed a variance-reduced version of UCB-Q, achieving an optimal sample complexity, and Li et al. (2021) further improved the performance by reducing the burn-in cost.

In addition to the worst-case regret bound, gap-dependent regret bounds often leverage benign properties of the environment and enjoys logarithmic regret bounds (Yang et al., 2021; Zheng et al., 2025). For instance, Yang et al. (2021) showed that UCB-Q has a logarithmic regret bound under the positive sub-optimality gap assumption, and Zheng et al. (2025) incorporated error decomposition to establish a gap-dependent bound for Q-learning with variance reduction techniques (Zhang et al., 2020; Li et al., 2021).

Model-based RL with Bayesian-based exploration. Extensive works have investigated the theoretical and empirical performance of Bayesian-based approaches exploration in model-based RL. One popular approach is posterior sampling for reinforcement learning (Strens, 2000; Osband et al., 2013; Agrawal and Jia, 2017; Zhang, 2022; Hao and Lattimore, 2022; Moradipari et al., 2023), where the policy is iteratively learned by sampling a model from its posterior distribution over MDP models. The approach has been shown to achieve the optimal regret bound when UCB on Q-functions are also incorporated (Tiapkin et al., 2022). In addition, several works (Osband et al., 2016b; Agrawal et al., 2021; Zanette et al., 2020) have investigated posterior sampling with linear function approximation.

Model-free RL with Bayesian-based exploration. Dann et al. (2021) sampled Q-functions directly from the posterior, but such an approach is computationally intractable. To address this, Tiapkin et al. (2024) introduced RandQL, the first tractable model-free posterior sampling-based algorithm, which encourages exploration through using randomized learning rates and achieves a regret bound of $\tilde{O}(SAH^3 + \sqrt{SAH^5T})$ when RandQL is staged. However, the slow policy update empirically leads to a significantly degenerated performance. This leaves a gap between the theoretical efficiency and practical performance in model-free RL with Bayesian-based exploration.

Notation. Throughout this paper, we define $\Delta(\mathcal{S})$ as the probability simplex over a set \mathcal{S} , and use $[H] := 1, \dots, H$ and $[T] := 1, \dots, T$ for positive integers $H, T > 0$. We denote $\mathbb{1}$ as the indicator function, which equals 1 if the specified condition holds and 0 otherwise. For any set \mathcal{D} , we write $|\mathcal{D}|$ to represent its cardinality (i.e., the number of elements in \mathcal{D}). The beta distribution with parameters α and β is denoted by $\text{Beta}(\alpha, \beta)$. Finally, we use the notations $\tilde{O}(\cdot)$ and $O(\cdot)$ to describe the order-wise non-asymptotic behavior, where the former omits logarithmic factors.

2 Problem Setup

Finite-horizon MDPs. Consider a tabular finite-horizon MDP $\mathcal{M}(\mathcal{S}, \mathcal{A}, \{P_h\}_{h=1}^H, \{r_h\}_{h=1}^H, H)$, where \mathcal{S} is the finite state space of cardinality S , \mathcal{A} is the action space of cardinality A , $P_h : \mathcal{S} \times \mathcal{A} \rightarrow \Delta(\mathcal{S})$ is the transition kernel and $r_h : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$ is the reward function at time step $h \in [H]$, and H is the number of steps within each episode. In each episode, the agent starts from an initial state $s_1 \in \mathcal{S}$ and then interacts with the environment for H steps. In each step $h \in [H]$, the agent observes the current state $s_h \in \mathcal{S}$, selects an action $a_h \in \mathcal{A}$, receives a reward $r_h(s_h, a_h)$, and transitions to the next state $s_{h+1} \sim P_h(\cdot | s_h, a_h)$.

Policy, value function and Q-function. We denote $\pi = \{\pi_h\}_{h=1}^H$ as the *policy* of the agent within an episode of H steps, where each $\pi_h : \mathcal{S} \rightarrow \Delta(\mathcal{A})$ specifies the action selection probability over the action space \mathcal{A} at the step $h \in [H]$. Given any finite-horizon MDP \mathcal{M} , we use the value function V_h^π (resp.Q-function) to denote the expected accumulative rewards starting from the state s (resp.the state-action pair (s, a)) at step h and following the policy π until the end of the episode:

$$\begin{aligned} \forall (h, s, a) \in [H] \times \mathcal{S} : \quad V_h^\pi(s) &= \mathbb{E}_\pi \left[\sum_{h'=h}^H r_{h'}(s_{h'}, a_{h'}) | s_h = s \right], \\ \forall (h, s, a) \in [H] \times \mathcal{S} \times \mathcal{A} : \quad Q_h^\pi(s, a) &= r_h(s, a) + \mathbb{E}_\pi \left[\sum_{h'=h+1}^H r_{h'}(s_{h'}, a_{h'}) | s_h = s, a_h = a \right]. \end{aligned}$$

By convention, we set $V_{H+1}^\pi(s) = 0$ and $Q_{H+1}^\pi(s, a) = 0$ for any $(s, a) \in \mathcal{S} \times \mathcal{A}$ and policy π . In addition, we denote $\pi^* = \{\pi_h^*\}_{h=1}^H$ as the *deterministic optimal policy*, which maximizes the value function (resp.Q-function) for all states (resp.state-action pairs) among all possible policies, i.e.

$$V_h^*(s) := V_h^{\pi^*}(s) = \max_\pi V_h^\pi(s), \quad Q_h^*(s, a) := Q_h^{\pi^*}(s, a) = \max_\pi Q_h^\pi(s, a), \quad (1)$$

where the existence of the optimal policy is well-established (Puterman, 2014).

Bellman equations. As the pivotal property of MDPs, the value function and Q-function satisfy the following Bellman consistency equations: for any policy π ,

$$\forall (h, s, a) \in [H] \times \mathcal{S} \times \mathcal{A} : \quad Q_h^\pi(s, a) = r_h(s, a) + P_{h,s,a} V_{h+1}^\pi, \quad (2)$$

where we use $P_{h,s,a} := P(\cdot | s, a) \in [0, 1]^{1 \times S}$ for any $(h, s, a) \in [H] \times \mathcal{S} \times \mathcal{A}$ to represent the transition probability (row) vector for the state-action pair (s, a) at h -th step. Similarly, we also have the following Bellman optimality equation regarding the optimal policy π^* :

$$\forall (h, s, a) \in [H] \times \mathcal{S} \times \mathcal{A} : \quad Q_h^*(s, a) = r_h(s, a) + P_{h,s,a} V_{h+1}^*. \quad (3)$$

Learning goal. In this work, our goal is to learn a policy that minimizes the total regret during T episodes, defined as

$$\text{Regret}_T = \sum_{t=1}^T \left(V_1^*(s_1) - V_1^{\pi^t}(s_1) \right), \quad (4)$$

in a computational efficient and scalable fashion. Here, π^t denotes the learned policy after the t -th episode.

3 Efficient and Agile Randomized Q-learning

In this section, we introduce a Bayesian variant of the Q-learning algorithm, referred to as **RANDOMIZEDQ**, which ensures the policy to be updated in an agile manner and enhances efficient exploration based on random sampling rather than adding explicit bonuses.

3.1 Motivation

Before describing the proposed algorithm in detail, we first review the critical role of learning rates in Q-learning to achieve polynomial sample complexity guarantees, and why such choices cannot be directly extended to the context of Bayesian-based exploration.

The effect of learning rates in Q-learning with UCB bonuses. Upon observing a sample transition (s, a, s') at the h -th step, the celebrated UCB-Q algorithm ([Jin et al., 2021](#)) updates the Q-values as:

$$Q_h(s, a) \leftarrow (1 - w_m)Q_h(s, a) + w_m[r_h(s, a) + V_{h+1}(s') + b_m],$$

where m is the number of visits to the state-action pair (s, a) at the h -th step, $w_m = \frac{H+1}{H+m}$ is the learning rate and b_m is the UCB-style bonus term to drive exploration. As detailed in [Jin et al. \(2021\)](#), such a learning rate of $O(H/m)$ is essential to ensure that the first earlier observations have negligible influence on the most recent Q-value updates.

Challenges in randomized Q-Learning. In the absence of bonus terms, the exploration is guided by assigning higher weights to important states and leveraging the inherent randomness of sampling from the posterior. As directly sampling Q-functions is computationally intractable ([Dann et al., 2021](#)), recent work ([Tiapkin et al., 2024](#)) encourages exploration by randomizing the learning rates according to Beta distribution with an expected order of $O(1/m)$. However, such a learning rate treats all episodes as equally informative, which can result in high bias and an exponential dependency of the sample complexity on the horizon H . To overcome the resulting exponential dependence on the horizon H , [Tiapkin et al. \(2024\)](#) resort to split the learning process, update the policy at exponentially slower frequencies, and reset the temporary Q values to ensure enough optimism. While this strategy mitigates the sample complexity issue, it suffers from practical inefficiencies due to discarding valuable data across stages, and is unsuitable to deploy in time-varying environments. This inefficiency is empirically demonstrated in [Tiapkin et al. \(2024, Appendix I\)](#) and Section 5.

Thus, it naturally raises the question: can we simply randomize learning rates with an expected order of $O(H/m)$, as used in in UCB-Q ([Jin et al., 2018](#))? Unfortunately, randomizing learning rates with an expected magnitude of $O(H/m)$ rapidly forgets the earlier episodes that includes the initialization and thus fails to maintain sufficient optimism.

3.2 Algorithm description

Motivated by these limitations, we propose an agile, bonus-free Q-learning algorithm for episodic tabular RL in online setting, referred to as **RANDOMIZEDQ** and summarized in Algorithm 1. The main idea behind **RANDOMIZEDQ** is to update the policy based on an *optimistic mixture of two Q-function ensembles*—each trained with a tailored distribution of learning rates—to balance between agile exploitation of recent observations and sufficiently optimistic exploration. Specifically, after initializing the counters, value and Q-functions as Line 1 in Algorithm 1, the updates at each step h of every episode t can be boiled down to the following key components.

Algorithm 1 RANDOMIZEDQ

Input: Inflation coefficient κ , ensemble size J , the number of prior transitions n_0 , initial state s_1 , optimistically-initial value function $V^0 = \{V_h^0\}_{h=1}^{H+1}$.

- 1: **Initialize:** $n_h(s, a), n_h^b(s, a), q_h(s, a) \leftarrow 0; \pi_h(s) = \arg \max_{a \in \mathcal{A}} r_h(s, a); \tilde{V}_h(s) \leftarrow V_h^0; \tilde{V}_h^b(s) \leftarrow (1+H)V_h^0;$
 $\tilde{Q}_h^j(s, a) \leftarrow r_h(s, a) + V_{h+1}^0; \tilde{Q}_h^b(s, a), \tilde{Q}_h^{b,j}(s, a) \leftarrow r_h(s, a) + (1+H)V_{h+1}^0$, for $(j, h, s, a) \in [J] \times [H] \times \mathcal{S} \times \mathcal{A}$.
- 2: **for** $t \in [T]$ **do**
- 3: **for** $h = 1, \dots, H$ **do**
- 4: Play $a_h = \arg \max_{a \in \mathcal{A}} \pi_h(s_h)$ and observe the next state $s_{h+1} \sim P_h(\cdot | s_h, a_h)$.
- 5: Set $m \leftarrow n_h(s_h, a_h)$ and $m^b \leftarrow m - n_h^b(s_h, a_h)$.
- 6: /* Update temporary Q-ensembles via randomized learning rates. */
- 7: **for** $j = 1, \dots, J$ **do**
- 8: Sample $w_m^j \sim \text{Beta}\left(\frac{H+1}{\kappa}, \frac{m+n_0}{\kappa}\right)$ and $w_m^{b,j} \sim \text{Beta}\left(\frac{1}{\kappa}, \frac{m^b+n_0}{\kappa}\right)$.
- 9: Update temporary Q-functions \tilde{Q}_h^j and $\tilde{Q}_h^{b,j}$ as
- 10:
$$\tilde{Q}_h^j(s_h, a_h) \leftarrow (1 - w_m^j)\tilde{Q}_h^j(s_h, a_h) + w_m^j \left(r_h(s_h, a_h) + \tilde{V}_{h+1}(s_{h+1}) \right), \quad (5)$$
- 11:
$$\tilde{Q}_h^{b,j}(s_h, a_h) \leftarrow (1 - w_m^{b,j})\tilde{Q}_h^{b,j}(s_h, a_h) + w_m^{b,j} \left(r_h(s_h, a_h) + \tilde{V}_{h+1}^b(s_{h+1}) \right). \quad (6)$$
- 12: **end for**
- 13: /* Update the policy Q-function by optimistic mixing. */
- 14: Update the policy $\pi_h(s_h) \leftarrow \arg \max_{a \in \mathcal{A}} Q_h(s_h, a)$.
- 15: /* Update \tilde{V}_h optimistically. */
- 16: Update the value function $\tilde{V}_h(s_h) \leftarrow \max_{j \in [J]} \tilde{Q}_h^j(s_h, \pi_h(s_h))$.
- 17: /* Update visit counters. */
- 18: Update the counter $n_h(s_h, a_h) \leftarrow n_h(s_h, a_h) + 1$ and $n_h^b(s_h, a_h) \leftarrow n_h^b(s_h, a_h) + 1$.
- 19: /* At the end of the stage: update \tilde{Q}_h^b , \tilde{V}_h^b and reset n_h^b , $\{\tilde{Q}_h^{b,j}\}$. */
- 20: **if** $n_h^b(s_h, a_h) = \lfloor (1+1/H)^q H \rfloor$ for the stage $q = q_h(s, a)$ **then**
- 21: Update $\tilde{Q}_h^b(s_h, a_h) \leftarrow \max_{j \in [J]} \tilde{Q}_h^{b,j}(s_h, a_h)$, $\tilde{V}_h^b(s_h) \leftarrow \min_{a \in \mathcal{A}} \tilde{Q}_h^b(s_h, a)$.
- 22: Reset $n_h^b(s_h, a_h) = n_h(s_h, a_h)$, $\tilde{Q}_h^{b,j}(s_h, a_h) \leftarrow r_h(s_h, a_h) + (1+H)V_h^0$.
- 23: Update the stage counter $q_h(s, a) \leftarrow q_h(s, a) + 1$.
- 24: **end if**
- 25: **end for**
- 26: **end for**

Two Q-ensembles for adaptation and exploration. To ensure the mixed Q-function with the learning rate scaled as $O(H/m)$, we tailor the probability distribution of the randomized learning rate as: $w_m^j \sim \text{Beta}\left(\frac{H+1}{\kappa}, \frac{m+n_0}{\kappa}\right)$, where m denotes the total number of visits to the state-action pair (s_h^t, a_h^t) just before current visit (cf. Line 5 in Algorithm 1), n_0 introduces pseudo-transitions to induce optimism, and $\kappa > 0$ controls the concentration level of the distribution. With these randomized learning rates in hand, the temporary Q-functions are updated in parallel via (5). As discussed in Section 3.1, such a learning rate could guarantee a polynomial dependency on the horizon H , but lead to rapidly forgetting the earlier episodes and assigning exponentially decreasing weights on the optimistic initialization. To increase the importance on the optimistic initialization, we also introduce another sequence of Q-ensembles updated via (6), with randomized learning rates sampled from $w_m^{b,j} \sim \text{Beta}\left(\frac{1}{\kappa}, \frac{m^b+n_0}{\kappa}\right)$, where m^b represents the number of visits during the previous stage.

Agile policy Q-function via optimistic mixing. Then, to promote optimism, the policy Q-function is computed via optimistic mixing:

$$Q_h^{t+1}(s, a) = \begin{cases} (1 - \frac{1}{H}) \max_{j \in [J]} \left\{ \tilde{Q}_h^{j,t+1}(s, a) \right\} + \frac{1}{H} \cdot \tilde{Q}_h^{\text{b},t+1}(s, a), & \text{if } (s, a) = (s_h^t, a_h^t), \\ Q_h^t(s, a), & \text{otherwise.} \end{cases}$$

where $\tilde{Q}_h^{\text{b},t+1}$ remains fixed throughout the current stage and is updated as the ensemble maximum only at the end of the stage (cf. Line 21 in Algorithm 1). Note that the first term—corresponding to the maximum over J temporary Q-values—is updated every step, which allows RANDOMIZEDQ performs agile policy updates rather than *exponentially slower*. Correspondingly, the policy π_h^{t+1} is updated greedily with respect to Q_h^{t+1} (cf. Line 14 in Algorithm 1). Such optimistic mixing allows RANDOMIZEDQ to remain responsive to new data and adapt the policy efficiently without requiring periodic resets.

Reset for bias mitigation and optimism restoration. To mitigate outdated data and ensure optimism, we reset the counter of visits and the temporary Q-ensembles $\tilde{Q}_h^{\text{b},j}$ according to the optimistic initialization V_h^0 (cf. Line 22 in Algorithm 1), when the number of visits in current stage exceeds a predefined threshold—specifically, when $n_h^{\text{b},t}(s_h^t, a_h^t) = \lfloor (1 + 1/H)^q H \rfloor$ for the q -th stage. Meanwhile, the staged value estimate \tilde{V}_h^{b} is reset conservatively, to help mitigate the potential bias introduced by stale data, as the value will be reused to update the temporary Q-values in the subsequent episode.

4 Theoretical Guarantee

In this section we provide both gap-independent and gap-dependent regret bounds for RANDOMIZEDQ, considering the worst-case scenario and favorable structural MDPs, respectively.

4.1 Gap-independent sublinear regret guarantee

To begin with, the following theorem shows that RANDOMIZEDQ has a \sqrt{T} -type regret bound, where the full proof is deferred to Appendix C.

Theorem 1. Consider $\delta \in (0, 1)$. Assume that $J = \lceil c \cdot \log(SAHT/\delta) \rceil$, $\kappa = c \cdot (\log(SAH/\delta) + \log(T))$, and $n_0 = \lceil c \cdot \log(T) \cdot \kappa \rceil$, where c is some universal constant. Let the initialized value function $V_h^0 = 2(H - h + 1)$ for any $h \in [H + 1]$. Then, with probability at least $1 - \delta$, Algorithm 1 guarantees that

$$\text{Regret}_T \leq \tilde{O} \left(\sqrt{H^5 SAT} \right).$$

Theorem 1 shows that RANDOMIZEDQ achieves a gap-independent regret bound of $\tilde{O} \left(\sqrt{H^5 SAT} \right)$, matching the guarantees of UCB-Q with Hoeffding-type bonuses (Jin et al., 2018) in episodic tabular MDPs. Moreover, this bound is minimax-optimal up to polynomial factors of H when compared to the known lower bound of $\Omega(\sqrt{H^3 SAT})$ (Jin et al., 2018; Domingues et al., 2021b).

Technical challenges. The primary challenge in analyzing RANDOMIZEDQ arises from several subtle requirements on the randomized learning rates. Specifically, these rates must:

- be sufficiently randomized to induce necessary optimism;
- avoid excessive randomness that could incur undesirable fluctuations;
- support efficient exploitation of the most recent observations to avoid introducing exponential dependence on the horizon H .

The subtle interplay among these conditions precludes the straightforward application of existing analytical techniques from the literature. For instance, the optimism may decay exponentially and be insufficient for sparse reward scenarios, so we re-inject the weighted optimistic values into the Q-ensembles at every stage

to ensure necessary optimism at every step. In addition, to bound undesirable fluctuations of randomized learning rates, prior work (Tiapkin et al., 2024) attempted to prove a concentration inequality based on Rosenthal’s inequality (Tiapkin et al., 2024, Theorem 6), which in turn requires a martingale property of the so-called *aggregated* learning rates. However, the martingale property in fact does not hold (detailed below), revealing a gap in their proof. We propose a new proof strategy to bridge this gap and to extend the concentration inequality to our setting. These challenges jointly necessitate a carefully constructed mixing scheme, refined control of fluctuation and favorable properties of learning rates to ensure that RANDOMIZEDQ attains near-optimal sample complexity with agile updates.

Identifying and fixing a technical gap in the proof of Tiapkin et al. (2024). While Tiapkin et al. (2024) established a comparable regret bound for Staged-RandQL, it turns out that analysis has a crucial technical gap. Specifically, central to the analysis is to study the concentration of the weighted sum of the *aggregated randomized learning rates*, defined as

$$W_{j,m}^0 = \prod_{k=0}^{m-1} (1 - w_k^j) \quad \text{and} \quad W_{j,m}^i = w_{i-1}^j \prod_{k=i}^{m-1} (1 - w_k^j), \quad \forall i \in [m],$$

which involves bounding the sum

$$\left| \sum_{i=0}^m \lambda_i (W_{j,m}^i - \mathbb{E}[W_{j,m}^i]) \right| \tag{8}$$

for fixed real numbers $\lambda_i \in [-1, 1]$, see, the proof of Lemma 4 in Tiapkin et al. (2024). To this end, Tiapkin et al. (2024) asserted that the partial sums $S_i = \sum_{k=0}^i \lambda_k (W_{j,m}^k - \mathbb{E}[W_{j,m}^k])$ form a martingale with respect to some filtration \mathcal{F}_i (cf. Proposition 7 there, which was invoked in the proof of Lemma 4 therein). The proof went on by a standard application of Rosenthal’s inequality (i.e., Theorem 6 therein). To prove the martingale property, it was claimed that $W_{j,m}^i$ is adapted to \mathcal{F}_i (cf. assumption of their Theorem 6), and $W_{j,m}^i$ is independent of \mathcal{F}_{i-1} (cf. assumption of their Proposition 7). Unfortunately, it is *impossible* to achieve both adaptedness and independence except for trivial cases (e.g., deterministic learning rates), regardless of choice of \mathcal{F}_i .² Indeed, if $\{\mathcal{F}_i\}$ is such a filtration, then $W_{j,m}^i$ being adapted means that all the randomness of $W_{j,m}^0, W_{j,m}^1, \dots, W_{j,m}^{i-1}$ is contained in \mathcal{F}_{i-1} . As $W_{j,m}^i$ is independent of \mathcal{F}_{i-1} , we see that $W_{j,m}^i$ is independent with all of $W_{j,m}^0, \dots, W_{j,m}^{i-1}$. By induction, we readily see that $W_{j,m}^0, \dots, W_{j,m}^m$ are jointly independent. However, since $\sum_{i=0}^m W_{j,m}^i = 1$ (Tiapkin et al., 2024, Lemma 3), such independence is not possible unless all aggregated learning rates $W_{j,m}^i, 0 \leq i \leq m$, are deterministic. Thus, Lemma 4 in Tiapkin et al. (2024) does not hold, thereby leaving a gap in the analysis. We fix this gap by introducing a reverse filtration that is tailored to the form of the aggregated weights, and study (8) using a backward martingale construction in contrast with partial sums with substantial new analyses. With this approach, we established the correct concentration inequality not only for their setting but also for ours.

Memory and computation complexity. As the number of ensembles is $J = \tilde{O}(1)$, the computational complexity is $O(H)$ per episode and the space complexity is $O(HSA)$, same as Tiapkin et al. (2024). However, we note that due to the use of optimistic mixing, RANDOMIZEDQ requires maintaining two ensembles, which effectively doubles the memory and computational cost compared to Staged-RandQL (Tiapkin et al., 2024).

4.2 Gap-dependent logarithmic regret guarantee

Note that such a \sqrt{T} -type regret bound holds for *any* episodic tabular MDPs, which might not be tight for environment with some benign structural properties. To this end, we further develop a gap-dependent regret bound, which improves the regret bound from sublinear to logarithmic under a strictly positive suboptimality gap condition, as follows.

²Tiapkin et al. (2024) chose a filtration $\{\mathcal{F}_i\}$ to which $W_{j,m}^i$ is actually not adapted.

Assumption 1 (Positive suboptimality gap). *For any $(s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]$, we denote the sub-optimality gap as $\Delta_h(s, a) := V_h^*(s) - Q_h^*(s, a)$ and assume that the minimal gap*

$$\Delta_{\min} \triangleq \min_{(s, a, h) \in \mathcal{S} \times \mathcal{A} \times [H]} \Delta_h(s, a) \mathbb{1}\{\Delta_h(s, a) \neq 0\} > 0.$$

Note that this assumption implies that there exist some strictly better actions (i.e., the optimal actions) outperform the others for every state. This assumption is mild, as the minimal suboptimality gap $\Delta_{\min} = 0$ only when the MDPs degenerates. Consequently, it is commonly fulfilled in environments with finite action spaces, such as Atari-games and control tasks, and it is also widely adopted in prior literature (Yang et al., 2021; Zheng et al., 2025).

Under this mild assumption, we have the following logarithmic gap-dependent regret bound, whose proof is deferred to Appendix D. To the best of our knowledge, this is the first guarantee that shows sampling-based Q-learning can also achieve the logarithmic regret in episodic tabular RL.

Theorem 2. *Consider $\delta \in (0, 1)$. Suppose all conditions in Theorem 1 and Assumption 1 hold. Then, with probability at least $1 - \delta$, Algorithm 1 guarantees that*

$$\mathbb{E}[\text{Regret}_T] \leq O\left(\frac{H^6 S A}{\Delta_{\min}} \log^5(S A H T)\right).$$

Note that the above logarithmic regret bound holds for RANDOMIZEDQ without assuming any prior knowledge on the minimal suboptimality gap Δ_{\min} during implementation. As shown in Simchowitz and Jamieson (2019), any algorithm with an $\Omega(\sqrt{T})$ regret bound in the worst case, has a $\log T$ -type lower bound of the expected gap-dependent regret. Also, our bound matches the expected regret for UCB-Q (Jin et al., 2018) under the same condition (i.e. Assumption 1) for episodic tabular MDPs (Yang et al., 2021), which is nearly tight in S, A, T up to the $\log(SAT)$ and H factors.

5 Experiments

In this section, we present the experimental results of RANDOMIZEDQ compared to baseline algorithms, using RLBERRY (Domingues et al., 2021a), in the following two environments. All the experiments are conducted on a machine equipped with 2 CPUs (Intel(R) Xeon(R) Gold 6244 CPU), running Red Hat Enterprise Linux 9.4, without GPU acceleration. The corresponding codes can be found at

<https://github.com/IrisHeWANG/RandomizedQ>

which is built upon the implementation of Tiapkin et al. (2024), using the RLBERRY library (Domingues et al., 2021a).

A grid-world environment. We first evaluate performance in a 10×10 grid-world environment as used in Tiapkin et al. (2024), where each state is represented as a tuple $(i, j) \in [10] \times [10]$, and the agent can choose from four actions: left, right, up, and down. The episode horizon is set to $H = 50$. At each step, the agent moves in the planned direction with probability $1 - \epsilon$ and to a random neighboring state with probability $\epsilon = 0.2$. The agent starts at position $(1, 1)$, and the reward is 1 only at state $(10, 10)$, with all other states yielding zero reward. We also examine the performance in a larger 20×20 grid-world environment, where the agent receives a reward of 1 only at state $(20, 20)$, with the episode horizon set to $H = 100$. Compared to the 10×10 setting, the reward is significantly sparser. The corresponding results are shown in Figures 1a and 1b, respectively.

A chain MDP. We also consider a chain MDP environment as described in Osband and Van Roy (2016), which consists of $L = 15$ states (i.e., the length of the chain) and two actions: left and right. The episode horizon is set to $H = 30$. With each action, the agent transits in the intended direction with probability 0.9, and in the opposite direction with probability 0.1. The agent starts in the leftmost state, which provides a reward of 0.05, while the rightmost state yields the highest reward of 1. Additionally, we evaluate performance in a longer chain MDP with $L = 30$ states and a horizon of $H = 50$. The corresponding results are shown in Figures 1c and 1d, respectively.

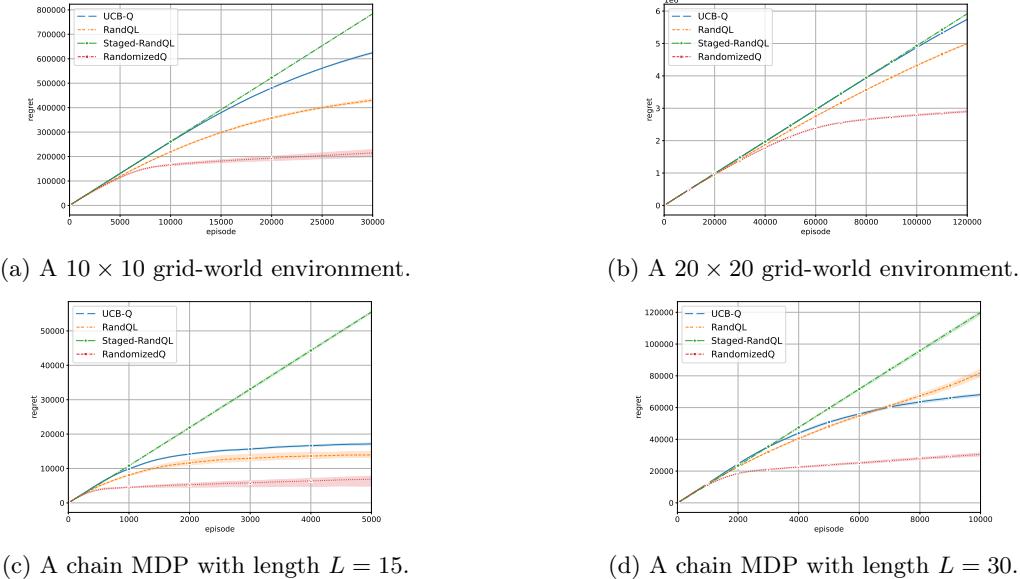


Figure 1: Comparison between RANDOMIZEDQ and baseline algorithms in the grid-world environment (cf. the first row) and the chain MDP (cf. the second row), where total regret is plotted against the number of episodes. RANDOMIZEDQ consistently achieves lower regret than UCB-Q, as well as both the standard randomized Q-learning (i.e., RandQL) and its stage-wise variant (i.e., Staged-RandQL), demonstrating superior sample efficiency and faster learning processes.

Baselines and experiment setups. We compare RANDOMIZEDQ with (1) UCB-Q: model-free Q-learning with bonuses (Jin et al., 2018) (2) Staged-RandQL (Tiapkin et al., 2024): the staged version of RandQL with theoretical guarantees (3) RandQL (Tiapkin et al., 2024): the randomized version of UCB-Q, without provable guarantees. For all algorithms with randomized learning rates in both environments, we let the number of ensembles $J = 20$, the inflation coefficient $\kappa = 1$, and the number of pseudo-transition $n_0 = 1/S$, where S corresponds to the size of the state space in different environments. For fair comparison, we repeat the experiments for 4 times and show the average along with the 90% confidence interval in the figures below.

Results. From Figure 1, RANDOMIZEDQ exhibits significantly improved performance in all the environments sizes, achieving substantially lower total regret. Unlike UCB-Q, which suffers from excessive over-exploration, and the Staged-RandQL that adapts the policy only at the end of each stage, RANDOMIZEDQ effectively balances exploration and exploitation through randomized learning rates and agile policy updates. We also observe that RANDOMIZEDQ performs even better than the empirical RandQL that lacks theoretical guarantees in prior work (Tiapkin et al., 2024), especially for larger environments with more sparse rewards. These results validate the effectiveness of sampling-driven updates without explicit bonus terms and highlight the benefit of avoiding stage-wise policy updates in model-free reinforcement learning.

6 Conclusion

In this work, we study the performance of Q-learning without exploration bonuses for episodic tabular MDPs in the online setting. We identify two key challenges in existing approaches: the additional statistical dependency introduced by randomizing learning rates, and the inefficiency of slow, stage-wise policy updates, as the bottlenecks of theoretical analysis and algorithm design. To address these challenges, we develop a novel randomized Q-learning algorithm with agile updates called RANDOMIZEDQ, which efficiently adapts the policy to newly observed data. Theoretically, we establish a sublinear worst-case and a logarithmic gap-dependent regret bounds. Empirically, our experiments show that RANDOMIZEDQ significantly outperforms than baseline algorithms in terms of total regret, due to the effective exploration and agile updates.

There are several promising directions for future research. For example, extending our analysis to function approximation settings—such as linear or neural representations—would significantly broaden the applicability of RANDOMIZEDQ (Melo and Ribeiro, 2007; Song and Sun, 2019). In addition, incorporating variance reduction techniques could further improve the regret bounds and potentially match the theoretical lower bounds (Zhang et al., 2020; Zheng et al., 2025).

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A Notation and Preliminary

Before proceeding, we first introduce the following notation with the dependency on the episode index t and its short-hand notation whenever it is clear from the context.

- $n_h^t(s, a)$, or the shorthand n_h^t : the number of previous visits to (s, a) at step h before episode t .
- $n_h^{b,t}(s, a)$, or the shorthand $n_h^{b,t}$: the number of previous visits to (s, a) at step h before current stage that the episode t belongs to.
- $q_h^t(s, a)$, or the shorthand q_h^t : the stage index of the i -th visit to (s, a) at step h and episode t .
- $\ell_h^i(s, a)$, or the shorthand ℓ^i : the episode index of the i -th visit to (s, a) at step h ; by convention $\ell^0 = 0$.
- $\ell_{q,h}^{b,i}(s, a)$, or the shorthand ℓ_q^i : the episode index of the i -th visit to (s, a) at step h during the stage q ; by convention, $\ell_0^0 = 0$ and $\ell_{q,h}^{b,0}(s, a)$ represents the episode when the q -th stage starts for (h, s, a) .
- $e_q = \lceil (1 + 1/H)^q H \rceil$: the length of the q -th stage; by convention, $e_{-1} = 0$.
- J : number of ensemble heads (temporary Q -functions) per episode.
- $\tilde{Q}_h^{j,t}(s, a)$: j -th temporary (ensemble) estimate of the optimal Q -value at the *beginning* of episode t , where the randomized learning rate follows $\text{Beta}(\frac{H+1}{\kappa}, \frac{n_h^t + n_0 - 1}{\kappa})$.
- $\tilde{Q}_h^{b,j,t}(s, a)$: j -th temporary (ensemble) estimate of the optimal Q -value at the *beginning* of episode t , where the randomized learning rate follows $\text{Beta}(\frac{1}{\kappa}, \frac{n_h^t - n_h^{b,t} + n_0 - 1}{\kappa})$.
- $\tilde{Q}_h^{b,t}(s, a)$: the optimistic approximation of the optimal Q -function updated at the end of each stage.
- $Q_h^t(s, a)$: the policy Q -function at the start of episode t ; its update at the visited pair (s_h^t, a_h^t) is

$$Q_h^t(s_h^t, a_h^t) = (1 - \frac{1}{H}) \max_{j \in [J]} \tilde{Q}_h^{j,t}(s_h^t, a_h^t) + \frac{1}{H} \tilde{Q}_h^{b,t}(s_h^t, a_h^t). \quad (9)$$

- $\tilde{V}_h^{\ell^i}, \tilde{V}_h^{\ell_q^i}$: optimistic value estimation of the optimal value function at episode ℓ^i and ℓ_q^i .

For analysis, we also introduce the following notation.

- s_0 : the optimistic pseudo-state s_0 with

$$r_h(s_0, a) = r_0(H+1) > 1, \quad p_h(s_0 | s, a) = \mathbf{1}\{s = s_0\}.$$

- $V_h^*(s_0)$: the cumulative return obtained by always staying at the optimistic state s_0 from step h , i.e., $V_h^*(s_0) = r_0(H+1)(H-h+1)$.

- n_0 : prior pseudo-transition count; thereby, each state-action pair (s, a) starts with n_0 prior pseudo-transitions, leading to

$$w_0^j \sim \text{Beta}((H+1)/\kappa, n_0/\kappa), \quad w_0^{b,j} \sim \text{Beta}(1/\kappa, n_0/\kappa), \quad j \in [J].$$

- $\mathcal{K}_{\inf}(p, \mu)$: the information-theoretic distance between some measure $p \in \mathcal{P}[0, b]$ and $\mu \in [0, b]$, defined as

$$\mathcal{K}_{\inf}(p, \mu) = \inf\{\text{KL}(p, q) : q \in \mathcal{P}[0, b], p \leq q, \mathbb{E}_{X \sim q}[X] \geq \mu\},$$

where $\mathcal{P}[0, b]$ denotes all probability measures supported on $[0, b]$.

- δ_x : Dirac measure concentrated at a single point x .
- $[n]$: the indexing shorthand; for a positive integer n , we write $[n] := \{1, 2, \dots, n\}$.
- $\|X\|_p$: the ℓ_p -norm of a vector $X \in \mathbb{R}^n$ where $p \geq 1$; formally defined as

$$\|X\|_p = \left(\sum_{i=1}^n |X_i|^p \right)^{1/p}.$$

- $(x)_k$: Pochhammer symbol, i.e., for $k \in \mathbb{N}$,

$$(x)_k = x(x+1) \cdots (x+k-1). \quad (10)$$

- $\mathbb{1}\{x \geq c\}$: an indicator function that equals 1 when $x \geq c$, and 0 otherwise.
- $|X|$: the cardinality of the set X .
- $A \lesssim B$: means $A \leq cB$ for some universal constant $c > 0$.

Beta distribution. We introduce the definition and important properties of Beta distribution ([Gupta and Nadarajah, 2004](#), Section 2), which is used in the follow-up analysis.

Definition 1 (Beta distribution). *A continuous random variable X is said to follow a Beta distribution with shape parameters $\alpha > 0$ and $\beta > 0$, written as*

$$X \sim \text{Beta}(\alpha, \beta),$$

if its probability density function is

$$f_X(x \mid \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1,$$

and $f_X(x) = 0$ otherwise, where the Beta function $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ serves as the normalizing constant.

Lemma 1 (Moments of the Beta distribution). *Let $X \sim \text{Beta}(a, b)$ with $a, b > 0$, and recall the Pochhammer symbol $(x)_r$ defined in (10). Then, for any positive integer r ,*

$$\mathbb{E}[X^r] = \frac{(a)_r}{(a+b)_r}. \quad (11)$$

In particular, the expectation and variance of X are

$$\mathbb{E}[X] = \frac{a}{a+b}, \quad \text{Var}(X) = \frac{ab}{(a+b)^2, (a+b+1)}. \quad (12)$$

We next collect a few auxiliary lemmas useful for the proof.

Lemma 2. For any $a, b \geq 1, \kappa \geq 0$, we have

$$\frac{a}{a+b} \leq \frac{a+\kappa}{a+b+\kappa} \leq (\kappa+1) \cdot \frac{a}{a+b}. \quad (13)$$

In addition, for any $r \in \mathbb{N}_+$,

$$\frac{a+r\kappa}{a+b+r\kappa} \leq r \cdot \frac{a+\kappa}{a+b+\kappa}. \quad (14)$$

Proof. The following proves (13) and (14), respectively. We start with the lower bound of (13). Define $f(t) = \frac{a+t}{a+b+t}$ for $t \geq 0$. Then $f'(t) = \frac{b}{(a+b+t)^2} > 0$, so f is increasing. Hence $f(0) = \frac{a}{a+b} \leq f(\kappa) = \frac{a+\kappa}{a+b+\kappa}$. Moving to the upper bound, it is equivalent to show

$$(a+\kappa)(a+b) \leq (\kappa+1)a(a+b+\kappa).$$

Expanding and rearranging leads to

$$(\kappa+1)a(a+b+\kappa) - (a+\kappa)(a+b) = \kappa(a^2 + ab + a\kappa - b) \geq 0,$$

which holds if $a, b \geq 1$ and $\kappa \geq 0$.

We now show (14). Let $r \in \mathbb{N}_+$ and

$$L = (a+r\kappa)(a+b+\kappa), \quad R = r(a+\kappa)(a+b+r\kappa).$$

We prove $L \leq R$ by first computing

$$\begin{aligned} R - L &= r(a+\kappa)(a+b+r\kappa) - (a+r\kappa)(a+b+\kappa) \\ &= (r-1) \left[a(a+b) + \kappa(a(r+1) + \kappa r) \right] \geq 0, \end{aligned}$$

because each bracketed term is non-negative and $r-1 \geq 0$. Dividing by the common positive factor $(a+b+r\kappa)(a+b+\kappa)$ yields $\frac{a+r\kappa}{a+b+r\kappa} \leq r \frac{a+\kappa}{a+b+\kappa}$. All statements are therefore proved. \square

Lemma 3 (Rosenthal inequality, Theorem 4.1 in [Pinelis \(1994\)](#)). Let X_1, \dots, X_n be a martingale-difference sequence adapted to a filtration $\{\mathcal{F}_i\}_{i=1,\dots,n}$:

$$\mathbb{E}[X_i | \mathcal{F}_{i-1}] = 0.$$

Define $\mathcal{V}_i = \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}]$. Then there exist universal constants c_1 and c_2 such that for any $p \geq 2$ the following holds

$$\mathbb{E}^{1/p} \left[\left| \sum_{i=1}^n X_i \right|^p \right] \leq C_1 p^{1/2} \mathbb{E}^{1/p} \left[\left| \sum_{i=1}^n \mathcal{V}_i \right|^{p/2} \right] + 2C_2 p \cdot \mathbb{E}^{1/p} \left[\max_{i \in [n]} |X_i|^p \right].$$

Lemma 4 (Corrected version of Lemma 12 in [Tiapkin et al. \(2024\)](#)).³ Let $\nu \in \mathcal{P}([0, b])$ be a probability measure over the segment $[0, b]$ and let $\bar{\nu} = (1-\alpha)\delta_{b_0} + \alpha \cdot \nu$ be a mixture between ν and a Dirac measure on $b_0 > b$, where $\alpha \in (0, 1)$. Then for any $\mu \in (0, b)$,

$$\mathcal{K}_{\inf}(\bar{\nu}, \mu) \leq \alpha \mathcal{K}_{\inf}(\nu, \mu). \quad (15)$$

Proof. From [Tiapkin et al. \(2024, Lemma 9\)](#), for any probability measure $\nu \in \mathcal{P}[0, b]$ and μ and any $\mu \in (0, B)$,

$$\mathcal{K}_{\inf}(\bar{\nu}, \mu) = \max_{\lambda \in [0, 1/(b_0 - \mu)]} \mathbb{E}_{X \sim \bar{\nu}} [\log(1 - \lambda(X - \mu))].$$

³We clarify an earlier oversight in [Tiapkin et al. \(2024, Lemma 12\)](#) by properly accounting for the Dirac measure's contribution, which was previously incorrectly separated after applying the variational formula—specifically, Lemma 9 in [Tiapkin et al. \(2024\)](#). Consequently, the right-hand side of (15) is now scaled by α , instead of the $1 - \alpha$ factor used in [Tiapkin et al. \(2024\)](#).

The support of $\bar{\nu}$ is contained in $[0, b_0]$, so for any $\lambda \in [0, 1/(b_0 - \mu)]$,

$$\mathbb{E}_{X \sim \bar{\nu}} [\log(1 - \lambda(X - \mu))] = (1 - \alpha) \log(1 - \lambda(b_0 - \mu)) + \alpha \mathbb{E}_{X \sim \nu} [\log(1 - \lambda(X - \mu))].$$

For every admissible λ we have $0 \leq \lambda(b_0 - \mu) \leq 1$, so $\log(1 - \lambda(b_0 - \mu)) \leq 0$. Hence

$$\mathcal{K}_{\inf}(\bar{\nu}, \mu) \leq \alpha \mathbb{E}_{X \sim \nu} [\log(1 - \lambda(X - \mu))].$$

Because $b_0 > b$, the interval $[0, 1/(b_0 - \mu)]$ is a subset of $[0, 1/(b - \mu)]$. Taking the maximum over the smaller interval leads to

$$\mathcal{K}_{\inf}(\bar{\nu}, \mu) \leq \alpha \max_{0 \leq \lambda \leq 1/(b - \mu)} \mathbb{E}_{X \sim \nu} [\log(1 - \lambda(X - \mu))] = \alpha \mathcal{K}_{\inf}(\nu, \mu).$$

□

B Reformulation of the Update Equation and Aggregated Weights

In this section, we rewrite the update of each temporary Q -function for every trajectory $t \in [T]$ and $j \in [J]$ by recursively unrolling the update (5) and (6), for each $(h, s, a) \in [H] \times S \times A$.

For the ease of notation, we denote $m := n_h^t(s, a)$.

Unrolled update for $\tilde{Q}_h^{j,t}$. For each $(j, t, h) \in [J] \times [T] \times [H]$, we can unroll (5) by

$$\begin{aligned} \tilde{Q}_h^{j,t}(s, a) &= \tilde{Q}_h^{j,\ell^m}(s, a) = (1 - w_{m-1}^j) \cdot \tilde{Q}_h^{j,\ell^{m-1}}(s, a) + w_{m-1}^j \left[r_h(s, a) + \tilde{V}_{h+1}^{\ell^m}(s_{h+1}^{\ell^m}) \right], \\ \tilde{Q}_h^{j,\ell^{m-1}}(s, a) &= (1 - w_{m-2}^j) \cdot \tilde{Q}_h^{j,\ell^{m-2}}(s, a) + w_{m-2}^j \left[r_h(s, a) + \tilde{V}_{h+1}^{\ell^{m-1}}(s_{h+1}^{\ell^{m-1}}) \right], \\ &\vdots \\ \tilde{Q}_h^{j,\ell^1}(s, a) &= (1 - w_0^j) \cdot \tilde{Q}_h^{j,\ell^0}(s, a) + w_0^j \left[r_h(s, a) + \tilde{V}_{h+1}^{\ell^1}(s_{h+1}^{\ell^1}) \right], \\ \tilde{Q}_h^{j,\ell^0}(s, a) &= r_h(s, a) + \tilde{V}_{h+1}^{\ell^0}(s_{h+1}^{\ell^0}), \end{aligned}$$

where we define $\tilde{V}_{h+1}^{\ell^0}(s) = \tilde{V}_{h+1}^0(s)$. For $m \geq 1$, let $W_{j,m} = (W_{j,m}^m, \dots, W_{j,m}^1, W_{j,m}^0)$ be the aggregated weights defined as

$$W_{j,m}^0 = \prod_{k=0}^{m-1} (1 - w_k^j) \quad \text{and} \quad W_{j,m}^i = w_{i-1}^j \prod_{k=i}^{m-1} (1 - w_k^j), \quad \forall i \in [m], \quad (16)$$

where w_k^j is sampled from Beta $(\frac{H+1}{\kappa}, \frac{k+n_0}{\kappa})$. Then, we have

$$\tilde{Q}_h^{j,t}(s, a) = r_h(s, a) + \sum_{i=0}^m W_{j,m}^i \left[\tilde{V}_{h+1}^{\ell^i}(s_{h+1}^{\ell^i}) \right]. \quad (17)$$

Unrolled update for $\tilde{Q}_h^{\flat,t}$. Suppose that q is the stage index of the episode t on (h, s, a) . We let e_q be the length of the q -th stage. Similar to the unrolling steps for $\tilde{Q}_h^{j,t}$, we define the corresponding aggregated weights as: For the q -th stage, let $W_{j,q}^{\flat} = (W_{j,e_q}^{\flat,0}, W_{j,e_q}^{\flat,1}, \dots, W_{j,e_q}^{\flat,e_q})$ be the aggregated weights at the defined as

$$W_{j,q}^{\flat,0} = \prod_{k=0}^{m-1} (1 - w_{k,q}^{\flat,j}) \quad \text{and} \quad W_{j,q}^{\flat,i} = w_{i-1}^{\flat,j} \prod_{k=i}^{m-1} (1 - w_{k,q}^{\flat,j}), \quad \forall i \in [e_q], \quad (18)$$

where $w_{k,q}^{\flat,j}$ is sampled from Beta $(1/\kappa, (k + n_0)/\kappa)$ during stage q .

For each $(t, h) \in [T] \times [H]$, as shown in (6), $\tilde{Q}_h^{\flat,t}(s, a)$ is updated via the most recent e_{q-1} samples before the q -th stage. Thus, for any $q \geq 1$, we have

$$\tilde{Q}_h^{\flat,t}(s, a) = r_h(s, a) + \max_{j \in [J]} \left\{ \sum_{i=0}^{e_{q-1}} W_{j,q-1}^{\flat,i} \left[\tilde{V}_{h+1}^{\flat,i}(s_{h+1}^{\flat,i}) \right] \right\}. \quad (19)$$

B.1 Properties of aggregated weights

In this section, we mainly discuss the properties of the aggregated weights $W_{j,m}$ for any $m \geq 1$.

To begin with, from (16), we can verify that the sum of the aggregated weights is equal to 1, i.e.,

$$\sum_{i=0}^m W_{j,m}^i = \prod_{k=0}^{m-1} (1 - w_k^j) + \sum_{i=1}^m w_{i-1}^j \prod_{k=i}^{m-1} (1 - w_k^j) = 1. \quad (20)$$

In the following proposition, we further show some desirable properties regarding the expectation and variance of the aggregated weights $W_{j,m}$.

Proposition 1. *The following properties hold for the aggregated weights $W_{j,m}$, $\forall j \in [J], m \geq 1$:*

(i) *The moment of the aggregated weights is given by:*

$$\mathbb{E}[(W_{j,m}^i)^d] = \left(\prod_{j=i+1}^m \frac{\left(\frac{n_0+j-1}{\kappa}\right)_d}{\left(\frac{H+n_0+j}{\kappa}\right)_d} \right) \cdot \frac{\left(\frac{H+1}{\kappa}\right)_d}{\left(\frac{H+n_0+i}{\kappa}\right)_d}; \quad (21)$$

(ii) *The upper bound of expectations and the sum of variances:*

$$\max_{i \in [m]} \mathbb{E}[W_{j,m}^i] \leq \frac{H+1}{H+n_0+m}, \quad \sum_{i=1}^m \text{Var}[W_{j,m}^i] \leq \frac{(H+1)\kappa}{H+n_0+m}; \quad (22)$$

(iii) *For every $i \geq 1$, we have*

$$\sum_{t=i}^{\infty} \mathbb{E}[W_t^i] \leq 1 + \frac{1}{H}. \quad (23)$$

Proof. We prove each item separately.

(i) Directly follows from Wong (1998, Section 2).

(ii) Similar to Jin et al. (2018, Lemma 4.1), we have that for $i \in [m]$

$$\begin{aligned} \mathbb{E}[W_{j,m}^i] &= \frac{H+1}{H+n_0+i} \left(\frac{n_0+i}{H+n_0+i+1} \frac{n_0+i+1}{H+n_0+i+2} \cdots \frac{n_0+m-1}{H+n_0+m} \right) \\ &= \frac{H+1}{H+n_0+m} \left(\frac{n_0+i}{H+n_0+i} \frac{n_0+i+1}{H+n_0+i+1} \cdots \frac{n_0+m-1}{H+n_0+m-1} \right) \\ &\leq \frac{H+1}{H+n_0+m}. \end{aligned}$$

Thus, $\max_{i \in [m]} \mathbb{E}[W_{j,m}^i] \leq \frac{H+1}{H+n_0+m}$ holds. From Lemma 2 and (21),

$$\begin{aligned} \text{Var}[W_{j,m}^i] &= \mathbb{E}[(W_{j,m}^i)^2] - \mathbb{E}[W_{j,m}^i]^2 \\ &= \mathbb{E}[W_{j,m}^i] \left(\left(\prod_{j=i+1}^m \frac{n_0+j-1+\kappa}{H+n_0+j+\kappa} \right) \cdot \frac{H+1+\kappa}{H+n_0+i+\kappa} - \mathbb{E}[W_{j,m}^i] \right) \\ &\leq \mathbb{E}[W_{j,m}^i] \left(\frac{H+1+\kappa}{H+n_0+m+\kappa} - \mathbb{E}[W_{j,m}^i] \right) \\ &\leq \kappa \mathbb{E}[W_{j,m}^i] \cdot \frac{H+1}{H+n_0+m}, \end{aligned}$$

and thus

$$\sum_{i=1}^m \text{Var}[W_{j,m}^i] \leq \kappa \cdot \frac{H+1}{H+n_0+m} \cdot \left(\sum_{i=1}^m \mathbb{E}[W_{j,m}^i] \right) \leq \frac{(H+1)\kappa}{H+n_0+m},$$

where the last inequality used $\sum_{i=1}^m \mathbb{E}[W_{j,m}^i] \leq \sum_{i=0}^m \mathbb{E}[W_{j,m}^i] = 1$ (recalling (20)).

(iii) Following Jin et al. (2018, equation (B.1)), it also holds for any positive integer n, k and $n \geq k$

$$\frac{n+n_0}{k} = 1 + \frac{n+n_0-k}{n+n_0+1} + \frac{(n+n_0-k)(n+n_0-k+1)}{(n+n_0+1)(n+n_0+2)} + \dots \quad (24)$$

which can be verified by induction. Specifically, we let the terms of the right-hand side be $x_0 = 1$, $x_1 = \frac{n+n_0-k}{n+n_0+1}$, \dots . Then, we will show $\frac{n+n_0}{k} - \sum_{j=0}^i x_j = \frac{n+n_0-k}{k} \prod_{j=1}^i \frac{n+n_0-k+j}{n+n_0+j}$ by induction.

- Base case when $i = 1$: It can easily verified that

$$\frac{n+n_0}{k} - 1 - \frac{n+n_0-k}{n+n_0+1} = \frac{(n+n_0-k)(n+n_0+1-k)}{k(n+n_0+1)}$$

- Suppose $i = r$, the claim holds, i.e., $\frac{n+n_0}{k} - \sum_{j=0}^r x_j = \frac{n+n_0-k}{k} \prod_{j=1}^r \frac{n+n_0-k+j}{n+n_0+j}$. Then, for $i = r+1$, we have

$$\begin{aligned} \frac{n+n_0}{k} - \sum_{j=0}^r x_j - x_{r+1} &= \frac{n+n_0-k}{k} \prod_{j=1}^r \frac{n+n_0-k+j}{n+n_0+j} - \prod_{j=1}^{r+1} \frac{n+n_0-k+j-1}{n+n_0+j} \\ &= \frac{n+n_0-k}{k} \prod_{j=1}^r \frac{n+n_0-k+j}{n+n_0+j} \left[1 - \frac{k}{n+n_0-k} \cdot \frac{n+n_0-k}{n+n_0+r+1} \right] \\ &= \frac{n+n_0-k}{k} \prod_{j=1}^r \frac{n+n_0-k+j}{n+n_0+j} \frac{n+n_0-k+r+1}{n+n_0+r+1} \\ &= \frac{n+n_0-k}{k} \prod_{j=1}^{r+1} \frac{n+n_0-k+j}{n+n_0+j}. \end{aligned}$$

By letting $i \rightarrow \infty$, we obtain (24). Thus, we have

$$\begin{aligned} \sum_{t=i}^{\infty} \mathbb{E}[W_{j,t}^i] &= \frac{H+1}{H+n_0+i} \left(1 + \frac{i+n_0}{H+n_0+i+1} + \frac{i+n_0}{H+n_0+i+1} \frac{i+n_0+1}{H+n_0+i+2} + \dots \right) \\ &= \frac{H+1}{H+n_0+i} \frac{H+n_0+i}{H} = 1 + \frac{1}{H}, \end{aligned}$$

where the second equality uses (24) with $n = i + H$ and $k = H$.

□

B.2 Concentration of aggregated weights

For notational convenience and clearness, we slightly abuse the notation by ignoring the dependency on the ensemble index j , while the following concentration lemma of the aggregated weights holds for any $j \in [J]$.

Lemma 5. Let $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_m$ be nonnegative real numbers such that $\lambda_i \leq 1$, $i = 1, 2, \dots, m$. Then, with probability at least $1 - \frac{\delta}{T}$, we have

$$\left| \sum_{i=0}^m \lambda_i (W_m^i - \mathbb{E}[W_m^i]) \right| \leq c_1 \sqrt{\frac{(H+1)\kappa^2 \log^3(T/\delta)}{H+n_0+m}} + c_2 \frac{(H+1)\kappa \log^2(T/\delta)}{H+n_0+m},$$

where c_1 and c_2 are some positive universal constants.

Proof. By definition, we can find independent random variables w_0, \dots, w_{m-1} such that $w_i \sim \text{Beta}(\frac{H+1}{\kappa}, \frac{n_0+i}{\kappa})$, and

$$W_m^0 = \prod_{k=1}^{m-1} (1 - w_k), \quad W_m^i = w_{i-1} \prod_{k=i}^{m-1} (1 - w_k), \quad 1 \leq i \leq m. \quad (25)$$

Let \mathcal{F}_{-i} be the σ -algebra generated by $w_{m-1}, w_{m-2}, \dots, w_{m-i}$, with the convention that $\mathcal{F}_{-0} = \mathcal{F}_0$ is the trivial σ -algebra. In the same spirit, denote

$$S_{-i} = \sum_{k=m-i+1}^m \lambda_k (W_m^k - \mathbb{E}[W_m^k]), \quad i = 0, 1, \dots, m. \quad (26)$$

For conceptual reason, we will use the notation $S_{-\infty}$ to denote the sum $\sum_{k=0}^m \lambda_k (W_m^k - \mathbb{E}[W_m^k])$, which corresponds to $S_{-(m+1)}$ in the above notation.

We view $\mathcal{F}_0 \subset \mathcal{F}_{-1} \subset \dots \subset \mathcal{F}_{-m}$ as a reverse filtration, and consider the backward martingale

$$M_{-i} := \mathbb{E}[S_{-\infty} | \mathcal{F}_{-i}], \quad i = 0, 1, \dots, m. \quad (27)$$

It is clear that $M_0 = \mathbb{E}[S_{-\infty}]$, while $M_{-m} = S_{-\infty}$. Therefore

$$\sum_{i=0}^m \lambda_i (W_m^i - \mathbb{E}[W_m^i]) = S_{-\infty} - \mathbb{E}[S_{-\infty}] = M_{-m} - M_0.$$

We may then apply the Rosenthal's inequality (i.e., Lemma 3) to obtain

$$\begin{aligned} (\mathbb{E}[|M_{-m} - M_0|^p])^{1/p} &\leq C\sqrt{p} \left(\mathbb{E}[\langle M \rangle_{-m}^{p/2}] \right)^{1/p} + Cp \left(\mathbb{E}[\max_{1 \leq i \leq m} |M_{-i} - M_{-(i-1)}|^p] \right)^{1/p} \\ &\leq C\sqrt{p} \left(\mathbb{E}[\langle M \rangle_{-m}^{p/2}] \right)^{1/p} + Cp \left(\sum_{i=1}^m \mathbb{E}[|M_{-i} - M_{-(i-1)}|^p] \right)^{1/p}, \end{aligned} \quad (28)$$

where

$$\langle M \rangle_{-m} := \sum_{i=1}^m \mathbb{E}[(M_{-i} - M_{-(i-1)})^2 | \mathcal{F}_{-(i-1)}].$$

To proceed, we calculate the martingale difference $M_{-i} - M_{-(i-1)}$ for any $i \in [m]$. To simplify the resulting expressions, we will denote

$$T_{-i} := \lambda_0 \prod_{j=0}^{m-i-1} (1 - w_j) + \sum_{k=1}^{m-i} \left(\lambda_k w_{k-1} \prod_{j=k}^{m-i-1} (1 - w_j) \right).$$

With this notation in hand, we can decompose $S_{-\infty}$ by

$$\begin{aligned} S_{-\infty} &= \lambda_{m-i+1} (W_m^{m-i+1} - \mathbb{E}[W_m^{m-i+1}]) + \sum_{k=0}^{m-i} \lambda_k (W_m^k - \mathbb{E}[W_m^k]) + \sum_{k=m-i+2}^m \lambda_k (W_m^k - \mathbb{E}[W_m^k]) \\ &= \lambda_{m-i+1} \left(w_{m-i} \prod_{k=m-(i-1)}^{m-1} (1 - w_k) - \mathbb{E} \left[w_{m-i} \prod_{k=m-(i-1)}^{m-1} (1 - w_k) \right] \right) \\ &\quad + \left(T_{-i} \prod_{k=m-i}^{m-1} (1 - w_k) - \mathbb{E} \left[T_{-i} \prod_{k=m-i}^{m-1} (1 - w_k) \right] \right) \\ &\quad + \sum_{k=m-i+2}^m \lambda_k (W_m^k - \mathbb{E}[W_m^k]), \end{aligned}$$

where only the first two terms involve the observation $w_{m-i} = \mathcal{F}_{-i} \setminus \mathcal{F}_{-(i-1)}$. Then for $i \in [m]$,

$$\begin{aligned} M_{-i} - M_{-(i-1)} &= \mathbb{E}[S_{-\infty} | \mathcal{F}_{-i}] - \mathbb{E}[S_{-\infty} | \mathcal{F}_{-(i-1)}] \\ &= \lambda_{m-i+1} (w_{m-i} - \mathbb{E}[w_{m-i}]) \prod_{k=m-(i-1)}^{m-1} (1 - w_k) \end{aligned} \quad (29)$$

$$\begin{aligned}
& + \mathbb{E}[T_{-i}] \prod_{k=m-(i-1)}^{m-1} (1-w_k) ((1-w_{m-i}) - \mathbb{E}[1-w_{m-i}]) \\
& = (\lambda_{m-i+1} - \mathbb{E}[T_{-i}]) \cdot (w_{m-i} - \mathbb{E}[w_{m-i}]) \prod_{k=m-(i-1)}^{m-1} (1-w_k),
\end{aligned} \tag{30}$$

where the last line is from $\mathbb{E}[1-w_{m-i}] = 1 - \mathbb{E}[w_{m-i}]$.

By our assumption $|\lambda_i| \leq 1$, $i = 1, \dots, m$, it can be readily checked that

$$|T_{-i}| \leq \prod_{j=0}^{m-i-1} (1-w_j) + \sum_{k=1}^{m-i} \left(w_{k-1} \prod_{j=k}^{m-i-1} (1-w_j) \right) = 1, \quad \forall i = \{1, \dots, m\}.$$

Consequently, the absolute value of the martingale difference can be bounded above by

$$|M_{-i} - M_{-(i-1)}| \leq 2|w_{m-i} - \mathbb{E}[w_{m-i}]| \prod_{k=m-(i-1)}^{m-1} (1-w_k). \tag{31}$$

Recall that $\langle M \rangle_{-m} := \sum_{i=1}^m \mathbb{E}[(M_{-i} - M_{-(i-1)})^2 | \mathcal{F}_{-(i-1)}]$. Together with (31), we have

$$\langle M \rangle_{-m} \leq 2 \sum_{i=1}^m \mathbb{E} \left(|w_{m-i} - \mathbb{E}[w_{m-i}]| \prod_{k=m-(i-1)}^{m-1} (1-w_k) \mid \mathcal{F}_{-(i-1)} \right)^2 \tag{32}$$

$$\begin{aligned}
& = 2 \sum_{i=1}^m \text{Var}[w_{m-i}] \prod_{k=m-(i-1)}^{m-1} (1-w_k)^2 \\
& = 2 \sum_{i=0}^{m-1} \text{Var}[w_i] \prod_{k=i+1}^{m-1} (1-w_k)^2 \\
& \leq 2 \sum_{i=0}^{m-1} \frac{\kappa(H+1)(n_0+i)}{(H+1+n_0+i)^2(H+1+n_0+i+\kappa)} \prod_{k=i+1}^{m-1} (1-w_k)^2,
\end{aligned} \tag{33}$$

where the second equality is due to the change of index, and the last inequality follows from (12). We may then apply triangle inequality to obtain

$$\begin{aligned}
\mathbb{E}^{2/p} [\langle M \rangle_{-m}^{p/2}] & \leq \sum_{i=0}^{m-1} 2 \frac{\kappa(H+1)(n_0+i)}{(H+1+n_0+i)^2(H+1+n_0+i+\kappa)} \mathbb{E}^{2/p} \left[\left(\prod_{k=i+1}^{m-1} (1-w_k)^2 \right)^{p/2} \right] \\
& = 2 \sum_{i=0}^{m-1} \frac{\kappa(H+1)(n_0+i)}{(H+1+n_0+i)^2(H+1+n_0+i+\kappa)} \left(\prod_{k=i+1}^{m-1} \mathbb{E}[(1-w_k)^p] \right)^{2/p},
\end{aligned}$$

for $p \geq 2$. Note that $1-w_k \sim \text{Beta}(\frac{n_0+k}{\kappa}, \frac{H+1}{\kappa})$, directly from the definition 1. Thus, By (11), we have

$$\mathbb{E} \prod_{k=i+1}^{m-1} (1-w_k)^p = \prod_{k=i+2}^m \frac{(\frac{n_0+k-1}{\kappa})_p}{(\frac{H+n_0+k}{\kappa})_p} \leq \prod_{r=0}^{p-1} \frac{n_0+i+1+r\kappa}{H+n_0+m+r\kappa} \leq p! \left(\frac{n_0+i+1+\kappa}{H+n_0+m+\kappa} \right)^p,$$

where the last inequality is from Lemma 2. Therefore, we can obtain

$$\begin{aligned}
\left(\mathbb{E}[\langle M \rangle_{-m}^{p/2}] \right)^{2/p} & \leq 2 \sum_{i=0}^{m-1} \frac{\kappa(H+1)(n_0+i)}{(H+1+n_0+i)^2(H+1+n_0+i+\kappa)} (p!)^{2/p} \left(\frac{n_0+i+1+\kappa}{H+n_0+m+\kappa} \right)^2 \\
& \leq 2\kappa p^2 \frac{H+1+\kappa}{(H+n_0+m+\kappa)^2} \sum_{i=0}^{m-1} \frac{n_0+i+\kappa}{H+1+n_0+i+\kappa}
\end{aligned}$$

$$\leq 2\kappa p^2 \frac{H+1+\kappa}{H+n_0+m+\kappa} \leq 2(\kappa+1)^2 p^2 \frac{H+1}{H+n_0+m}$$

where the first and last inequality are due to Lemma 2 (i.e., (13)), the second inequality uses the facts that $p! \leq p^p$ and $\frac{n_0+i+1+\kappa}{H+1+n_0+i+\kappa} \leq 1$ for every $i = 0, \dots, m-1$, and the third inequality is from the fact that the summation term is less than m .

Therefore,

$$\left(\mathbb{E}[\langle M \rangle_{-m}^{p/2}] \right)^{1/p} \leq 2p(\kappa+1) \sqrt{\frac{(H+1)}{H+n_0+m}}. \quad (34)$$

We turn to bound $\mathbb{E}[|M_{-i} - M_{-(i-1)}|^p]$ for $i \in [m]$. It is clear from (31) that

$$\begin{aligned} \mathbb{E}[|M_{-i} - M_{-(i-1)}|^p] &\leq \mathbb{E} \left[\left| 2|w_{m-i} - \mathbb{E}[w_{m-i}]| \prod_{k=m-(i-1)}^{m-1} (1-w_k) \right|^p \right] \\ &\leq 8^p \mathbb{E}[w_{m-i}^p] \prod_{k=m-(i-1)}^{m-1} \mathbb{E}[(1-w_k)^p] \\ &= 8^p \mathbb{E}[(W_m^{m-i+1})^p] \end{aligned}$$

where the second inequality uses $|w_{m-i} - \mathbb{E}[w_{m-i}]| \leq 2^{p+1} w_{m-i}^p$, and the equality is from (25) where $W_m^{m-i+1} = w_{m-i} \prod_{k=m-(i-1)}^{m-1} (1-w_k)$ for $i \in [m]$.

From Proposition 1, we have that for any $i \in [m]$,

$$\begin{aligned} \mathbb{E}[(W_m^i)^p] &= \left(\prod_{j=i+1}^m \frac{(\frac{n_0+j-1}{\kappa})_p}{(\frac{H+n_0+j}{\kappa})_p} \right) \cdot \frac{(\frac{H+1}{\kappa})_p}{(\frac{H+n_0+i}{\kappa})_p} \\ &\leq \prod_{r=0}^{p-1} \left(\frac{H+1+r\kappa}{H+n_0+i+r\kappa} \cdot \prod_{j=i+1}^m \frac{n_0+j-1+r\kappa}{H+n_0+j+r\kappa} \right) \\ &\leq \prod_{r=0}^{p-1} \frac{r\kappa(H+1)}{H+n_0+m} \leq p! \left(\frac{(H+1)\kappa}{H+n_0+m} \right)^p \end{aligned}$$

Thus,

$$\left(\sum_{i=1}^m \mathbb{E}[|M_{-i} - M_{-(i-1)}|^p] \right)^{1/p} \leq \left(8^p \sum_{i=1}^m \mathbb{E}[(W_m^{m-i+1})^p] \right)^{1/p} \leq 8m^{1/p} p \frac{(H+1)\kappa}{H+n_0+m}. \quad (35)$$

Substituting (34) and (35) into (28) leads to

$$\mathbb{E}^{1/p}[|M_{-m} - M_0|^p] \leq 2C \sqrt{\frac{p^3(\kappa+1)^2(H+1)}{H+n_0+m}} + 8C \cdot m^{1/p} p^2 \frac{(H+1)\kappa}{H+n_0+m}.$$

Note that by definition, $m = n_h^t(s, a) \leq T$ always holds for any $(h, t, s, a) \in [H] \times [T] \times \mathcal{S} \times \mathcal{A}$. Let $p = \lceil \log(T/\delta) \rceil \geq 2$ and thus $m^{1/p} \leq e$ since $m^{1/p} \leq e^{(\log T)/p}$. Finally, by Markov inequality with $t = 2eC\sqrt{\frac{(H+1)(\kappa+1)^2 \log^3(T/\delta)}{H+n_0+m}} + 8e^2C \frac{(H+1)\kappa \log^2(T/\delta)}{H+n_0+m}$, we have

$$\mathbb{P}\left[\left| \sum_{i=0}^m \lambda_i (W_m^i - \mathbb{E}[W_m^i]) \right| \geq t \right] \leq \left(\frac{\mathbb{E}^{1/p} \left[\left| \sum_{i=0}^m \lambda_i (W_m^i - \mathbb{E}[W_m^i]) \right|^p \right]}{t} \right)^p \leq \left(\frac{1}{e} \right)^{\lceil \log(T/\delta) \rceil} \leq \frac{\delta}{T},$$

which finishes the proof. \square

In addition, we also have the following lemma regarding the concentration of W_q^b for any stage $q \geq 0$. We omit its proof since it is similar to Lemma 5.

Lemma 6. Let $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{e_q}$ be nonnegative real numbers such that $\lambda_i \leq 1$, $i = 1, 2, \dots, e_q$. Then, with probability at least $1 - \delta/T$, we have

$$\left| \sum_{i=0}^{e_q} \lambda_i \left(W_q^{\flat, i} - \mathbb{E}[W_q^{\flat, i}] \right) \right| \leq c_1^{\flat} \sqrt{\frac{\kappa^2 \log^3(T/\delta)}{n_0 + e_q}} + c_2^{\flat} \frac{\kappa \log^2(T/\delta)}{n_0 + e_q},$$

where c_1^{\flat} and c_2^{\flat} are some positive universal constants.

C Analysis: Gap-independent Regret Bound

In this section, we present the detailed proof of Theorem 1. Before proceeding, we first rewrite the update of policy Q-function by unrolling the updates of temporary Q-functions (i.e., equations (17) and (19)) as

$$\forall (h, t) \in [H] \times [T] : Q_h^t(s_h^t, a_h^t) = \left(1 - \frac{1}{H}\right) \max_{j \in [J]} \left\{ \tilde{Q}_h^{j,t}(s_h^t, a_h^t) \right\} + \frac{1}{H} \cdot \tilde{Q}_h^{\flat,t}(s_h^t, a_h^t) \quad (36)$$

$$= r_h(s_h^t, a_h^t) + \left(1 - \frac{1}{H}\right) \max_{j \in [J]} \left\{ \sum_{i=0}^{n_h^t} W_{j,n_h^t}^i \tilde{V}_{h+1}^{\ell^i}(s_{h+1}^{\ell^i}) \right\} \\ + \frac{1}{H} \cdot \max_{j \in [J]} \left\{ \sum_{i=0}^{e_{q-1}} W_{j,q-1}^{\flat,i} \tilde{V}_{h+1}^{\flat,\ell_{q-1}^i}(s_{h+1}^{\ell_{q-1}^i}) \right\}, \quad (37)$$

where in the notations we omit the dependency of q regarding the step h and the episode t for simplicity. The properties and the concentration inequality of the aggregated weights (i.e., W_{j,n_h^t} and $W_{j,q}^{\flat}$), proved in Appendix B.1 and B.2, will play a crucial role in the following analysis.

C.1 Proof of Theorem 1

To control the total regret, we first present the following lemma regarding the optimism of the policy Q-function Q_h^t for any $(t, h, s, a) \in [T] \times [H] \times \mathcal{S} \times \mathcal{A}$, where the detailed proof is postponed to Appendix C.2.

Lemma 7 (Optimism). Consider $\delta \in (0, 1)$. Assume that $J = \lceil c \cdot \log(SAHT/\delta) \rceil$, $\kappa = c \cdot (\log(SAH/\delta) + \log(T))$, and $n_0 = \lceil c \cdot \log(T) \cdot \kappa \rceil$, where c is some universal constant. Let $V_h^0 = 2(H-h+1)$. Then, for any $(t, h, s, a) \in [T] \times [H] \times \mathcal{S} \times \mathcal{A}$, with probability at least $1 - \delta$, the following event holds

$$Q_h^t(s, a) \leq Q_h^t(s, a).$$

Define

$$V_h^t(s_h^t) := Q_h^t(s_h^t, \pi_h^t(s_h^t)) = \max_{a \in \mathcal{A}} Q_h^t(s_h^t, a). \quad (38)$$

Note that Lemma 7 implies that

$$V_1^*(s_1^t) = \max_a Q_1^*(s_1^t, a) \leq \max_a Q_1^t(s_1^t, a) = Q_1^t(s_1^t, a_1^t) = V_1^t(s_1^t). \quad (39)$$

Thus, we can decompose the total regret as

$$\text{Regret}_T = \sum_{t=1}^T (V_1^* - V_1^{\pi^t})(s_1^t) \leq \sum_{t=1}^T (V_1^t - V_1^{\pi^t})(s_1^t) = \sum_{t=1}^T \delta_1^t,$$

where we denote the performance gap as $\delta_h^t = (V_h^t - V_h^{\pi^t})(s_h^t)$ for every $(t, h) \in [T] \times [H]$.

Note that $Q_h^{\pi^t}(s_h^t, a_h^t) = V_h^{\pi^t}(s_h^t)$. Then, for any fixed episode $t \in [T]$ and step $h \in [H]$, we decompose the performance gap δ_h^t :

$$\delta_h^t \leq (Q_h^t - Q_h^*)(s_h^t, a_h^t) + (Q_h^* - Q_h^{\pi^t})(s_h^t, a_h^t)$$

$$\begin{aligned}
&\leq (Q_h^t - Q_h^\star)(s_h^t, a_h^t) + P_{h, s_h^t, a_h^t} \left(V_{h+1}^\star - V_{h+1}^{\pi^t} \right) \\
&\leq (Q_h^t - Q_h^\star)(s_h^t, a_h^t) + \underbrace{(V_{h+1}^t - V_{h+1}^{\pi^t})(s_{h+1}^t)}_{=: \delta_{h+1}^t} - \underbrace{(V_{h+1}^t - V_{h+1}^\star)(s_{h+1}^t)}_{=: \xi_{h+1}^t} \\
&\quad + \underbrace{P_{h, s_h^t, a_h^t} \left(V_{h+1}^\star - V_{h+1}^{\pi^t} \right) - \left(V_{h+1}^\star - V_{h+1}^{\pi^t} \right) (s_{h+1}^t)}_{=: \tau_{h+1}^t},
\end{aligned}$$

where the second inequality is from the Bellman equations (2) and (3). Together with the update (36),

$$\begin{aligned}
\delta_h^t &\leq \left(1 - \frac{1}{H} \right) \underbrace{\max_{j \in [J]} \left\{ \tilde{Q}_h^{j,t}(s_h^t, a_h^t) - Q_h^\star(s_h^t, a_h^t) \right\}}_{=: \zeta_h^t} + \frac{1}{H} \underbrace{\left\{ \tilde{Q}_h^{\flat,t}(s_h^t, a_h^t) - Q_h^\star(s_h^t, a_h^t) \right\}}_{=: \zeta_h^{\flat,t}} \\
&\quad + \delta_{h+1}^t - \xi_{h+1}^t + \tau_{h+1}^t.
\end{aligned} \tag{40}$$

The main idea of the proof is to show that the performance gap δ_h^t can be upper bounded by some quantities from the next step $h+1$, and correspondingly, the total regret can be controlled by rolling out the performance gap over all episodes and steps.

Next, the following lemmas present a recursive bound for ζ_h^t and $\zeta_h^{\flat,t}$, respectively. The proofs are deferred to Appendix C.3 and C.4.

Lemma 8 (Recursive bound for ζ_h^t). *Consider $\delta \in (0, 1)$. For any $i \in [m]$, let $\alpha_m^0 = \prod_{k=1}^m \frac{n_0+k-1}{H+n_0+k}$ and $\alpha_m^i := \frac{H+1}{H+n_0+i} \prod_{k=i+1}^m \frac{n_0+k-1}{H+n_0+k}$, where $m = n_h^t(s_h^t, a_h^t)$. Then, for any $(t, h, s, a) \in [T] \times [H] \times \mathcal{S} \times \mathcal{A}$, with probability $1 - \delta$, we have*

$$\zeta_h^t \leq \alpha_m^0 V_{h+1}^0 + \sum_{i=1}^m \alpha_m^i \left(\tilde{V}_{h+1}^{\ell^i} - V_{h+1}^\star \right) (s_{h+1}^{\ell^i}) + \tilde{b}_h^t, \tag{41}$$

where $\tilde{b}_h^t = \tilde{O} \left(\sqrt{\frac{(H+1)^3}{H+n_0+m}} + \frac{(H+1)^2}{H+n_0+m} \right)$.

Lemma 9 (Recursive bound for $\zeta_h^{\flat,t}$). *Consider $\delta \in (0, 1)$. For any $(t, h, s, a) \in [T] \times [H] \times \mathcal{S} \times \mathcal{A}$, we let $q = q_h^t(s, a)$ represent the index of the current stage and e_{q-1} be the length of the prior stage. Then for any $t \in [T]$, with probability at least $1 - \delta$, we have*

$$\zeta_h^{\flat,t} \leq \left(\sum_{i=1}^{e_{q-1}} \frac{1}{e_{q-1}} \left(\tilde{V}_{h+1}^{\flat, \ell_{q-1}^i} - V_{h+1}^\star \right) (s_{h+1}^{\ell_{q-1}^i}) + \tilde{b}_h^{\flat,t} \right) \cdot \mathbb{1}(q_h^t \geq 1) + \left(\frac{1/H+1}{1/H} \right) \cdot V_{h+1}^0 \cdot \mathbb{1}(q_h^t = 0), \tag{42}$$

where $\tilde{b}_h^{\flat,t} = \tilde{O} \left(\sqrt{\frac{(H+1)^4}{e_{q-1}}} + \frac{(H+1)^2}{e_{q-1}} \right)$.

We denote $\tilde{\xi}_h^t = (\tilde{V}_h^t - V_h^\star)(s_h^t)$ and $\tilde{\xi}_h^{\flat,t} = (\tilde{V}_h^{\flat,t} - V_h^\star)(s_h^t)$. Combining Lemma 8 and Lemma 9 with (40), we have that for any $(h, t, s, a) \in [H] \times [T] \times \mathcal{S} \times \mathcal{A}$ within the stage $q_h^t \geq 1$,

$$\delta_h^t \leq \alpha_{n_h^t}^0 V_{h+1}^0 + \left(1 - \frac{1}{H} \right) \sum_{i=1}^{n_h^t} \alpha_{n_h^t}^i \tilde{\xi}_{h+1}^{\ell^i} + \frac{1}{e_{q-1} H} \sum_{i=1}^{e_{q-1}} \tilde{\xi}_{h+1}^{\flat, \ell_{q-1}^i} + \tilde{b}_h^t + \frac{\tilde{b}_h^{\flat,t}}{H} + (\delta_{h+1}^t - \tilde{\xi}_{h+1}^t + \tau_{h+1}^t),$$

and during the initial stage $q_h^t = 0$

$$\delta_h^t \leq \alpha_{n_h^t}^0 V_{h+1}^0 + \left(1 - \frac{1}{H} \right) \sum_{i=1}^{n_h^t} \alpha_{n_h^t}^i \tilde{\xi}_{h+1}^{\ell^i} + \tilde{b}_h^t + \left(1 + \frac{1}{H} \right) \cdot V_{h+1}^0 + (\delta_{h+1}^t - \tilde{\xi}_{h+1}^t + \tau_{h+1}^t).$$

Note that any episode $t \in [T]$ during the initial stage $q_h^t = 0$, satisfies $1/H \leq 1/n_h^t$. Define $b_h^t = \tilde{b}_h^t + \mathbb{1}\{q_h^t \geq 1\} \cdot \frac{\tilde{b}_h^{b,t}}{H} + \frac{(H+1)V_{h+1}^0}{n_h^t}$. Thus,

$$\begin{aligned} \sum_{t=1}^T \delta_h^t &\leq \sum_{t:q_h^t=0}^T \delta_h^t + \sum_{t:q_h^t \geq 1}^T \delta_h^t \\ &\leq \sum_{t=1}^T \alpha_m^0 V_{h+1}^0 + \left(1 - \frac{1}{H}\right) \cdot \sum_{t=1}^T \sum_{i=1}^m \alpha_m^i \tilde{\xi}_{h+1}^{\ell^i} + \frac{\mathbb{1}\{q_h^t \geq 1\}}{e_{q-1} H} \sum_{t=1}^T \sum_{i=0}^{e_{q-1}} \tilde{\xi}_{h+1}^{b, \ell_{q-1}^{b,i}} \\ &\quad + \sum_{t=1}^T (b_h^t + \delta_{h+1}^t - \tilde{\xi}_{h+1}^t + \tau_{h+1}^t). \end{aligned} \tag{43}$$

The first term on the right-hand-side of (43) can be bounded by

$$\sum_{t=1}^T \alpha_{n_h^t}^0 V_{h+1}^0 = \sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} \sum_{n_h^t=1}^{n_h^T(s,a)} \alpha_{n_h^t}^0 V_{h+1}^0 \leq \sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} \sum_{m=1}^{\infty} \alpha_{n_h^t}^0 V_{h+1}^0 \leq \frac{n_0 S A V_{h+1}^0}{H-1}, \tag{44}$$

where the last inequality is from

$$\sum_{m=1}^{\infty} \alpha_{n_h^t}^0 = \frac{n_0}{H+n_0} \left(1 + \frac{n_0+1}{H+n_0+1} + \dots\right) = \frac{n_0}{H+n_0} \frac{H+n_0}{H-1} \leq \frac{n_0}{H-1}.$$

From Proposition 1, we have $\sum_{m=i}^{\infty} \alpha_m^i \leq (1 + \frac{1}{H})$. Thus, we control the second term of (43) by

$$\sum_{t=1}^T \sum_{i=1}^{n_h^t} \alpha_{n_h^t}^i \tilde{\xi}_{h+1}^{\ell^i} \leq \sum_{t'=1}^T \tilde{\xi}_{h+1}^{t'} \sum_{m=n_h^{t'}}^{\infty} \alpha_m^{n_h^{t'}} \leq (1 + \frac{1}{H}) \sum_{t=1}^T \tilde{\xi}_{h+1}^t. \tag{45}$$

Following Zhang et al. (2020), the third term can be bounded by

$$\begin{aligned} \sum_{t=1}^T \sum_{i=0}^{e_{q-1}} \frac{\mathbb{1}\{q_h^t \geq 1\}}{e_{q-1} H} \tilde{\xi}_{h+1}^{b, \ell_{q-1}^{b,i}} &= \sum_{t'=1}^T \sum_{t=1}^T \frac{\mathbb{1}\{q_h^t \geq 1\}}{e_{q-1} H} \sum_{i=1}^{e_{q-1}} \tilde{\xi}_{h+1}^{b, \ell_{q-1}^{b,i}} \mathbb{1}\{\ell_{q-1}^{b,i} = t'\} \\ &\leq \frac{H+1}{H^2} \sum_{t=1}^T \mathbb{1}\{q_h^t \geq 1\} \cdot \tilde{\xi}_{h+1}^t, \end{aligned} \tag{46}$$

where the inequality utilizes the fact that $\sum_{i=1}^{e_{q-1}} \mathbb{1}\{\ell_{q-1}^{b,i} = t'\} \leq 1$ for any fixed $t' \in [T]$.

Moreover, it is easy to verify that $(1 - \frac{1}{H})\tilde{\xi}_h^t + \frac{1}{H}\tilde{\xi}_h^{b,t} \leq \xi_h^t$, since

$$\begin{aligned} (1 - \frac{1}{H})\tilde{V}_h^t(s_h^t) + \frac{1}{H} \cdot \tilde{V}_h^{b,t}(s_h^t) &\leq (1 - \frac{1}{H}) \max_{j \in [J]} \{\tilde{Q}_h^{j,t}(s_h^t, a_h^t)\} + \frac{1}{H} \cdot \tilde{Q}_h^{b,t}(s_h^t, a_h^t) \\ &\leq Q_h^t(s_h^t, \pi_h^t(s_h^t)) = V_h^t(s_h^t), \end{aligned} \tag{47}$$

where the first inequality follows from Line 16 and 21 in Algorithm 1, and the second inequality is from (36).

By substituting (44)-(46) to (43), we have

$$\begin{aligned} \sum_{t=1}^T \delta_h^t &\leq 4n_0 S A + (1 + \frac{1}{H}) \sum_{t=1}^T \left((1 - \frac{1}{H})\tilde{\xi}_{h+1}^t + \frac{1}{H}\tilde{\xi}_{h+1}^{b,t} \right) + \sum_{t=1}^T (b_h^t + \delta_{h+1}^t - \tilde{\xi}_{h+1}^t + \tau_{h+1}^t) \\ &\leq 4n_0 S A + (1 + \frac{1}{H}) \sum_{t=1}^T \xi_{h+1}^t + \sum_{t=1}^T (b_h^t + \delta_{h+1}^t - \tilde{\xi}_{h+1}^t + \tau_{h+1}^t) \end{aligned}$$

$$\leq 4n_0SA + (1 + \frac{1}{H}) \sum_{t=1}^T \delta_{h+1}^t + \sum_{t=1}^T (b_h^t + \tau_{h+1}^t),$$

where the last line is from the fact that $\xi_{h+1}^t \leq \delta_{h+1}^t$, since $V^* \geq V^\pi$ for any policy π .

By unrolling the above inequality until $h = 1$, we obtain

$$\sum_{t=1}^T \delta_1^t \leq \tilde{O} \left(SAH + \sum_{t=1}^T \sum_{h=1}^H (b_h^t + \tau_{h+1}^t) \right). \quad (48)$$

Recall that $b_h^t = \tilde{b}_h^t + \mathbb{1}\{q_h^t \geq 1\} \cdot \tilde{b}_h^{b,t}/H + \frac{(H+1)V_{h+1}^0}{n_h^t}$. Before proceeding, we note that $\frac{e_q}{\sqrt{e_{q-1}}} \leq 2\sqrt{e_{q-1}}$ for any $q \geq 1$ and we denote $Q_{h,s,a} = q_h^{T+1}(s,a)$ for any $(h,s,a) \in [H] \times \mathcal{S} \times \mathcal{A}$ such that $\sum_{(s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H]} \sum_{q=1}^{Q_{h,s,a}} e_{q-1} \leq TH$. Also, let $Q = \max_{(h,s,a) \in [H]} Q_{h,s,a} \leq \frac{\log(T/H)}{\log(1+\frac{1}{H})} \leq 4H \log(T/H)$, where the last inequality is due to the fact that $\log(1 + \frac{1}{H}) \geq \frac{1}{4H}$ for $H \geq 1$. Thus, one has

$$\begin{aligned} \sum_{t=1}^T \sum_{h=1}^H \frac{\mathbb{1}\{q_h^t \geq 1\}}{\sqrt{e_{q-1}}} &\leq \sum_{(s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H]} \sum_{q=1}^{Q_{h,s,a}} \frac{e_q}{\sqrt{e_{q-1}}} \leq 2 \sum_{(s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H]} \sum_{q=1}^{Q_{h,s,a}} \sqrt{e_{q-1}} \\ &\leq 4\sqrt{SAH^2 \log(T/H)} \sqrt{\sum_{(s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H]} \sum_{q=1}^{Q_{h,s,a}} e_{q-1}} \\ &\leq O(1)\sqrt{SAH^2 \log(T \cdot TH)} \leq \tilde{O}(\sqrt{SAH^3 T}) \end{aligned}$$

where the penultimate line is from the Cauchy-Schwarz inequality. In addition, we have

$$\begin{aligned} \sum_{t=1}^T \sum_{h=1}^H (\tilde{b}_h^t + \frac{(H+1)V_{h+1}^0}{n_h^t}) &\leq \tilde{O}(1) \sum_{(s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H]} \sum_{m=1}^{n_h^T(s,a)} \sqrt{\frac{H^3}{m}} \\ &\leq \tilde{O}(1)\sqrt{SAH \cdot T/SA} \sqrt{\sum_{(s,a,h) \in \mathcal{S} \times \mathcal{A} \times [H]} \sum_{m=1}^{\frac{T}{SA}} \frac{H^3}{m}} \\ &\leq \tilde{O}(\sqrt{SAH^5 T}) \end{aligned}$$

where the penultimate inequality holds since the left-hand side is maximized when $n_h^T(s,a) = \frac{T}{SA}$ for every $(h,s,a) \in [T] \times \mathcal{S} \times \mathcal{A}$. Then, with the above inequalities in hand, one has

$$\sum_{h=1}^H \sum_{t=1}^T b_h^t = \sum_{h=1}^H \sum_{t=1}^T \left(\tilde{b}_h^t + \frac{(H+1)V_{h+1}^0}{n_h^t} + \sum_{h=1}^H \sum_{t=1}^T \frac{\mathbb{1}\{q_h^t \geq 1\}}{H} \cdot \tilde{b}_h^{b,t} \right) \leq \tilde{O}(\sqrt{SAH^5 T}) \quad (49)$$

Moreover, we have

$$\tau_{h+1}^t = P_{h,s_h^t, a_h^t} \left(V_{h+1}^* - V_{h+1}^{\pi^t} \right) - \left(V_{h+1}^* - V_{h+1}^{\pi^t} \right) (s_{h+1}^t)$$

is a martingale-difference sequence with respect to the filtration $\mathcal{F}_{t,h}$ that contains all the random variables before the step-size $h+1$ at the t -th episode. By Hoeffding's inequality, we have

$$|\sum_{h=1}^H \sum_{t=1}^T \tau_{h+1}^t| \leq \tilde{O}(\sqrt{H^3 T}) \quad (50)$$

with probability $1 - \delta$.

Finally, substituting (49) and (50) into (48) and rescaling δ to $\delta/4$ complete the proof.

C.2 Proof of Lemma 7

For any $(s, a, h, t) \in \mathcal{S} \times \mathcal{A} \times [H] \times [T]$, we let $m = n_h^t(s, a)$ and $q = q_h^t(s, a)$ for simplicity. From (37), one has

$$Q_h^t(s, a) = (1 - \frac{1}{H}) \max_{j \in [J]} \left\{ \sum_{i=0}^n W_{j,n}^i \tilde{V}_{h+1}^{\ell^i}(s_{h+1}^{\ell^i}) \right\} + \frac{1}{H} \tilde{Q}_h^{\flat, t}(s, a), \quad (51)$$

where $\{\ell^i\}_{i=0}^m$ represent the episode index of the i -th visit before t , and

$$\tilde{Q}_h^{\flat, t}(s, a) = r_h(s, a) + \max_{j \in [J]} \left\{ \sum_{i=0}^{e_{q-1}} W_{j,q-1}^{\flat, i} \tilde{V}_{h+1}^{\flat, \ell_{q-1}^i}(s_{h+1}^{\ell_{q-1}^i}) \right\}.$$

Before proceeding, we first claim that for any $(t, h, s, a) \in [T] \times [H] \times \mathcal{S} \times \mathcal{A}$, we have $Q_h^t(s, a) \geq Q_h^*(s, a)$, if the following relationship holds

$$\max_{j \in [J]} \left\{ \sum_{i=0}^{e_{q-1}} W_{j,q-1}^{\flat, i} V_{h+1}^*(s_{h+1}^{\ell_{q-1}^i}) \right\} \geq H \cdot P_{h,s,a} V_{h+1}^* + (H - h + 1), \quad (52)$$

where we leave the detailed proof of this claim to the end of this subsection.

Next, the following lemma shows that (52) holds with high probability, which implies the optimism of RANDOMIZEDQ and completes the proof of Lemma 7. The detailed proof of the following lemma is postponed to Appendix C.5.

Lemma 10. Consider $\delta \in (0, 1)$. Assume that $J = \lceil c \cdot \log(SAHT/\delta) \rceil$, $\kappa = c \cdot (\log(SAH/\delta) + \log(T))$, and $n_0 = \lceil c \cdot \log(T) \cdot \kappa \rceil$, where c is some universal constant. Let $V_h^0 = 2(H - h + 1)$. Then, for any $(h, s, a) \in [H] \times \mathcal{S} \times \mathcal{A}$ and any stage $q \geq 1$, the equation (52) holds with probability at least $1 - \delta$.

Proof of the claim: Assuming that (52) holds for any $(h, t) \in [H] \times [T]$, we will first show by induction that

$$\tilde{Q}_h^{\flat, t}(s, a) \geq H \cdot P_{h,s,a} V_{h+1}^* + (H - h + 1)$$

holds correspondingly, for any $(s, a, h, t) \in \mathcal{S} \times \mathcal{A} \times [H] \times [T]$. To begin with, when $h' = H + 1$, $\tilde{Q}_{H+1}^{\flat, t}(s, a) = H \cdot P_{H+1,s,a} V_{H+1}^* = 0$ holds naturally. When $h' = h + 1 \leq H$, suppose that $\tilde{Q}_{h+1}^{\flat, t}(s, a) \geq H \cdot P_{h+1,s,a} V_{h+1}^* + H - h' + 1 \geq H - h' + 1$, for any $(t, s, a) \in [T] \times \mathcal{S} \times \mathcal{A}$. By this hypothesis, we also have $\tilde{V}_{h+1}^{\flat, t}(s_{h+1}^t) = \min_a \tilde{Q}_{h+1}^{\flat, t}(s_{h+1}^t, a) \geq H - h' + 1 \geq V_{h+1}^*(s_{h+1}^t)$. By induction, when $h' = h$, we have

$$\begin{aligned} \tilde{Q}_h^{\flat, t}(s, a) &= r_h(s, a) + \max_{j \in [J]} \left\{ \sum_{i=0}^{e_{q-1}} W_{j,q-1}^{\flat, i} \tilde{V}_{h+1}^{\flat, \ell_{q-1}^i}(s_{h+1}^{\ell_{q-1}^i}) \right\} \\ &\geq r_h(s, a) + \max_{j \in [J]} \left\{ \sum_{i=0}^{e_{q-1}} W_{j,q-1}^{\flat, i} V_{h+1}^*(s_{h+1}^{\ell_{q-1}^i}) \right\} \end{aligned}$$

Thus, if (52) holds, we have $\tilde{Q}_h^{\flat, t}(s, a) \geq H P_{h,s,a} V_{h+1}^* + (H - h + 1)$. Note that $r_h(s, a) \leq 1$ for any $(h, s, a) \in [H] \times \mathcal{S} \times \mathcal{A}$. Thus, the equation (51) becomes

$$\begin{aligned} Q_h^t(s, a) &\geq (1 - \frac{1}{H}) r_h(s, a) + \frac{H - h + 1}{H} + P_{h,s,a} V_{h+1}^* \\ &\geq r_h(s, a) + P_{h,s,a} V_{h+1}^* = Q_h^*(s, a), \end{aligned}$$

for any $(h, s, a) \in [H] \times \mathcal{S} \times \mathcal{A}$.

C.3 Proof of Lemma 8

Denote $m = n_h^t(s, a)$ as the number of visits on (h, s, a) before the t -th episode. Let $\alpha_m^0 = \prod_{k=1}^m \frac{n_0+k-1}{H+n_0+k}$ and $\alpha_m^i := \frac{H+1}{H+n_0+i} \prod_{k=i+1}^m \frac{n_0+k-1}{H+n_0+k}$. From (17), for any $(s, a, h, t) \in \mathcal{S} \times \mathcal{A} \times [H] \times [T]$, we have

$$\tilde{Q}_h^{j,t}(s, a) = r_h(s, a) + \sum_{i=0}^m W_{j,m}^i \tilde{V}_{h+1}^{\ell^i}(s_{h+1}^{\ell^i}) \leq r_h(s, a) + 2H \cdot \sum_{i=0}^m W_{j,m}^i \frac{\tilde{V}_{h+1}^{\ell^i}(s_{h+1}^{\ell^i})}{2H}.$$

Note that $\frac{\tilde{V}_{h+1}^{\ell^i}(s_{h+1}^{\ell^i})}{2H} \leq 1$ for any $i = 0, \dots, m$. Thus, we can apply Lemma 5 and Proposition 1

$$\begin{aligned} \max_{j \in [J]} \tilde{Q}_h^{j,t}(s, a) &\leq r_h(s, a) + 2H \left(\sum_{i=0}^m \mathbb{E}[W_{j,m}^i] \frac{\tilde{V}_{h+1}^{\ell^i}(s_{h+1}^{\ell^i})}{2H} + \frac{c_1}{2} \sqrt{\frac{(H+1)\kappa^2 \log^3(2SAHTJ/\delta)}{H+n_0+m}} \right. \\ &\quad \left. + \frac{c_2}{2} \frac{(H+1)\kappa \log^2(2SAHTJ/\delta)}{H+n_0+m} \right) \\ &\leq r_h(s, a) + \alpha_m^0 \tilde{V}_{h+1}^0 + \sum_{i=1}^m \alpha_m^i \tilde{V}_{h+1}^{\ell^i}(s_{h+1}^{\ell^i}) \\ &\quad + c_1 \sqrt{\frac{(H+1)^3 \kappa^2 \log^3(2SAHTJ/\delta)}{H+n_0+m}} + c_2 \frac{(H+1)^2 \kappa \log^2(2SAHTJ/\delta)}{H+n_0+m}, \end{aligned}$$

with probability at least $1 - \delta/2$, where c_1, c_2 are universal constants. In addition, we denote \mathcal{F}_i as the filtration containing all the random variables before the episode $\ell_h^i(s, a)$, such that $(\alpha_m^i V_{h+1}^*(s_{h+1}^{\ell^i}) - P_{h,s,a} V_{h+1}^*)$ is a martingale difference sequence w.r.t. $\{\mathcal{F}_i\}_{i \geq 0}$ for any $i \leq m$. Following Jin et al. (2018) and by Hoeffding's inequality and Proposition 1 (i.e., (22)), with probability at least $1 - \delta/2$, we have

$$\begin{aligned} \left| \sum_{i=1}^m \alpha_m^i \left(V_{h+1}^*(s_{h+1}^{\ell^i}) - P_{h,s,a} V_{h+1}^* \right) \right| &\leq c_3 H \sqrt{\sum_{i=1}^m (\alpha_m^i)^2 \log(2SAHT/\delta)} \\ &\leq c_3 \sqrt{\frac{(H+1)^3 \kappa}{H+n_0+m} \log(2SAHT/\delta)} \end{aligned}$$

for any $(t, h, s, a) \in [T] \times [H] \times \mathcal{S} \times \mathcal{A}$ and some universal constant $c_3 > 0$. Note that $\tilde{V}_{h+1}^{\ell^i}(s_{h+1}^{\ell^i}) \leq V_h^0$. Thus,

$$\begin{aligned} \zeta_h^t &\leq \alpha_m^0 V_{h+1}^0 + \sum_{i=1}^m \alpha_m^i \left(\tilde{V}_{h+1}^{\ell^i} - V_{h+1}^* \right) (s_{h+1}^{\ell^i}) + \sum_{i=1}^m \alpha_m^i \left(V_{h+1}^*(s_{h+1}^{\ell^i}) - P_{h,s,a} V_{h+1}^* \right) \\ &\quad + (c_1 + c_3) \sqrt{\frac{(H+1)^3 \kappa^2 \log^3(2SAHTJ/\delta)}{H+n_0+m}} + c_2 \frac{(H+1)^2 \kappa \log^2(2SAHTJ/\delta)}{H+n_0+m} \\ &\leq \alpha_m^0 V_h^0 + \sum_{i=1}^m \alpha_m^i \left(\tilde{V}_{h+1}^{\ell^i} - V_{h+1}^* \right) (s_{h+1}^{\ell^i}) + \tilde{b}_h^t, \end{aligned}$$

where we let $\tilde{b}_h^t = \tilde{O} \left(\sqrt{\frac{(H+1)^3}{H+n_0+m}} + \frac{(H+1)^2}{H+n_0+m} \right)$ and $m = n_h^t(s, a)$ for any $(h, t) \in [H] \times [T]$.

C.4 Proof of Lemma 9

Note that during the initial stage, for any $(h, t, s, a) \in [H] \times [T] \times \mathcal{S} \times \mathcal{A}$ within the stage $q_h^t(s_h^t, a_h^t) = 0$, we have

$$\tilde{Q}_h^{\flat,t}(s, a) - Q_h^*(s, a) \leq (H+1)V_h^0.$$

For any $(h, t, s, a) \in [H] \times [T] \times \mathcal{S} \times \mathcal{A}$ with $q_h^t(s_h^t, a_h^t) \geq 1$, following the similar procedure in Appendix C.3 and applying Lemma 6 leads to

$$\begin{aligned}\tilde{Q}_h^{\flat, t}(s, a) - Q_h^*(s, a) &\leq \max_{j \in [J]} \left\{ \sum_{i=0}^{e_{q-1}} W_{j, q-1}^{\flat, i} \tilde{V}_{h+1}^{\flat, \ell_{q-1}^{\flat, i}}(s_{h+1}^{\ell_{q-1}^{\flat, i}}) \right\} - Q_h^*(s, a) \\ &\leq \frac{1}{e_{q-1}} \sum_{i=1}^{e_{q-1}} \left(\tilde{V}_{h+1}^{\flat, \ell_{q-1}^{\flat, i}} - V_{h+1}^* \right) (s_{h+1}^{\ell_{q-1}^{\flat, i}}) + \tilde{b}_h^{\flat, t},\end{aligned}$$

with probability $1 - \delta$, where $\tilde{b}_h^{\flat, t} = \tilde{O} \left(\sqrt{\frac{(H+1)^4}{e_{q-1}}} + \frac{(H+1)^2}{e_{q-1}} \right)$.

Combining the above we have

$$\begin{aligned}\zeta_h^{\flat, t} &\leq \mathbb{1}(q_h^t(s, a) \geq 1) \left(\sum_{i=1}^{e_{q-1}} \frac{1}{e_{q-1}} \left(\tilde{V}_{h+1}^{\flat, \ell_{q-1}^{\flat, i}} - V_{h+1}^* \right) (s_{h+1}^{\ell_{q-1}^{\flat, i}}) + \tilde{b}_h^{\flat, t} \right) \\ &\quad + \mathbb{1}(q_h^t(s, a) = 0) \cdot (H+1)V_h^0,\end{aligned}$$

which completes the proof.

C.5 Proof of Lemma 10

We first let $\mathcal{E}^*(\delta)$ be the event containing all $(h, s, a) \in [H] \times \mathcal{S} \times \mathcal{A}$ with the stage $q := q_h^t(s, a)$, such that

$$\mathcal{K}_{\inf} \left(\frac{1}{e_q} \sum_{i=1}^{e_q} \delta_{V_{h+1}^*(s_{h+1}^{\ell_q^i})}, P_{h, s, a} V_{h+1}^* \right) \leq \frac{\beta^*(\delta, e_q)}{e_q},$$

where $\beta^*(\delta, n) := \log(2SAH/\delta) + 3\log(e\pi(2n+1))$. The following lemma shows that $\mathcal{E}^*(\delta)$ holds with probability $1 - \frac{\delta}{2}$

Lemma 11 (Lemma 4 in Tiapkin et al. (2024)). *Consider $\delta \in (0, 1)$. With probability $1 - \frac{\delta}{2}$, the following event holds*

$$\mathcal{K}_{\inf} \left(\frac{1}{e_q} \sum_{i=1}^{e_{q-1}} \delta_{V_{h+1}^*(s_{h+1}^{\ell_q^i})}, P_{h, s, a} V_{h+1}^* \right) \leq \frac{\beta^*(\delta, e_q)}{e_q}, \quad \forall (t, h, s, a) \in [T] \times [H] \times \mathcal{S} \times \mathcal{A}, \quad (53)$$

where $q = q_h^t(s, a)$ and $\beta^*(\delta, n) := \log(2SAH/\delta) + 3\log(e\pi(2e_q + 1))$.

To construct an anti-concentration inequality bound of the weighted sum $W_{j, q}^{\flat, i} V_{h+1}^*(s_{h+1}^{\ell_q^i})$, we first recall the following two lemmas provided in Tiapkin et al. (2024).

Lemma 12 (Lemma 3 in Tiapkin et al. (2024)). *For any stage $q \geq 0$ and $j \in [J]$, the aggregated weights $W_{j, q}^{\flat}$ follows a standard Dirichlet distribution $\text{Dir}(n_0/\kappa, 1/\kappa, \dots, 1/\kappa)$.*

Lemma 13 (Lemma 10 in Tiapkin et al. (2024)). *For any $\alpha = (\alpha_0 + 1, \alpha_1, \dots, \alpha_m) \in \mathbb{R}_{++}^{m+1}$, define $\bar{p} \in \Delta_m$ such that $\bar{p}(\ell) = \alpha_\ell / \bar{\alpha}$, $\ell = 0, \dots, m$, where $\bar{\alpha} = \sum_{j=0}^m \alpha_j$. Also define a measure $\bar{\nu} = \sum_{i=0}^m \bar{p}(i) \cdot \delta_{f(i)}$. Let $\varepsilon \in (0, 1)$. Assume that $\alpha_0 \geq c_0 + \log_{17/16}(2(\bar{\alpha} - \alpha_0))$ for some universal constant c_0 . Then for any $f : \{0, \dots, m\} \rightarrow [0, b_0]$ such that $f(0) = b_0$, $f(j) \leq b \leq b_0/2$, $j \in [m]$, and any $\mu \in (0, b)$*

$$\mathbb{P}_{w \sim \text{Dir}(\alpha)}[wf \geq \mu] \geq (1 - \varepsilon) \mathbb{P}_{g \sim \mathcal{N}(0, 1)} \left[g \geq \sqrt{2\bar{\alpha} \mathcal{K}_{\inf}(\bar{\nu}, \mu)} \right].$$

By applying Lemma 13 with $\alpha_0 = n_0/\kappa - 1$, $\alpha_i = 1/\kappa$, $\forall i \in [e_q]$, $r_0 = 2$, $b_0 = 2(H+1)(H-h+1)$, and $\bar{\nu}_q = \frac{n_0 - \kappa}{e_q + n_0 - \kappa} \delta_{V_{h+1}^*(s_0)} + \sum_{i=1}^{e_q} \frac{1}{e_q + n_0 - \kappa} \delta_{V_{h+1}^*(s_{h+1}^{\ell_q^i})}$, we have that conditioned on the event $\mathcal{E}^*(\delta)$ holds, we have

$$\mathbb{P} \left(\sum_{i=0}^{e_q} W_{j, q}^{\flat, i} V_{h+1}^*(s_{h+1}^{\ell_q^i}) \geq H \cdot P_{h, s, a} V_{h+1}^* + (H-h+1) \middle| \mathcal{E}^*(\delta) \right)$$

$$\geq \frac{1}{2} \left(1 - \Phi \left(\sqrt{\frac{2(e_q + n_0 - \kappa) \mathcal{K}_{\inf}(\bar{\nu}_q, P_{h,s,a} V_{h+1}^*)}{\kappa}} \right) \right) \geq \frac{1}{2} \left(1 - \Phi \left(\sqrt{\frac{2\beta^*(\delta, T)}{\kappa}} \right) \right).$$

where Φ denotes the CDF of the standard normal distribution. Here the last inequality is from Lemma 4 and Lemma 11.

Then, by selecting $\kappa = 2\beta^*(\delta, T)$, we ensure a constant probability of optimism:

$$\mathbb{P} \left(\sum_{i=0}^{e_q} W_{j,q}^{\flat,i} V_{h+1}^*(s_{h+1}^{\ell_q^{\flat,i}}) \geq H \cdot P_{h,s,a} V_{h+1}^* + (H-h+1) \mid \mathcal{E}^*(\delta) \right) \geq \frac{1 - \Phi(1)}{2} \triangleq \gamma.$$

Now, choosing $J = \lceil \log \left(\frac{2SAHT}{\delta} \right) / \log(1/(1-\gamma)) \rceil = \lceil c_J \cdot \log(2SAHT/\delta) \rceil$ ensures:

$$\begin{aligned} & \mathbb{P} \left(\max_{j \in [J]} \left\{ \sum_{i=0}^{e_q} W_{j,q}^{\flat,i} V_{h+1}^*(s_{h+1}^{\ell_q^{\flat,i}}) \right\} \geq H \cdot P_{h,s,a} V_{h+1}^* + (H-h+1) \mid \mathcal{E}^*(\delta) \right) \\ & \geq 1 - (1-\gamma)^J \geq 1 - \frac{\delta}{2SAHT}. \end{aligned}$$

By applying a union bound and taking expectation on $\mathcal{E}^*(\delta)$ from Lemma 11, we conclude the proof.

D Analysis: Gap-dependent Regret Bound

We begin by decomposing the total regret using the suboptimality gaps defined in Assumption 1. Following Yang et al. (2021), we obtain:

$$\begin{aligned} \text{Regret}_T &= \sum_{t=1}^T (V_1^* - V_1^{\pi^t})(s_1^t) = \sum_{t=1}^T \left(V_1^*(s_1^t) - Q_1^*(s_1^t, a_1^t) + (Q_1^* - Q_1^{\pi^t})(s_1^t, a_1^t) \right) \\ &= \sum_{t=1}^T \Delta_1(s_1^t, a_1^t) + \sum_{t=1}^T \mathbb{E}_{s_2^t \sim P_{1,s_1^t, a_1^t}} \left[(V_2^* - V_2^{\pi^t})(s_2^t) \right] \\ &= \dots = \mathbb{E} \left[\sum_{t=1}^T \sum_{h=1}^H \Delta_h(s_h^t, a_h^t) \mid a_h^t = \pi_h^t(s_h^t) \right], \end{aligned}$$

where the expectation is taken with respect to the underlying transition kernel. Before proceeding, we introduce the following lemma that characterizes the learning error of the Q-functions, which will be used to control the suboptimality gaps. The proof is deferred to Appendix D.1.

Lemma 14. Let $\mathcal{E} = \{ \forall (s, a, h, t) \in \mathcal{S} \times \mathcal{A} \times [H] \in [T] : 0 \leq (Q_h^t - Q_h^*)(s, a) \leq \alpha_{n_h^t}^0 V_{h+1}^0 + (1 + \frac{1}{H}) (V_{h+1}^t - V_{h+1}^*) (s_{h+1}^t) + B_h^t \}$, where B_h^t is defined in (58). The event \mathcal{E} holds with probability at least $1 - 1/T$.

In addition, we define the operator $\text{clip}[x|c] := x \cdot \mathbb{1}_{x \geq c}$ for some constant $c \geq 0$, which is commonly used in prior work (Simchowitz and Jamieson, 2019; Yang et al., 2021; Zheng et al., 2025). By Lemma 14, we have $V_h^*(s_h^t) = \max_a Q_h^*(s_h^t, a) \leq \max_a Q_h^t(s_h^t, a) = Q_h^t(s_h^t, a_h^t)$ such that $\Delta_h(s_h^t, a_h^t) = \text{clip}[V_h^*(s_h^t) - Q_h^*(s_h^t, a_h^t) \mid \Delta_{\min}] \leq \text{clip}[(Q_h^t - Q_h^*)(s_h^t, a_h^t) \mid \Delta_{\min}]$.

Thus, by definition, the expected total regret can be written as

$$\begin{aligned} \mathbb{E}[\text{Regret}_T] &= \mathbb{P}(\mathcal{E}) \cdot \mathbb{E} \left[\sum_{t=1}^T \sum_{h=1}^H \text{clip}[(Q_h^t - Q_h^*)(s_h^t, a_h^t) \mid \Delta_{\min}] \mid \mathcal{E} \right] \\ &\quad + \mathbb{P}(\mathcal{E}^c) \cdot \mathbb{E} \left[\sum_{t=1}^T \sum_{h=1}^H \text{clip}[(Q_h^t - Q_h^*)(s_h^t, a_h^t) \mid \Delta_{\min}] \mid \mathcal{E}^c \right] \end{aligned}$$

$$\leq (1 - \frac{1}{T})\mathbb{E} \left[\sum_{t=1}^T \sum_{h=1}^H \text{clip}[(Q_h^t - Q_h^*)(s_h^t, a_h^t) | \Delta_{\min}] | \mathcal{E} \right] + \frac{1}{T} \cdot TH^2. \quad (54)$$

Next, we control the first term in (54) by categorizing the suboptimality gaps into different intervals. Specifically, we split the interval $[\Delta_{\min}, H]$ into N disjoint intervals, i.e., $\mathcal{I}_n := [2^{n-1}\Delta_{\min}, 2^n\Delta_{\min}]$ for $n \in [N-1]$ and $\mathcal{I}_N := [2^{N-1}\Delta_{\min}, H]$. Denote the counter of state-action pair for each interval as $C_n := |\{(t, h) : (Q_h^t - Q_h^*)(s_h^t, a_h^t) \in \mathcal{I}_n\}|$. We then upper bound (54) as follows:

$$\mathbb{E}[\text{Regret}_T] = (1 - \frac{1}{T}) \sum_{n=1}^N 2^n \Delta_{\min} C_n + H^2. \quad (55)$$

The following lemma shows that the counter is bounded in each interval, conditioned on event \mathcal{E} .

Lemma 15. *Under \mathcal{E} , we have that for every $n \in [N]$, $C_n \leq O(\frac{H^6 SA\kappa^2 \log^3(SAHT)}{4^n \Delta_{\min}^2})$.*

The proof is postponed to Appendix D.2. Thus, (55) becomes

$$\mathbb{E}[\text{Regret}_T] \leq O\left(\frac{H^6 SA\kappa^2 \log^3(SAHT)}{\Delta_{\min}}\right),$$

Noting that $\kappa = O(\log(SAHT))$, we complete the proof.

D.1 Proof of Lemma 14

We begin by applying Lemma 7, which guarantees that for an ensemble size $J = \lceil c_J \cdot \log(4SAHT^2) \rceil$, it holds with probability at least $1 - \frac{1}{2T}$ that

$$Q_h^k(s, a) \geq Q_h^*(s, a), \quad \forall (h, t, s, a) \in [H] \times [T] \times \mathcal{S} \times \mathcal{A},$$

Recalling the definition of Q_h^t from (36), we have, again with probability at least $1 - \frac{1}{2T}$,

$$\begin{aligned} & (Q_h^t - Q_h^*)(s_h^t, a_h^t) \\ &= (1 - \frac{1}{H}) \max_{j \in [J]} \left\{ \tilde{Q}_h^{j,t}(s_h^t, a_h^t) - Q_h^*(s_h^t, a_h^t) \right\} + \frac{1}{H} \left\{ \tilde{Q}_h^{b,t}(s_h^t, a_h^t) - Q_h^*(s_h^t, a_h^t) \right\} \\ &\leq \alpha_{n_h^t}^0 V_{h+1}^0 + (1 + \frac{1}{H}) (V_{h+1}^t - V_{h+1}^*) (s_{h+1}^t) + B_h^t \end{aligned} \quad (56)$$

where the final inequality follows by applying Lemma 8, Lemma 9, and (45)-(47). Here,

$$\begin{aligned} B_h^t &\leq c' \left(\sqrt{\frac{H^3 \kappa^2 \log^3(SAHT^2)}{n_h^t}} + \frac{H^2 \kappa \log^2(SAHT^2)}{n_h^t} \right. \\ &\quad \left. + \sqrt{\frac{H^2 \kappa^2 \log^3(SAHT^2)}{e_{q_h^t-1}}} \cdot \mathbb{1}\{q_h^t \geq 1\} + \frac{H \kappa \log^2(SAHT^2)}{e_{q_h^t-1}} \cdot \mathbb{1}\{q_h^t \geq 1\} \right), \end{aligned} \quad (57)$$

where c' is a positive constant. To simplify the expression in (57), we note

$$\frac{n_h^t(s, a)}{e_{q_h^t(s, a)-1}} \leq \frac{\sum_{i=1}^{q_h^t(s, a)} e_i}{e_{q_h^t(s, a)-1}} = 1 + \frac{\sum_{i=1}^{q_h^t(s, a)-2} e_i}{e_{q_h^t(s, a)-1}} + \frac{e_{q_h^t(s, a)}}{e_{q_h^t(s, a)-1}} \leq 2 + \frac{1}{H} + 4H \leq 8H,$$

where the second inequality uses Zheng et al. (2025, Lemma D.3) and the staged update rule, i.e., $e_{q_h^t} = (1 + \frac{1}{H})e_{q_h^t-1}$.

Thus, we could rewrite (57) as

$$B_h^t \leq c' \left(\sqrt{\frac{H^3 \kappa^2 \log^3(SAHT^2)}{n_h^t}} + \frac{H^2 \kappa \log^2(SAHT^2)}{n_h^t} \right). \quad (58)$$

D.2 Proof of Lemma 15

We first partition each interval \mathcal{I}_n according to the step index h . Specifically, for every $n \in [N]$ and $h \in [H]$, we define:

$$\begin{aligned} w_{n,h}^t &:= \mathbb{1}\{(Q_h^t - Q_h^*)(s_h^t, a_h^t) \in \mathcal{I}_n\}, \quad \forall t \in [T], \\ C_{n,h} &:= \sum_{t=1}^T w_{n,h}^t. \end{aligned} \tag{59}$$

Note that $w_{n,h}^t \in \{0, 1\}$ for all t , since it is an indicator function. By definition, we have $C_n = \sum_{h=1}^H C_{n,h}$, and furthermore, for every $n \in [N]$ and $h \in [H]$,

$$2^{n-1} \Delta_{\min} C_{n,h} \leq \sum_{t=1}^T w_{n,h}^t (Q_h^t - Q_h^*)(s_h^t, a_h^t) \tag{60}$$

To control the right-hand side of (60), we provide the following lemma, which upper bounds the weighted sum of Q-value errors. The proof is postponed to Appendix D.3.

Lemma 16. *Under event \mathcal{E} , for any $h \in [H]$ and $n \in [N]$, the weights $\{w_{n,h}^t\}_{t=1}^T$ defined in (59) satisfy:*

$$\begin{aligned} \sum_{t=1}^T w_{n,h}^t (Q_h^t - Q_h^*)(s_h^t, a_h^t) &\leq 4n_0 SAH + O\left(\sqrt{C_{n,h} SAH^5 \kappa^2 \log^3(SAHT)}\right) \\ &\quad + O(SAH^3 \kappa \log^2(SAHT) \log(1 + C_{n,h})). \end{aligned}$$

Combining this with (60), we obtain the following bound:

$$C_{n,h} \leq O\left(\frac{H^5 SA \kappa^2 \log^3(SAHT)}{4^n \Delta_{\min}^2}\right).$$

Summing over $h \in [H]$, we conclude:

$$C_n = \sum_{h=1}^H C_{n,h} \leq O\left(\frac{H^6 SA \kappa^2 \log^3(SAHT)}{4^n \Delta_{\min}^2}\right).$$

D.3 Proof of Lemma 16

Under event \mathcal{E} , we recall the following upper bound:

$$(Q_h^t - Q_h^*)(s, a) \leq \alpha_{n_h^t}^0 V_{h+1}^0 + (1 + \frac{1}{H}) (V_{h+1}^t - V_{h+1}^*) (s_{h+1}^t) + B_h^t, \tag{61}$$

where

$$B_h^t \leq c' \underbrace{\sqrt{\frac{H^3 \kappa^2 \log^3(8SAHT^2)}{n_h^t}}}_{B_{h,1}^t} + c' \underbrace{\frac{H^2 \kappa \log^2(8SAHT^2)}{n_h^t}}_{B_{h,2}^t}$$

For the first term on the right-hand side of (61), we follow the argument in (44) and obtain:

$$\sum_{t=1}^T w_{n,h}^t \alpha_{n_h^t}^0 V_{h+1}^0 \leq \frac{n_0 S A V_{h+1}^0}{H-1} \leq 4n_0 S A$$

For the second term, we note that $V_{h+1}^*(s_{h+1}^t) \geq Q_{h+1}^*(s_{h+1}^t, a_{h+1}^t)$ by the definition of the optimal policy in (1), and $V_{h+1}^t(s_{h+1}^t) = Q_{h+1}^t(s_{h+1}^t, a_{h+1}^t)$ by (38). Thus,

$$\sum_{t=1}^T w_{n,h}^t (1 + \frac{1}{H}) (V_{h+1}^t - V_{h+1}^*) (s_{h+1}^t) \leq (1 + \frac{1}{H}) \sum_{t=1}^T w_{n,h}^t (Q_{h+1}^t - Q_{h+1}^*) (s_{h+1}^t, a_{h+1}^t)$$

We now recursively unroll the Q-value difference over future steps, yielding:

$$\begin{aligned}
& \sum_{t=1}^T w_{n,h}^t (Q_h^t - Q_h^*) (s_h^t, a_h^t) \\
& \leq 4n_0 SA + (1 + \frac{1}{H}) \sum_{t=1}^T w_{n,h}^t (Q_{h+1}^t - Q_{h+1}^*) (s_{h+1}^t, a_{h+1}^t) + w_{n,h}^t B_h^t \\
& \leq \dots \leq 4en_0 SAH + e \sum_{h'=h}^H w_{n,h'}^t B_{h'}^t,
\end{aligned} \tag{62}$$

where we utilize $(1 + \frac{1}{H})^H \leq e$.

Similar to the steps in Appendix C, we bound $\sum_{t=1}^T w_{n,h}^t B_h^t$ by controlling each term separately. We first define $C_{n,h,s,a} := \sum_{m=1}^{n_h^T(s,a)} w_{n,h}^{l^m}$, to count the number of (h, s, a) -sample such that its learning error (i.e., $(Q_h^t - Q_h^*)(s, a)$ for $t \in [T]$) lies in the n -th interval, during the T episodes. We have

$$\begin{aligned}
\sum_{t=1}^T w_{n,h}^t B_{h,1}^t & \leq \sqrt{H^3 \kappa^2 \log^3(8SAHT^2)} \sum_{t=1}^T \frac{w_{n,h}^t}{\sqrt{n_h^t}} \\
& \lesssim \sqrt{H^3 \kappa^2 \log^3(SAHT^2)} \sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} \sum_{m=1}^{n_h^T(s,a)} \frac{w_{n,h}^{l^m}}{\sqrt{m}} \\
& \lesssim \sqrt{H^3 \kappa^2 \log^3(SAHT^2)} \sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} \sum_{m=1}^{C_{n,h,s,a}} \frac{1}{\sqrt{m}} \\
& \lesssim \sqrt{H^3 \kappa^2 \log^3(SAHT^2)} \sqrt{SA \sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} C_{n,h,s,a}} \\
& \lesssim \sqrt{C_{n,h} SAH^3 \kappa^2 \log^3(SAHT^2)},
\end{aligned} \tag{63}$$

where the third inequality holds since the left-hand side is maximized when the first $C_{n,h,s,a}$ visits belongs to the n -th interval, the penultimate line uses Cauchy-Schwartz inequality, and the last line is due to the fact that $\sum_{s,a} C_{n,h,s,a} \leq C_{h,h}$.

Similarly,

$$\begin{aligned}
\sum_{t=1}^T w_{n,h}^t B_{h,2}^t & \leq \sum_{t=1}^T w_{n,h}^t \frac{H^2 \kappa \log^2(8SAHT^2)}{n_h^t} \\
& \lesssim H^2 \kappa \log^2(SAHT^2) \sum_{t=1}^T \frac{w_{n,h}^t}{n_h^t} \\
& \lesssim H^2 \kappa \log^2(SAHT^2) \sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} \sum_{m=1}^{C_{n,h,s,a}} \frac{1}{m} \\
& \lesssim H^2 \kappa \log^2(SAHT^2) \sum_{(s,a) \in \mathcal{S} \times \mathcal{A}} \log(1 + C_{n,h,s,a}) \\
& \lesssim SAH^2 \kappa \log^2(SAHT^2) \log(1 + C_{n,h}).
\end{aligned} \tag{64}$$

Finally, substituting (63) and (64) into (62) yields

$$\sum_{t=1}^T w_{n,h}^t (Q_h^t - Q_h^*) (s_h^t, a_h^t)$$

$$\begin{aligned}
&\leq 4n_0SA + \left(1 + \frac{1}{H}\right) \sum_{t=1}^T w_{n,h}^t (Q_{h+1}^t - Q_{h+1}^*) (s_{h+1}^t, a_{h+1}^t) + w_{n,h}^t B_h^t \\
&\leq \dots \leq 4en_0SAH + O\left(\sqrt{C_{n,h}SAH^5\kappa^2 \log^3(SAHT^2)}\right) \\
&\quad + O\left(SAH^3\kappa \log^2(SAHT^2) \log(1 + C_{n,h})\right),
\end{aligned}$$

which completes the proof.