**The Effect of Move Shapes in the *Lights Out* Game**

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## Abstract

Variations of the original Lights Out game are studied for patterns of winnability. Varieties of move shapes and grid sizes are considered, and general formulas for winnability are obtained. When H is an $m \times n$ grid with a move shape **p**, we study something called the puzzle space (H,**p**). (H,**p**) is a vector space of patterns, called even-dominating sets, for the graph. Both linear algebra (mod 2) and ad hoc methods are used in generating all even-dominating sets.

## 1. Introduction

*Lights Out*, a game by Tiger Electronics, is a solitaire, electronic, hand-held game that is also available for online play over the Internet. The game consists of a $5 \times 5$ grid of squares that can, at any time, be lit or unlit. The status of a square refers to whether it is lit. Depending on the form of the game, the squares that are lit can be shown as having a different color than the unlit squares, or as bearing some signifying mark, like an ×, or ✓. A game begins with a randomly generated pattern of lit squares, or an *initial coloring*. A player is allowed to "move" at any square by pressing that square. Once a square is moved at, the status of the square itself is changed, along with the status of each of its non-diagonal adjacent neighbor squares. If one of these squares was lit, it becomes unlit, and vice versa. Since the objective of the game is to have all squares simultaneously unlit in the least number of moves, one moves at various squares until the game has been won. Figure 1 shows an example initial coloring and Figure 2 illustrates which squares one could move at in order to win the game. (The original Tiger Electronics' game only generates **winnable** initial colorings to make the game fun, but in fact, it is easy to generate unwinnable initial colorings.)

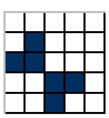

Figure 1

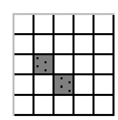

Figure 2

The Lights Out game is popular, judging from the vast collection of websites concerning the game. Children, puzzle lovers, and mathematicians alike enjoy playing or studying the game. Mathematicians have become increasingly interested in the underlying mathematical concepts of the game. Specifically, graph theoreticians have observed that there are graph theory terms that are appropriate for describing certain aspects of the original Lights Out game. For instance, the $5 \times 5$ grid can be represented as a graph, where the squares now are vertices. Figure 3 shows the $5 \times 5$ grid of the original Lights

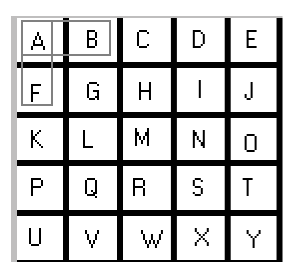


Figure 3

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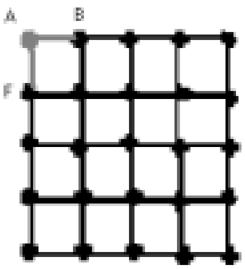


Figure 4

Out game and Figure 4 shows the graph theory interpretation of Figure 3. Pairs of squares that were adjacent are now pairs of vertices that are directly connected via an edge.

The original Lights Out game has been studied extensively. Puzzlers and mathematicians have analyzed game-winning algorithms and investigated questions of optimization and winnability; refer to [1, 2, 3, 4]. A game is considered *winnable* if there exists a sequence of squares (or vertices) such that when all of these squares are moved at, the game is won. (It can easily be verified that the order of the sequence of moves is irrelevant.) If it is not possible to win a game, that is, make every square unlit, one might seek to minimize the number of remaining lit squares. Mathematicians and puzzle solvers have already answered these questions regarding the original Lights Out game. The natural progression of this research leads to altering some of the parameters of the game. The underlying graph of the grid can be changed from a simple $5 \times 5$ grid to a larger $m \times n$ grid or any other connected graph. For example, if the game were in the form of a Cartesian product, say a cube graph, the game would be reminiscent of a Rubiks Cube. Altering the *move shape*, or roughly the pattern describing how moving at a vertex affects its neighbors, would also affect patterns of winnability. Figure 5 is the move shape of the original Lights Out game and Figure 6 is an alternate move shape. Although the move shape can be characterized by a $3 \times 3$ grid, the move shape is not a game. The move shape grid merely describes the affect of moving at a square. Moving at the center square changes the status of each lit square. (Figure 6 is also known as the Sigma Plus Rules on the original Lights Out game, because when playing a game, clicking on a square does not change its own status.)

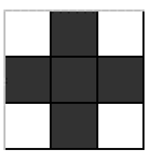


Figure 5

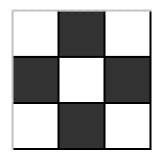


Figure 6

In order to study the winnability of a game, one can use linear algebra by analyzing the game's adjacency matrix of the graph involved. An *adjacency matrix* M for a game on a graph G is a matrix describing how the vertices of G are related. For the vertices 1, 2, . . . , $n$ of G, there are $n$ rows and $n$ columns corresponding to those vertices in that order. The $i,j^{th}$ entry of M is 1 if either $i$ is adjacent to $j$ or if $i = j$, and is 0 otherwise. The graph's adjacency matrix, M, will be symmetric, because for all $x$ and y that belong to V(G), $x$ being adjacent to $y$ implies that $y$ is adjacent to $x$.

In order to illustrate the usefulness of a linear algebraic approach towards researching the winnability patterns of certain games, the following fact is stated. If M is invertible, it follows that the game is always winnable, regardless of its initial coloring. Furthermore, the sequence of moves required to win the game will be essentially unique.

To use linear algebra to investigate the winnability of a game on graph G, create a column vector **b** describing the initial coloring of the game. In **b**, each lit vertex of G is denoted by 1 and each unlit vertex by 0. This leads to mod 2 arithmetic because when one moves at a lit square (a square denoted by 1), the square becomes unlit (a square denoted by 0); when moving at square $i$, the new coloring is the mod 2 sum $\mathbf{b} \oplus \mathbf{c}_i$ where $\mathbf{c}_i$ is the $i^{th}$ column vector of M. In order to win the game, one must have each vertex of G be unlit and denoted by 0. So, each vertex that is initially lit should have its status changed an odd number of times and each vertex that is initially unlit should have its status changed an even number of times.

By solving the equation M**x** = **b** for **x,** with mod 2 arithmetic, where M is the adjacency matrix and **b** is the initial coloring vector, any resulting solution for **x** is a vector that describes a solution to the game. When M is invertible (mod 2), this result is **unique**, because the only solution for the equation is **x** = $M^{-1}$ **b** where $M^{-1}$ is the inverse of M. The entries of **x**, a vector describing the solution to the game, will be either 0 or 1. A number 1 in the $i^{th}$ entry of the **x** will indicate to move at the $x_i$ square one time, and a 0 in the $i^{th}$ entry indicates otherwise.

To repeat, if M is invertible, then there is a solution to every game, regardless of its initial coloring. Similarly, if $M^{-1}$ does not exist, then for some initial colorings of G, there are no solutions. When $M^{-1}$ does not exist, there are columns of M whose sum (mod 2) is the zero vector, or equivalently, there are columns that are linearly dependent.

The reader will be reminded that the *rank* of M, rank(M), is the largest number of linearly independent columns that can be selected from among the columns of M. Alternatively, rank(M) is the number of vertices in G, less the maximum number of columns that can be removed from M simultaneously such that each is a linear combination or the remaining columns. There are $2^{|V(G)|}$ many initial colorings possible. But only the initial colorings that are winnable correspond to those initial coloring column vectors **b** such that M**x** = **b** has at least one solution for **x**. That is, b corresponds to a winnable initial coloring if and only if **b** belongs to the span of the columns of M. When one selects any rank(M) many linearly independent columns, the number of winnable initial colorings is $2^{\text{rank(M)}}$. There are $2^{\text{rank(M)}}$ winnable initial colorings because of the possible linear combinations of

$$a_1\mathbf{c}_1 \oplus a_2\mathbf{c}_2 \oplus \ldots \oplus a_r\mathbf{c}_r$$

Here each $a_i$ is either 1 or 0, $\boldsymbol{c}_i$ is the $i^{th}$ linearly independent column, and $r$ is rank(M).

So, the proportion of winnable initial configurations out of all initial configurations is

$$\frac{2^{\text{rank(M)}}}{2^{|V(G)|}}$$

or equivalently,

$$(1/2)^{|V(G)|\text{-rank(M)}}$$

Note that $|V(G)|$ - rank(M) is what is called the *nullity*, or *co-rank*, of M. This is the same as the number of all-zero rows in the row reduced echelon form of M. Once M has been row reduced to its echelon form, one can simply count its number of all zero columns, thus making inspection a viable method for researching.

In summary, randomly selected initial colorings have probability $(1/2)^{\text{nullity(M)}}$ of being winnable. This is why we are led to be interested in determining the nullity of M for various graphs corresponding to various move shapes on a grid of squares. If one can find linear dependencies among the column vectors, then one can demonstrate that the number of winnable colorings decreases. This is because the linear dependencies demonstrate that the nullity of M is larger than what it previously was known to be.

Related to the concept of the nullity of M, is that of an even-dominating set. An *even-dominating set*, or EDS of a graph, is a collection of vertices such that when all of these vertices are moved at, there is no net effect on the graph's coloring. The empty set is the trivial even-dominating set. One possible way to generate EDSs is by inspection of the graph. In order to generate EDSs by inspection, more terminology is needed. An element $x$ of V(G) is *dominated* by an element $y$ of V(G) if moving at $y$ changes $x$'s status. Dominance is a symmetric property. An element $x$ of V(G) is *even-dominated* by a subset E of V(G) if the number of elements of E that dominate $x$ is even. If every element x of V(G) is even-dominated by a subset E of V(G), then E is an EDS. So, by checking that every element of V(G) is even-dominated by E, one can verify that E is an EDS. Figures 10 and 11 are EDSs of a 5 x 5 grid. An interested reader may verify this.

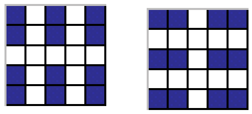


Figure 10

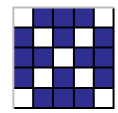


Figure 11

In describing EDSs, it is important to note that the collection of EDSs forms a vector space; it is closed under addition and scalar multiplication (the latter being trivial, since 0 and 1 are the only scalars!). The dimension of this vector space is the nullity of M and is also the number of linearly independent EDSs that span the space of all EDSs for the graph. Note that only two of the three EDSs in Figures 10 and 11 are basis elements. That is, any third EDS is a linear combination of the other two. Because the sum of the rank of a game's adjacency matrix and the dimension of the EDS vector subspace is $|\mathrm{V(G)}|$, if one knows any two of these parameters of the game, then one also knows the third.

This paper uses both inspection and linear algebra to produce EDSs of graphs so that patterns of winnability of games with these graphs can be analyzed. For a rigorous explanation of even-dominating sets, the reader is referred to [2].

## 2. Findings

Let G be a graph with $x \in$ V(G). There is a specified description for the set of all vertices whose status changes when one moves at x. In the original Lights Out game, moving at $x$ affected $x$ and its non-diagonal adjacent squares. This paper only discusses move shapes where adjacent squares of $x$ are affected. Because of the restriction on the location of squares that are in a move shape, a $3 \times 3$ grid illustrates a move shape. When numbered as in Figure 12, one can see that in the original Lights Out game, moving at square 5 affects squares 2, 4, 5, 6, and 8. The reader is reminded that the original Lights move shape is also illustrated in Figure 5. A move shape is also described as a vector **p**, where $p_i$, the $i^{th}$ square in the move shape grid, is denoted by the number 1 if and only if the $i^{th}$ square is affected in the move shape. Otherwise, the $p_i^{th}$ entry is denoted by the number 0. If H is an $m \times n$ grid, with $m \leq$ n, then (H, **p**) is called a puzzle space.

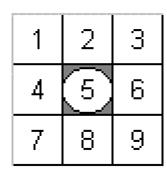


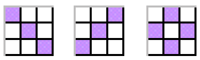

Figure 12

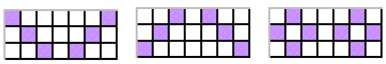
The move shapes studied in this paper are symmetric about the diagonal, thus forcing the winnability patterns of a puzzle space with an $m \times n$ grid to be the same as those of a puzzle space with an $n \times m$ grid. How does a puzzle space's move shape affect its winnability? How do the values of $m$ and $n$ affect a puzzle space's winnability? From these questions, we attain the following findings:

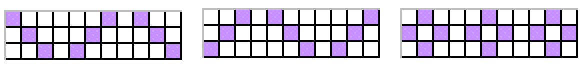
**Conjecture 1:** Let (H,**p**) be a puzzle space where **p** = (0,1,0,1,0,1,0,1,0), $m$ is odd and $n$ = 2k +1 for some k ∈ **N**. If S is a basis for the space of EDSs of (H,**p**), then

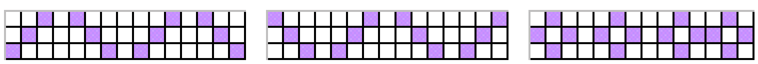
$$|S| = \begin{cases} m & \text{if } 2k+1 = \ell(m+1)-1, \text{ for some } \ell \in \mathbf{N} \\ 1 & \text{otherwise} \end{cases}.$$

Comment: It appears that the basis elements of the set of all EDSs are of the following pattern:

$3 \times 3$ 

$3 \times 7$ 

$3 \times 11$ 

$3 \times 15$ 

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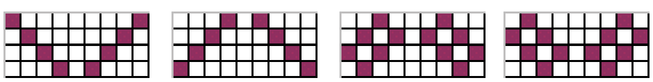
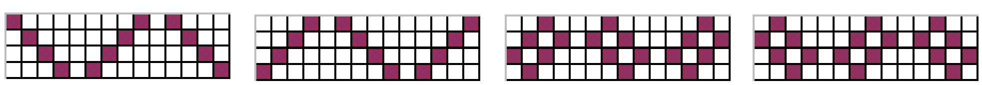
$4 \times 9$ 

$4 \times 14$ 

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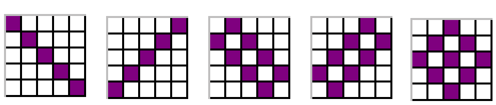
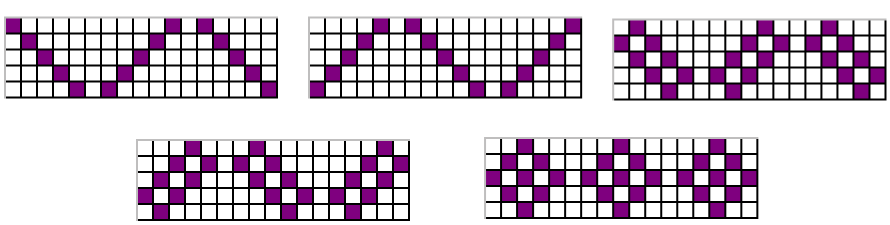
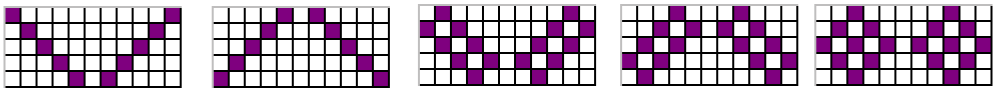
$5 \times 5$ 

$5 \times 11$ 

$5 \times 17$ 

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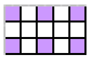
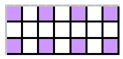
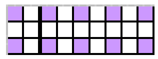
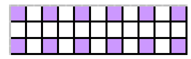
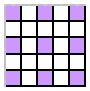
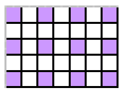
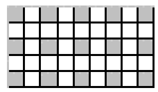
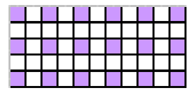
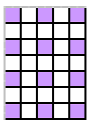
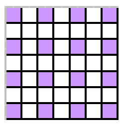
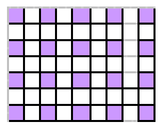
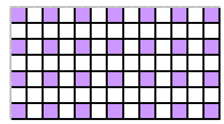
$3 \times 5$  $3 \times 7$  $3 \times 9$  $3 \times 11$ 

$5 \times 5$  $5 \times 7$  $5 \times 9$  $5 \times 11$ 

$7 \times 5$  $7 \times 7$  $7 \times 9$  $7 \times 11$ 

**Conjecture 2:** Let (H,**p**) be a puzzle space where **p** = (0,1,0,1,0,1,0,1,0), $m$ is even and $n = k(m+1) - 1$ for some $k \in \mathbf{N}$. If S is a basis for the space of EDSs of (H,**p**), then $|S| \geq n$ .

**Theorem 3:** *Let (H,**p**) be a puzzle space where **p** = (1,0,0,0,0,0,0,0,0). If S is a basis for the space of EDSs of (H,**p**), then* $|S| = m + n - 1$.

**Proof:** Let H be an $m \times n$ graph with a move shape of **p** = (1,0,0,0,0,0,0,0,0). The adjacency matrix M, for H is of the form

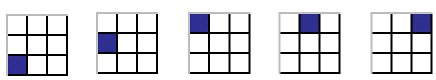
$$\begin{bmatrix} \mathbf{0} & \mathbf{I'} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I'} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I'} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I'} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \mathbf{I'} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

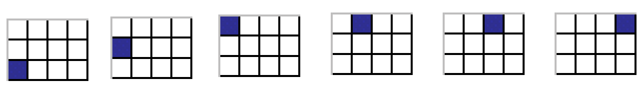
$$\text{where } \mathbf{I'} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and } \mathbf{0'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

both are $n \times n$ matrices.

Notice that M is in row-reduced echelon form. Since $|S|$ is also the nullity of M, $|S|$ can be calculated by counting the number of all- zero rows in M. There are $mn$- $(m-1)n$ all- zero rows in the bottom most rows of M, and each nth row is also all- zero. Therefore, $|S| = n + (m-1)$ as required. □

Comment: The basis elements of the set of all EDSs are of the following pattern:

3×3 

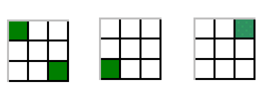
3×4 

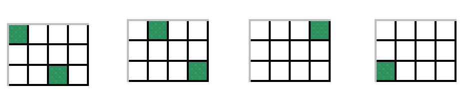
and so on . . .

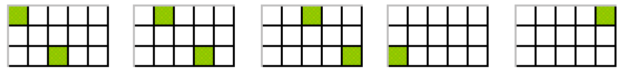
**Conjecture 4:** Let (H,**p**) be a puzzle space where **p** = (1,0,0,0,0,0,0,0,1), $m = 3k_1 + r_1$, and $n = 3k_1 + r_1$ , for some $k_i$, $r_i \in \mathbf{N}$ with i $\in$ {1, 2}. If S is a basis for the space of EDSs of (H,**p**), then

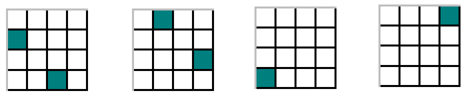
$$|S| = \begin{cases} k_1 + k_2 + 1 & \text{if } r_2 = 0 \text{ or } 2 \\ k_1 + k_2 + 2 & \text{if } r_2 = 1 \end{cases} .$$

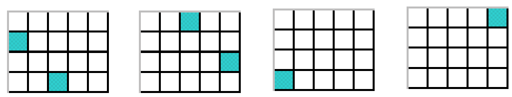
Comment: It appears that the basis elements of the set of all EDSs are of the following pattern:

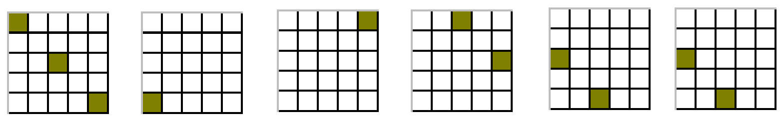
3×3 

3×4 

3×5 

4×4 

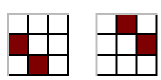
4×5 

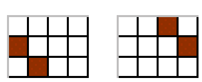
5×5 

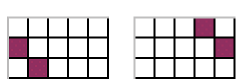
**Conjecture 5:** Let (H,**p**) be a puzzle space where **p** = (1,0,0,0,1,0,0,0,1), $m = 3k_1 + r_1$, and $n = 3k_1 + r_1$ , for some $k_i$, $r_i \in \mathbf{N}$ with $i \in \{1, 2\}$. If S is a basis for the space of EDSs of (H,**p**), then

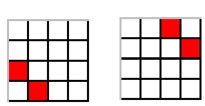
$$|\mathrm{S}| = \begin{cases} \dfrac{2m}{3} & \text{if } m \equiv 0 \pmod 3 \\ \dfrac{2(m-1)}{3} & \text{if } m \equiv 1 \pmod 3 \\ n-2 & \text{if } m \equiv 2 \pmod 3 \end{cases} .$$

Comment: It appears that the basis elements of the set of all EDSs are of the following pattern:

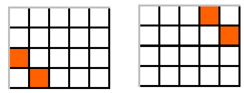
3×3 

3×4 

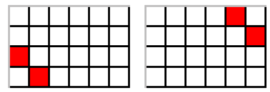
3×5 

4×4 

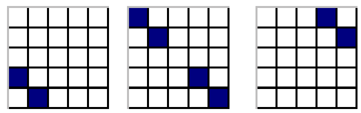
$4 \times 5$



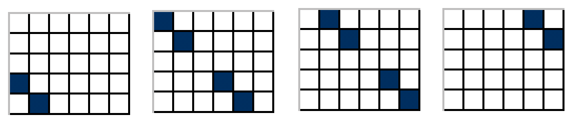
$4 \times 6$



$5 \times 5$



$5 \times 6$



and so on. . .

**Conjecture 6:** Let (H,**p**) be a puzzle space where **p** = (1,1,0,1,0,1,0,1,1) and let (H,**q**) be a puzzle space where **q** = (0,1,0,1,0,1,0,1,0) If S is a basis for the space of EDSs of (H,**p**), and T is a basis for the space of EDSs of (H,**q**), then |S| = |T|.

**Conjecture 7:** Let (H,**p**) be a puzzle space where **p** = (1,1,1,1,1,1,1,1,1) and S is a basis for the space of EDSs of (H,**p**).

If $m \equiv 2 \pmod 3$, then

$|S| = n$ if $n \equiv 0$ or $1 \pmod 3$

$m + n - 1$ if $n \equiv 2 \pmod 3$

If $m \equiv 0$ or $1 \pmod 3$ and $n \equiv 2 \pmod 3$ and $n \equiv 2 \pmod 3$, then $|S| = n$.

If $m \equiv 0$ or $1 \pmod 3$ and $n \equiv 0$ or $1 \pmod 3$ and $n \equiv 2 \pmod 3$, then $|S| = 0$.

**Conjecture 8:** Let (H,**p**) be a puzzle space where **p** = (1,0,1,0,0,0,1,0,1) and S is a basis for the space of EDSs of (H,**p**). If $m$ is odd and $n$ is odd, then $|S| = n + 2$. If $m$ is odd and $n$ is even, then $|S| = n$. If $m$ is even and $n$ is odd, then $|S| = m$. If $m$ is even and $n$ is even, then $|S| = 0$.

**Theorem 8:** *Let (H,**p**) be a puzzle space where **p** = (1,0,1,0,0,0,1,0,1). If S is a basis for the space of EDSs of (H,**p**), then |S| = m.*

**Proof:** Let H be an $m$ x $n$ grid with a move shape of **p** = (0,1,0,1,0,0,0,0,0,). The adjacency matrix M, for H is of the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{I'} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I'} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I'} & \mathbf{I} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I'} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{I} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \mathbf{I'} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{I'} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{0'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and **I**, the identity matrix, are all $n \times n$ matrices.

A diligent reader can verify that the only rows of M that are linearly independent are as follows:

The $mn^{th}$ row is the zero vector

The $(m-1)n^{th}$ row is the $mn - 1^{st}$ row

The $(m-2)\,n^{th}$ row is the sum of the $mn - n - 1^{st}$ row and the $mn - 2^{nd}$ row

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

The $(m-k)\,n^{th}$ row is the sum of the $(mn - kn) + (n-1)^{th}$ row and the $(mn - kn) + 2(n-1)^{th}$ row and the . . . and the $(mn - kn) + k(n-1)^{th}$ rows, where $k \in \{1, 2, \ldots, m\}$.

Since there are at least m linearly dependent vectors of M, rank(M) $\leq mn - m$. Let M* denote the matrix formed from M by deleting rows numbered

$$mn,\ (m-1)n,\ (m-2)n,\ \ldots,\ n.$$

Since M* is an upper triangular matrix, rank(M*) equals the number of rows, which is $mn - m$. This shows that because rank(M) $\geq$ rank(M*) $= mn - m$, rank(M) $\geq mn - m$. This forces rank(M) $= mn - m$ and therefore, $|S| = m$ as required. □

**Theorem 10:** *Let (H,**p**) be a puzzle space where **p** is symmetric about its left diagonal, and has $p_5 = 1$, $p_6 = p_7 = 0$. If S is a basis for the space of EDSs of (H,**p**), then $|S| = 0$.*

**Proof:** We will exhibit an algorithm that can be used to solve every initial coloring of H.
Step 1. Locate the lowest, right-most lit square and move at it.
Step 2. Repeat Step 1 until the game has been won.

**Corollary 11:** *Let (H,**p**) be a puzzle space where S is a basis for the space of EDSs of (H,**p**). If **p** = (1,1,0,0,1,0,0,0,0), **p** = (0,1,0,1,1,0,0,0,0), **p** = (0,0,0,0,1,0,0,0,0), **p** = (1,1,0,1,1,0,0,0,0), or **p** = (1,0,0,0,1,0,0,0,0), then $|S| = 0$.* ▯

## 3. Discussion

The ultimate goal of this paper has been to introduce the subject of move shape variations. This paper motivates a discussion about the connection between move shapes. What does it mean to have two move shapes produce the same patterns of winnability? When two move shapes are the mirror images of each other, how are their patterns of winnability similar?

It can be shown that the move shapes form a group under °, the operation of composition. If **p** and **q** are two moves shapes that are symmetric about the diagonal, then $\mathbf{p} \circ \mathbf{q} = \mathbf{p} \oplus \mathbf{q}$, where the $i^{th}$ element of $\mathbf{p} \oplus \mathbf{q}$ is the mod 2 sum of the $i^{th}$ elements of **p** and **q**. The identity element for the group is the zero vector, and the inverse of a move shape **p** is **p**, itself. Is there a correlation between different groups of move shapes? What is the connection between move shapes that are composites of other move shapes? These questions and a bevy of others can be investigated as part of a continuation of this research.

## 4. Conclusion

The results presented here have several salient products. The applications of the linear algebraic aspect of this research are obvious, and have, to some extent, been used. However, these results also have relevance to other aspects of teaching.

Pattern recognition is a necessary skill for any age. One can use the patterns found in EDSs to aid in pattern recognition and description. A more sophisticated approach towards EDS pattern evaluation can be an exercise in writing and understanding mathematical notation. The Lights Out game is interesting and can be a unique motivation for one to gain better facilitation with describing sets and performing linear algebra.

Every child is introduced to the notion of modular arithmetic when he or she learns how to tell time. The results of this paper force one to become acquainted with another concrete example of the usefulness of modular arithmetic.

The moves required to win a game are commutative and associative, in that the order in which the moves are performed is unimportant. This aspect of the game clarifies some of the basic abstract qualities of algebra. Therefore, the game can be used to illustrate elementary concepts in a lower-level undergraduate mathematics class.

Along with the classroom applications, this paper can also be used to introduce the complexities associated with error-correcting codes.

The results of this paper can motivate a deeper investigation of the essence of the problem: how does changing the parameters of the Lights Out game affect it? There are definitely connections between move shapes and winnability, some of which have been illustrated in this paper. The next logical step in researching the game would include a careful investigation of the algebraic properties of the different puzzle spaces, and a venture into the winnability of games plated on other graphs, such as Cartesian products of "MOPs" and grids.

It is also appropriate to discuss suggestions for further attempts of similar problem solving. While Matlab Software program was indispensable for the research conducted for this paper because it performed basic matrix manipulations in mod 2, further research would require a superior computer program to achieve results that are more sophisticated. If one could write a computer program that would be able to generate nested matrices, row-reduce these matrices into echelon form, and compute inverse matrices using any modular arithmetic, then one could research in a more efficient manner.

The results of this paper highlight the intricacies of the original Lights Out game because they show the drastic differences that result from changing certain variables.

## 5. Acknowledgements

Dr. Dan Pritikin managed to do the impossible, make me eat my words! With his help, I *have* become very interested in this subject and *will* continue to research and improve my findings. For this, I thank him. I also am grateful to Professor Moira Miller for her diligent guidance and aid. And, kudos for Charles Hague, graduate assistant, who helped me as well. Finally, I would like to thank Dr. Stephen Wright, for without his program for rref calculations, (mod 2), I would not have been nearly as successful in my endeavors.

## 6. References

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[4] Jaap's Puzzle Page, http://www.org2.com/jaap/puzzles/lights.htm#variants .