

Average Reward Adjusted Deep Reinforcement Learning: Near-Blackwell-Optimal Policies for Order Release

Manuel Schneckenreither¹, Stefan Haeussler¹, and Juanjo Peiró²

¹ Department of Information Systems, Production and Logistics Management,
University of Innsbruck, Austria

`manuel.schneckenreither@uibk.ac.at, stefan.haeussler@uibk.ac.at`

² Departament d'Estadística i Investigació Operativa, Universitat de València, Spain
`juanjo.peiro@uv.es`

Abstract. [MS: proof-of-concept work] An essential task in manufacturing planning and control is to determine when to release orders to the shop floor. One key parameter is the lead time which is the planned time that elapses between the release of an order and its completion. Lead times are normally determined based on the actual duration orders previously took to traverse through the production system (flow times). Traditional order release models assume static lead times, although it has been shown that they should be set dynamically to react the dynamic operational characteristics of the system. Therefore, we present an order release model which sets lead times dynamically by using an reinforcement learning approach. Therefore, we provide an in-depth analysis of reinforcement learning to show that average reward reinforcement learning is a better approach for operations research applications as discounted reinforcement learning. Additionally we present an average reward reinforcement learning algorithm which infers near-Blackwell-optimal policies. We use a simulation model of a [MS: todo] two-stage flow-shop to compare the algorithm to well-known order release mechanisms. We show that in the current version our proposed model using reinforcement learning outperforms some, but not all other tested approaches.

Keywords: operations research, production planning, order release, machine learning, reinforcement learning

[MS: ideas: Similar Selçuk (2013) detect that a high update frequency has a negative effect and thus should be reduced. They conclude that the lead times ought to control the WIP level, instead of the due date reliability as changes are directly seen (Knollmann and Windt, 2013). For a review of analytical works in the context of the LTS see Bendul and Knollmann (2016).]

1 Introduction

An important goal in Manufacturing Planning and Control (MPC) systems is to achieve short and predictable flow times, especially where high flexibility in

meeting customer demand is required. Besides achieving short flow times, one should also maintain high output and due-date performance while keeping the work-in-process level low. One approach to address this problem is to collect all incoming orders in an order-pool and periodically decide which orders to release to the shop floor. Once orders are released, costs start to accumulate as planned orders materialise as actual jobs in the production system. The main challenge is to find a good compromise of balancing the shop floor and timely completion of jobs. Although the performance of such systems can be measured manifold the most overarching objective is to maximise profits by adequately assigning holding and lateness costs.

One of the key modeling parameters for order release mechanisms is the *lead time*, which refers to the planned time that elapses between the release of an order and its arrival in the finished goods inventory. This planning parameter is often based on the observable time an order needs to traverse the production system, which in contrast is denoted as the *flow time*. Flow times consist of processing, setup, control, transport, and waiting times, whereas the latter is the governing factor (Zäpfel, 1982, p.223). Waiting times are a result from queuing (e.g. jobs queue before and after processing), depend heavily on the amount of jobs in the system (WIP) and thus are relatively difficult to estimate, which makes the setting of favorable lead times so difficult (e.g., Tatsiopoulos and Kingsman 1983; Wiendahl 1995). Most state-of-the-art order release mechanisms use static or fixed lead times to address the order release problem, and thus neglect the nonlinear relationship between resource utilisation and flow times, which is well known from practice and queuing theory (Pahl et al. 2007).

One way to address this nonlinear interaction effects is to set lead times dynamically. Intuitively the order release problem is solved by perfectly matching the lead times to the flow times, but the corresponding optimisation problem faces “sampling issues” meaning that the flow times depend on the lead times. An extreme scenario of this problem is the so called “lead time syndrome”, which describes a vicious cycle where increasing flow times perpetually inflate the lead times which leads to worse performance (see e.g., Knollmann and Windt 2013; Mather and Plossl 1978; Selcuk et al. 2006). Thus, setting lead times dynamically harbors optimisation potential (Hoyt, 1978), but may also substantially degrade the system performance. Schneckenreither et al. (2020) have established following categorisation of dynamic lead time management approaches:

- **Reactive lead time management** approaches set lead times by reacting on earlier flow times (e.g., Enns and Suwanruji 2004; Selcuk et al. 2006). Note that the forecast is always based on *past* data as the most recent system changes cannot be reflected by flow times until the corresponding orders arrive in the FGI, which might take several periods.
- **Proactive lead time management** may incorporate *past* data in conjunction with the *current* system state to set lead times (e.g., Bertrand 1983; Chung and Huang 2002). Put differently, these methods aim to find a function that provides lead times based on the current state of the production system and possibly information from the past.

- Predictive lead time management may not only incorporate *past* data and the *current* system state to set lead times, but also utilises the anticipated *future* system state to detect arising issues of future periods and react accordingly (e.g. Paternina-Arboleda and Das 2001; Schneckenreither 2019; Schneckenreither et al. 2020). Thus, it extends proactive lead time management from a flow time forecasting or simple lead time setting technique to a lead time management approach that integrates the future behaviour of the system when setting lead times. This allows reasoning of the system dynamics, as for instance triggering the lead time system, and thus such an algorithm can react accordingly to find a more farsighted optimal lead time update.

We hypothesis that dynamic lead time management approaches need to aim for a predictive design in order to be able to compete with state-of-the-art order release methods from literature. However, there only exist three papers that propose a predictive lead time management algorithm in literature.

The first approach by Paternina-Arboleda and Das (2001) introduces a predictive order release model by using reinforcement learning in a single product, four-station serial flow line and compare its performance (WIP costs) with conventional order release policies (e.g., Kanban and CONWIP). The algorithm of Paternina-Arboleda and Das (2001) decides on whether or not to release an order after each system change, the completion of an operation of any order at any stage or a new order arrival, and assumes that any unsatisfied demand is lost. Thus, they use a continuous order release method although in practice order release decisions often need to be made on a periodical basis, e.g. daily (see Enns and Suwanruji 2004; Gelders and Van Wassenhove 1982). They outperform existing control policies with their tabular based reinforcement learning agent. Then, Schneckenreither (2019) use several different reinforcement learning algorithms to make periodic order release decisions for a flow shop production system. The algorithm directly sets lead times for each product type, which are then used to release the orders in the order pool. They show that their approach outperforms static order release mechanisms by yielding lower costs, lateness and standard deviation of lateness, but conclude that research using average reward reinforcement learning methods harbor optimisation potential for the order release problem. And finally, Schneckenreither et al. (2020) present a flow time estimation procedure to set lead times dynamically using an artificial neural network, which is used to forecast flow times. By implementing a rolling horizon order release simulation of the proceeding periods they lift their approach to a predictive lead time management approach, which is able to detect backorders of future periods and reacts by releasing orders earlier. Nonetheless, their method is unable to foresee the triggering of the lead time syndrome and therefore they introduce an upper lead time bound to prevent the negative effects of the lead time syndrome. Their model outperforms static and reactive lead time management approaches, especially for scenarios with high utilisation and high variability in processing times.

As reinforcement learning stems from dynamic programming the future system state is by design considered as a main driver of decision making in the

current period. Its goal is to find the best stationary policy for a given problem. The advantage of reinforcement learning over dynamic programming is that (i) the problem space is explored by an agent and thus only expectantly interesting parts of the problem space need to be assessed and (ii) the knowledge (acquisition) of transition probabilities becomes unnecessary as the states are evaluated by consecutively observed states solely.

Over the past decades reinforcement learning has been applied to various problems, for which astonishing results have been reported. E.g. only recently Mnih et al. (2015) have presented a novel value-iteration reinforcement learning agent which exceeds human-level abilities in playing many classic Atari 2600 games (Bellemare et al., 2012). Further, Mnih et al. (2016) present improved results with asynchronous actor-critic reinforcement learning. Also games like Go (Silver et al., 2016) and Chess (Silver et al., 2017) have been mastered with superhuman performance by *tabula rasa* reinforcement learning agents. Furthermore, the method has also been applied in the setting of manufacturing system, e.g. to improve the ramp-up process (Doltsinis et al., 2012), in locally selecting appropriate dispatching rules (Wang and Usher, 2005; Zhang and Dietterich, 1995) or scheduling (Waschneck et al., 2018; Zhang and Dietterich, 1995). However, all these applications use discounted reinforcement learning and are either designed to investigate a rather simple MDP, e.g. by selecting heuristics instead of optimising the underlying problem itself, or by mapping the actual objective in a reward function that is approximately 0 on average over time. This is due to the fact that state value is largely composed of a term defined by the policies' average reward value (Blackwell, 1962; Miller and Veinott, 1969) which would otherwise dilute the state values and thus decrease the solution quality, as can for instance be observed in Schneckenreither and Haeussler (2019) and Gijbrecchts et al. (2018).

Therefore, most applications that incorporate and directly reflect costs or profit in the reward function use average reward reinforcement learning. Aydin and Öztemel (2000) use it in the setting of scheduling, while in a series of papers Mahadevan et al. investigated several problem domains starting with simple MDPs (Mahadevan, 1996b,c). After these foundational works for average reward reinforcement learning they introduced a continuous time average reward reinforcement learning algorithm named SMART (Mahadevan et al., 1997). Applications of SMART reach from the optimisation of queuing systems (Mahadevan, 1996a,d), maintenance of an inventory system (Das et al., 1999) to optimising transfer line in terms of maximizing demand, while keeping inventory levels of unfinished product as low as possible (Mahadevan and Theocharous, 1998). However, in practise usually decision have to be made on a daily basis (Enns and Suwanruji, 2004; Gelders and Van Wassenhove, 1982). Therefore, we refrain from this adaption and concentrate on standard MDPs only. Furthermore, often continuous-time semi-MDP problems can be converted through uniformisation into equivalent discrete time instances (see Bertsekas et al., 1995; Puterman, 1994).

Like discounted reinforcement learning also average reward reinforcement learning is based on an oracle function, in our case the accumulated costs of a period, to assess the decisions taken by the agent. By repeatedly choosing different actions the agent examines the problem space and rates possible actions for any observed state. The advantage of average reward reinforcement learning over the widely applied discounted reinforcement learning framework is that the underlying optimisation technique is able to find better policies. This yields from the fact that in standard discounted reinforcement learning method the states are assessed independently and by a single value. To be more precise, average reward reinforcement learning splits up the evaluation of the average reward per step, a bias value that specifies the amount of collected rewards to reach the optimal path when starting in a suboptimal state and an error term which defines the number of steps to reach the optimal path (Howard, 1964; Mahadevan, 1996b; Puterman, 1994). In contrast to that in the standard discounted framework one single scalar value consisting of the addition of these subterms is estimated, where however the average reward is scaled by $1/(1-\gamma)$. The discount factor γ is usually set very close to 1, e.g. 0.99, which leads to the fact that this subterm dominates the other two. Further, in commonly applied unichain (and thus ergodic) MDPs the average reward per step is equal for all states. Thus, independently assessing it is not only computationally unwisely, but also problematic when iteratively annealed as done in reinforcement learning. Therefore, we adapt an algorithm that uses a scalar value for the estimation of the average reward over all states, and estimates for every state-action pair that incorporate the bias and error term. The later values are adjusted by the average reward.

Thus, as opposed to the commonly applied discounted reinforcement learning algorithm we use an average reward adjusted reinforcement learning algorithm to adaptively release orders based on the assessed state values of the production system. In contrast to the aforementioned works on average reward reinforcement learning our algorithm incorporates the optimisation of not only the average reward over time, but also the bias values, which is an important adaption in highly stochastic systems. Only the work by Mahadevan (1996a) integrates this second-level refinement optimisation. However, their algorithm requires the selection of a reference state to prevent an infinite increase of state values. This results from the lack of feedback in the iterative process of optimising the different refinement optimisation levels. Furthermore, they assess the average reward independently for each state.

In summary, we propose a novel average reward adjusted reinforcement learning algorithm to assess orders for their release to the shop floor. This is the first average reward adjusted reward learning algorithm that operates using a artificial neural network as function approximation. To ensure scalability we adapt a multi-agent approach, where each agent optimises the lead time management of a single product type. The agents do so by assessing the expected costs for the possible releases imposed by adapting the lead time. According to the learned estimates each agent sets a planned lead time for the corresponding product type and with that releases orders into the production system. The highly optimised

value-iteration algorithm uses improved n-step reinforcement learning with small action-based replay memories and worker agents to infer order release policies which outperform [MS: todo] .

Structure. The rest of the paper is structured as follows. The next section introduced average reward reinforcement learning and presents the used algorithm (see also Schneckenreither 2020). Section 3 describes the simulation model we use to evaluate the approach. [MS: todo]

2 Average Reward Adjusted Reinforcement Learning

This section briefly reintroduces the most important concepts of average reward adjusted reinforcement learning, elaborates on optimality criteria and provides insights of the underlying algorithm. For a more extensive introduction to average reward reinforcement learning we refer to Schneckenreither (2020).

Like Miller and Veinott (1969) we are considering problems that are observed in a sequence of points in time labeled $1, 2, \dots$ and can be modelled using a finite set of states \mathcal{S} , labelled $1, 2, \dots, |\mathcal{S}|$, where the size $|\mathcal{S}|$ is the number of elements in \mathcal{S} . At each point t in time the system is in a state $s_t \in \mathcal{S}$. Further, by choosing an action a_t of a finite set of possible actions A_s the system returns a reward $r_t = r(s_t, a_t)$ and transitions to another state $s_{t+1} \in \mathcal{S}$ at time $t + 1$ with conditional probability $p(s_{t+1}, r_t \mid s_t, a_t)$. That is we assume that reaching state s_{t+1} from state s_t with reward r_t depends solely on the previous state s_t and chosen action a_t . In other words, we expect the system to possess the Markov property (Sutton et al., 1998, p.63). Reinforcement learning processes that possess the Markov property are referred to as Markov decision processes (MDPs) (Sutton et al., 1998, p.66).

Thus, the action space is defined as $F = \times_{s=1}^{|\mathcal{S}|} A_s$, where A_s is a finite set of possible actions. A *policy* is a sequence $\pi = (f_1, f_2, \dots)$ of elements $f_t \in F$. Using the policy π means that if the system is in state s at time t the action $f_t(s)$, i.e. the s -th component of f_t , is chosen. A stationary policy $\pi = (f, f, \dots)$ does not depend on time. In the sequel we are concerned with stationary policies only.

2.1 Discounted Reinforcement Learning

In the widely applied discounted framework the value of a state $V_\gamma^{\pi_\gamma}(s)$ is defined as the expected discounted sum of rewards under the stationary policy π_γ when starting in state s . Note that the policy π_γ depends on the selected discount factor. That is

$$V_\gamma^{\pi_\gamma}(s) = \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{t=0}^{N-1} \gamma^t R_t^{\pi_\gamma}(s) \right],$$

where $0 \leq \gamma < 1$ is the discount factor and $R_t^\pi(s) = \mathbb{E}_\pi[r(s_t, a_t) \mid s_t = s, a_t = a]$ the reward received at time t upon starting in state s by following

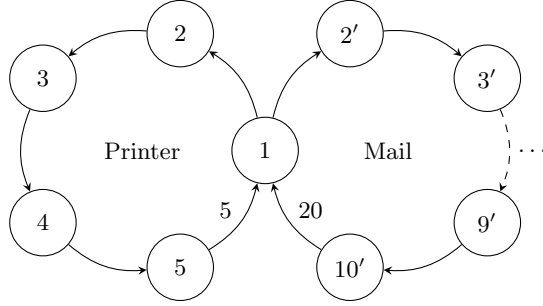


Fig. 1: A MDP with a single action choice in state 1, i.e. two different deterministic policies (Mahadevan, 1996c, adapted).

policy π (Mahadevan, 1996b). The aim in the discounted framework is to find an optimal policy π_γ^* , which when followed, maximises the state value for all states s as compared to any other policy π_γ : $V_\gamma^{\pi_\gamma^*} - V_\gamma^{\pi_\gamma} \geq 0$. This criteria is usually referred to as γ -optimality as the discount factor γ is fixed. However, this also means that the actual value set for γ determines the best achievable policy. For instance, as can be seen in Figure 1 setting $\gamma < 0.8027$ the printer-loop is preferred over the mail-loop, although the mail-loop accumulates more reward over time, i.e. selecting the mail-loop is a sub-optimal choice. Observe that the average reward received per step equals 1 for the printer-loop and 2 for the mail-loop. Thus, the (Blackwell-)optimal policy is to choose the mail-loop. However, if $\gamma < 3^{-\frac{1}{5}} \approx 0.8027$ an agent using discounted reinforcement learning chooses the printer loop as going the printer loop has a higher evaluation.

Therefore, in the seek of a more general optimality criteria for determining the best stationary policy π for a given problem, we introduce n -discount-optimality.

Definition 1 (n -Discount-Optimality). *Due to Veinott (1969) for a MDP a policy π^* is n -discount-optimal for $n = -1, 0, 1, \dots$ for all states $s \in \mathcal{S}$ with discount factor γ if and only if*

$$\lim_{\gamma \rightarrow 1} (1 - \gamma)^{-n} (V_\gamma^{\pi^*}(s) - V_\gamma^\pi(s)) \geq 0.$$

Only if a policy is n -discount-optimal for all $n < m$ it can be m -discount optimal (Puterman, 1994; Veinott, 1969). If a policy is -1 -discount-optimal it is called gain-optimal, if it is 0-discount-optimal it is also called bias-optimal. Furthermore, if a policy is ∞ -discount-optimal then it is said to be Blackwell-optimal (Blackwell, 1962). That is, for Blackwell-optimal policies π^* there exists a discount factor $\gamma^* < 1$ such that $V_\gamma^{\pi^*}(s) \geq V_\gamma^\pi(s)$ for all $\gamma \geq \gamma^*$ and under all policies π (Blackwell, 1962; Mahadevan, 1996d). Informally that means there exists a discount factor $\gamma < 1$ which finds Blackwell-optimal policies. However, (i) in more complex, i.e. real world, MDPs this value can be arbitrary close to 1 and (ii) the difference in state values may be very small, which (iii) due to the need of state value function approximation likely causes errors when choosing

among different actions in the discounted framework. Schneckenreither (2020) establishes a practically relevant definition for near-Blackwell-optimality.

Definition 2. *If an algorithm infers for any MDP bias-optimal policies, and for a given MDP can in theory be configured to infer Blackwell-optimal policies, but in practise this ability is naturally limited due to the finite accuracy of floating-point representation of modern computer systems, it is said to be near-Blackwell-optimal. An according to a near-Blackwell-optimal algorithm inferred Blackwell-optimal policy is called near-Blackwell-optimal.*

Note that standard discounted RL does not meet the requirements of near-Blackwell-optimality, as it generally does not infer bias-optimal policies. Only by chance the resulting policy could be bias-optimal. The corresponding Bellman equation defined as $V_{\gamma}^{\pi}(s) = \mathbb{E}_{\pi}[r + \gamma V_{\gamma}^{\pi}(s')]$ (see e.g. Sutton et al., 1998, p.70) provides a way to assess the state values using two consecutively observed states s, s' and the observed reward r returned by the system. When this formula is used as an update rule it provides an algorithm that converges the state values by iteratively adapting the state value estimates. There are four major issues with discounted RL, which results in the fact that it is inapplicable for many problems of operations research.

1. In general standard discounted RL can only infer suboptimal policies as the discount factor is strictly less than one, i.e. $\gamma < 1$.
2. It is very difficult to specify a desired balance between the short-term and long-term (average) rewards. This results from the fact that the long-term rewards are scaled exponentially, where the factor depends on the chosen γ value.
3. Episodic MDPs with an average reward per step that is non-zero cannot be solved correctly, as the average reward is ignored in the terminal states.
4. The average reward is independently assessed for each state, even though it is the dominating part for all state values and thus is shared between more than one or, for unichain MDPs, even all states. This usually leads to an exponentially increased number of learning steps required for policies to converge.

Especially due to the fourth issue standard discounted RL is not well applicable in the operations research domain. This mainly results from the fact that a continuous assessing of the state values using costs or profit is often required, which leads to intractable long learning phases with high exploration rate even for small sized problems. But as high exploration rates produce errors in the state value estimations (Miller and Veinott, 1969) finding a well working parameterisation is difficult. Average reward reinforcement learning overcomes these issues by separately assessing the subterms of the state values directly.

2.2 Average Reward Reinforcement Learning

Due to Howard (1964) the average reward $\rho^\pi(s)$ of a policy π and a starting state s is defined as

$$\rho^\pi(s) = \lim_{N \rightarrow \infty} \frac{\mathbb{E}[\sum_{t=0}^{N-1} R_t^\pi(s)]}{N}.$$

In the common case of unichain MDPs, in which only a single set of recurrent states exists, the average reward $\rho^\pi(s)$ is equal for all states s (Mahadevan, 1996b; Puterman, 1994). In the sequel we focus on unichain MDPs in this work and thus may simply refer to it as ρ^π . A policy π^* that maximises the average reward $\rho^{\pi^*}(s) - \rho^\pi(s) \geq 0$ in every state s as compared to any other policy π is called gain-optimal. Mahadevan (1996a) shows that n -discount-optimality for $n = -1$ describes gain-optimality.

Further, for an unichain aperiodic³ MDP problem, such as ergodic MDPs are, the average adjusted sum of rewards or bias value is defined as

$$V^\pi(s) = \lim_{N \rightarrow \infty} \mathbb{E}[\sum_{t=0}^{N-1} (R_t^\pi(s) - \rho^\pi)] ,$$

where again $R_t^\pi(s)$ is the reward received at time t , starting in state s and following policy π . Note that the bias values are bounded due to the subtraction of the average reward. Thus the bias value can be seen as the rewards that additionally sum up in case the process starts in state s . A policy π^* that is gain-optimal is also bias-optimal, if it maximises the bias values $V^{\pi^*}(s) - V^\pi(s) \geq 0$ in every state s and compared to every other policy π . Bias-optimality is given by setting $n = 0$ in the definition of n -discount-optimality (Mahadevan, 1996a).

Especially for highly probabilistic systems bias-optimality is important. To clarify this consider Figure 2 which again consists of two possible deterministic policies with the only choice in state 1. Both policies have the same average reward of 1. However, only taking the A loop is bias-optimal, as its policy π_A leads to bias values $V^{\pi_A}(1) = 0.5$, $V^{\pi_A}(0) = -0.5$ and $V^{\pi_A}(2) = 1.5$, while policy π_B which selects the B loop generates bias values $V^{\pi_B}(1) = -0.5$, $V^{\pi_B}(0) = -1.5$ and $V^{\pi_B}(2) = 0.5$. This yields from the fact that the actions with non-zero rewards are selected earlier in policy π_A . Consider starting in state 1. Under the policy π_A the reward sequence is $(2, 0, 2, 0, \dots)$, while for the other policy π_B it is $(0, 2, 0, 2, \dots)$.

2.3 The Laurent Series Expansion of Discounted State Values

Miller and Veinott (1969) established the link between discounted RL state values $V_\gamma^\pi(s)$ and average reward RL values $\rho^\pi(s)$ and $V^\pi(s)$ using the Laurent

³ In the periodic case the Cesaro limit of degree 1 is required to ensure stationary state transition probabilities and thus stationary bias values (Puterman, 1994). Therefore to ease readability we concentrate on unichain and aperiodic MDPs.



Fig. 2: A MDP with two gain-optimal deterministic policies, π_A going left in 1 is also bias-optimal, while π_B is not (Adapted from Mahadevan, 1996b).

series expansion as

$$V_\gamma^\pi(s) = \frac{\rho^\pi(s)}{1-\gamma} + V^\pi(s) + e_\gamma^\pi(s),$$

where the error term $e_\gamma^\pi(s)$ exists of infinitely many subterms and Puterman (1994) shows that $\lim_{\gamma \rightarrow 1} e_\gamma^\pi(s) = 0$. Note how the first term depending on the average reward $\rho^\pi(s)$ converges to infinity as γ increases.

However, an important insight is the connection between n -discount-optimality and the Laurent series expansion of $V_\gamma^\pi(s)$. Each addend corresponds to one step in n -discount-optimality. That is for (-1) -discount-optimality the addend describes (a scaled version of) the term to maximise for gain-optimality, for 0-discount-optimality the term to maximise for bias-optimality, and finally n -discount-optimality for $n \geq 1$ requires to maximise the error term $e_\gamma^\pi(s)$. The later only exists as γ is strictly less than 1. Thus this term incorporates the number of expected steps and their corresponding reward on the path to the Blackwell-optimal policy. Put differently, it optimises the expected reward collected to reach the Blackwell optimal policy according to the occurrence on the paths, where shorter paths and those which collect the rewards sooner are preferred.

These addends, where $n = -1, 0, \dots$ denote the coefficients of the Laurent series expansion and thus correspond to the n of the definition of n -discount-optimality, can be reformulated to following constraint problem (Miller and Veinott, 1969; Puterman, 1994, p.346).

$$\rho^\pi(s) - \mathbb{E}[\rho^\pi(s)] = 0 \quad \text{for } n = -1 \quad (1)$$

$$\rho^\pi(s) + V^\pi(s) - \mathbb{E}[V^\pi(s)] = R^\pi(s) \quad \text{for } n = 0 \quad (2)$$

$$W_{n-1}^\pi(s) + W_n^\pi(s) - \mathbb{E}[W_n^\pi(s)] = 0 \quad \text{for } n \geq 1, \text{ where } W_0^\pi(s) = V^\pi(s) \quad (3)$$

The error term $e_\gamma^\pi(s)$ is given by $e_\gamma^\pi(s) = \sum_{n=1}^{\infty} W_n^\pi(s)$ and the expected reward $R^\pi(s) = \mathbb{E}_\pi[r(s, a)]$. Puterman (1994, p.343ff) shows that due to the given degree of freedom if $n = -1, 0, \dots, M$ constraints are satisfying the above conditions for all states s , then only $\rho^\pi(s), V^\pi(s), W_1^\pi, \dots, W_{M-1}^\pi$ are unique, whereas W_M^π is offset by the vector u where for the transition probability matrix P the vector u is characterised by $(I - P)u = 0$. Note that u is determined by the number of closed irreducible classes of P , that is for ergodic MDPs u is determined by a single constant. Average reward learning is based on the above formulation.

A major problem occurring at average reward RL is that the bias values are not uniquely defined without solving the first set of constraints defined by the error term addends (see Mahadevan, 1996d; Puterman, 1994, p.346). Our algorithm, based on the tabular version of Schneckenreither (2020), does not require the exact solution for $V^\pi(s)$, but a solution which is offset suffices. Clearly this observation reduces the required iteration steps tremendously as finding the exact solution, especially for large discount factors, is tedious. Therefore, we allow to set $\gamma = 1$, which induces $X_\gamma^\pi(s) = V^\pi(s) + u$, where u is for unichain MDPs a scalar value independent of s , i.e. equivalent for all states of the MDP (Puterman, 1994, p.346). If we are interested in correct bias values, i.e. γ is sufficiently close but strictly less than 1, our approach is a tremendous advantage over average reward RL as it reduces the number of iterative learning steps by requiring only a single constraint per state plus one for the scalar average reward value. That is, for an MDP with N states only one more constraint ($N + 1$) has to be solved in ARAL as compared to (at least) $2N + 1$ nested constraints for average reward RL. Therefore, it is cheap to compute $X_\gamma^\pi(s)$, while it is rather expensive to find the correct values of $V^\pi(s)$ directly, especially in an iterative manner as RL is.

2.4 Average Reward Adjusted Reinforcement Learning

Thus, average reward adjusted RL can be seen as a specialised version of average reward RL, where it targets unichain MDPs only. In the sequel we briefly present the needed formalism of average reward adjusted RL. For more comprehensive version we refer to Schneckenreither (2020).

Definition 3. *The average reward adjusted (discounted) state value $X_\gamma^\pi(s)$ of a state s under policy π and with discount factor $0 \leq \gamma \leq 1$ is given as $X_\gamma^\pi(s) := V^\pi(s) + e_\gamma^\pi(s)$.*

With the Laurent Series expansion this reformulates to $X_\gamma^\pi(s) = V_\gamma^\pi(s) - \frac{\rho^\pi}{1-\gamma}$. Therefore, $X_\gamma^\pi(s)$ describes the discounted state value adjusted by the average reward term. The Bellman Equation is given by $X_\gamma^\pi(s) = \mathbb{E}_\pi[r_t + \gamma X_\gamma^\pi(s_{t+1}) - \rho^\pi \mid s_t = s]$. The corresponding Bellman optimality equation then is given by $X_\gamma^{\pi^*}(s) = \max_a \mathbb{E}_{\pi^*}[r_t + \gamma X_\gamma^{\pi^*}(s_{t+1}) - \rho^\pi \mid s_t = s]$. By turning this formula into an update rule it builds the foundation of our algorithm.

Average Reward Adjusted Deep RL Algorithm. The model-free average reward adjusted RL algorithm is based on the tabular version of Schneckenreither (2020) and depicted in Algorithm 1. The algorithm operates on a target network θ_X^T and a worker network θ_X^W as in Mnih et al. (2015). In the action selection process a ϵ -sensitive lexicographic order $a = (a_1, \dots, a_n) \prec_\epsilon (b_1, \dots, b_n) = b$ is used. It is defined as $a \prec_\epsilon b$ if and only if $|a_j - b_j| \leq \epsilon$ for all $j < i$ and $|a_i - b_i| > \epsilon$. Note that the resulting sets of actions may not be disjoint, but taking the maximum as required in our algorithm is straight-forward and thus cheap to compute. The first action selection criteria selects the actions which maximise the bias values,

Algorithm 1 Basic near-Blackwell-optimal deep RL for unichain MDPs

- 1: Initialise state s_0 and network parameters θ_X^T, θ_X^W randomly, set an exploration rate $0 \leq p_{\text{learn}} < p_{\text{expl}} \leq 1$, exponential smoothing learning rates $0 < \alpha, \gamma < 1$, and discount factors $0 < \gamma_0 < \gamma_1 \leq 1$, where $\gamma_1 = 1$ is usually a good choice.
 - 2: **while** the stopping criterion is not fulfilled **do**
 - 3: With probability p_{expl} choose a random action and probability $1 - p_{\text{expl}}$ one that fulfills $\max_a \preceq_\epsilon (X_{\gamma_1}^\pi(s_t, a; \theta_X^W), X_{\gamma_0}^\pi(s_t, a; \theta_X^W))$. Let a_t^{rd} indicate if the action was chosen randomly.
 - 4: Carry out action a_t , observe reward r_t and resulting state s_{t+1} . Store the experience $(s_t, a_t, a_t^{\text{rd}}, r_t, s_{t+1})$ in the experience replay memory M .
 - 5: Sample random mini-batch of m experiences from M
 - 6: Reset gradients: $d\theta_X^W \leftarrow 0$
 - 7: **for** each experience i with $(s_t, a_t^{\text{rd}}, a_t, r_t, s_{t+1})$ **do**
 - 8: **if** a non-random action was chosen or $p_{\text{expl}} > p_{\text{learn}}$ **then**
 - 9: $\rho^\pi \leftarrow (1 - \alpha)\rho^\pi + \alpha[r_t + \max_a X_{\gamma_1}^\pi(s_{t+1}, a; \theta_X^W) - X_{\gamma_1}^\pi(s_t, a_t; \theta_X^W)]$
 - 10: $y_{i, \gamma_0} \leftarrow r_t + \gamma_0 \max_a X_{\gamma_0}^\pi(s_{t+1}, a; \theta_X^T) - \rho^\pi$
 - 11: $y_{i, \gamma_1} \leftarrow r_t + \gamma_1 \max_a X_{\gamma_1}^\pi(s_{t+1}, a; \theta_X^T) - \rho^\pi$
 - 12: Sum gradients on $(y_{i, \cdot} - X_{\gamma_1}^\pi(s_{t+1}, a; \theta_X^W))^2$ wrt. parameters θ_X^W in $d\theta_X^W$
 - 13: Perform update with average of the gradients $d\theta_X^W/m$ on θ_X^W
 - 14: Every C steps exponentially set target network: $\theta_X^T \leftarrow (1 - \gamma)\theta_X^T + \gamma\theta_X^W$
 - 15: Set $s \leftarrow s', t \leftarrow t + 1$ and decay parameters
-

the second the error term. Equation 2 of the average reward RL constraints is used to estimate the average reward based on the [MS: reformulate avg RL constraints using s' etc.?] currently inferred state values of two consecutive states and the returned reward. This infers a solid average reward prediction of the current policy in few steps only and thus quickly leads to better policies, as ones with higher gain are preferred. Schneckenreither (2020) and Tadepalli and Ok (1998) find that updating the average reward by Equation 2 is superior as compared to exponentially smoothing the actual observed rewards. This makes sense as the later rather evaluates past policies.

The average reward adjusted state value, which is approximated based on the Bellman optimality equation, is estimated twice. Once to infer the bias values with a very high discount factor γ_1 , e.g. $\gamma_1 = 1.0$, and a second time with a discount factor $\gamma_0 < \gamma_1$. The later is used to more selectively choose actions in case more than one action leads to bias-optimal policies. The predetermined discount value γ_0 is a measure for farsightedness. Lower values prefer to partially collect reward in fewer steps and higher values are used to prefer actions for which the full bias is collected earlier. The corresponding update values are saved to calculate the gradients based on the quadratic errors. Every C steps the target network smoothly adapts the parameters of the worker network.

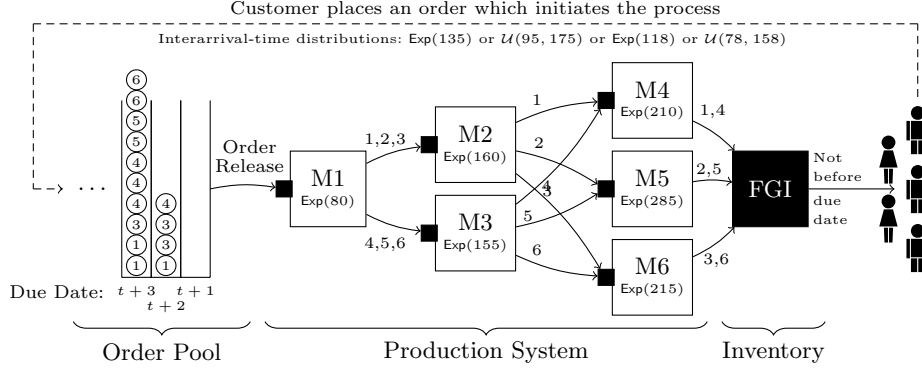


Fig. 3: Production System of the Simulation Model with routing, processing time distributions and demand interarrival time distributions.

3 Experimental Evaluation

This section introduces the simulation model and the parameterisation of the algorithm. To provide a proof-of-concept for the proposed algorithm we adapt the flow shop presented in Schneckenreither et al. (2020). However, our algorithm operates on discrete lead times on a product type basis and thus is not directly comparable to their order based lead time management approach. Nonetheless, this section is largely based on the corresponding section of Schneckenreither et al. (2020).

3.1 Simulation Model

We adapt the simulation model of a make-to-order (MTO) flow shop with six different products and six work-centers of Schneckenreither et al. (2020) and examine the system under moderate and high utilisation in combination with moderate or high variability in demand and processing times. This section provides the details of the simulation model and the experimental setup.

The simulation model is depicted in Figure 3, where the customers initiate the process by placing orders as indicated by the dashed line. The manufacturer produces six different product types, each of which is identified by a natural number $1, 2, \dots, 6$ and has a different routing through the production system. The routing setup is provided by corresponding labels of the edges. For instance, product types $1 - 3$ are routed from machine $M1$ to machine $M2$, while the other products are transferred to machine $M3$. The production system consists of three production stages with diverging material flow and no return visits. The incoming orders are uniformly distributed among the product types. A period lasts 960 minutes ($= 16$ hours) which represent two 8-hour shifts. At the beginning of each period orders can be released into the production system assuming material availability for all orders at their release date. Upon release,

the orders are placed in the buffer at machine $M1$. Buffers and inventories are plotted as filled squares. Thus, each workstation is equipped with a buffer which queues the orders until processing is started. All buffers use a first-come-first-serve dispatching rule. Furthermore, each workstation processes only one order at a time and early deliveries are prohibited. Orders completed ahead of their due date remain in the finished goods inventory (FGI) until they are due and sent to the customer.

Processing Times. The machine processing time distributions are specified under the corresponding node labels of the machines. To simulate high variance the processing times for all machines are drawn from an exponential distribution. There is one bottleneck machine ($M5$) and therefore we refer to product types 2 and 5, those routed through machine $M5$, as bottleneck products whereas products 1, 3, 4, 6 are non-bottleneck products.

Demand. The incoming orders are placed in the order pool with a due date slack of 10 periods, that is each order arriving within period t will be due at the end of period $t + 10$. The due date slack of 10 was chosen to (i) ensure sufficient time between the first occurrence of orders in the order pool and the latest possible release of orders and (ii) to have a clear cause and effect relationship between the order release decision and the cost performance in the analysis. To simulate moderate, medium and high variability of the demand process the interarrival time between consecutive incoming orders is either drawn from an exponential (Exp) or an uniform (\mathcal{U}) distribution, which are adjusted to yield the desired bottleneck utilisation level of either 70% with Exp(135) and $\mathcal{U}(95, 175)$, 80% with Exp(118) and $\mathcal{U}(78, 158)$, or 90% with Exp(105) and $\mathcal{U}(65, 145)$. These values were chosen as they comprise the non-linear relationship between flow times and high utilisation levels.

Order Release. Before the order release decision is made, all orders in the order pool are sorted by due date. This portrays the implementation of a sequencing rule. According to this sequence, orders are considered for release in the beginning of each period starting with the highest priority order.

In our model, orders are released by specifying lead times $LT_1, LT_2, \dots, LT_6 \geq 1$, where the index corresponds to the product type and $LT_i \in \mathbb{N}$. By setting a lead time LT_i a *planned release date* is computed for each order j of product type i in the order pool given by

$$PRD_j = DD_j - LT_i, \quad (4)$$

where DD_j denotes the due date of the order. For dynamic order releases the lead times LT_i may vary from period to period, whereas in static order release methods the lead times are predetermined and fixed (Kim and Bobrowski 1995; Ragatz and Mabert 1988). For both approaches a job of product type i with lead time LT_i in period t and due date DD_j is released at the end of period t if and only if $t \geq PRD_j$. Thus, in the dynamic setting the planned release date

PRD_j of order j can be updated several times before it is actually released to the production system. However, once an order is released its release cannot be revoked. Hence, if the set lead time corresponds with the actual flow time the product is finished in the same period as it is shipped. If the lead times are set either too long or short, the order has to wait in the FGI until it is due or the order is late and backorder (BO) costs occur.

Costs. The cost parameters are set by assuming an increase in value from raw material (1 Dollar per order and period) to the final product (4 Dollar per order and period) and the backorder costs are set very high (16 Dollar per order and period) due to the MTO environment. All costs are assessed based on the production system state at the end of each period.

3.2 Conventional Order Release Rule

As external benchmark for comparison we use different parameterized backward infinite loading (BIL) techniques (Ackerman, 1963). BIL uses Equation 4 with a predetermined and fixed lead times LT_i .

3.3 Algorithm Setup

Markov Decision Process. The underlying MDP is unichain and looks as follows. Both, state space \mathcal{S} and action space \mathcal{A} are discrete. For the *state space* we adapt the result of Knollmann and Windt (2013), which state that the lead times ought to control the WIP level, instead of the due date reliability as changes are directly seen. Thus, any state $s \in \mathcal{S}$ of the state space is composed of the following information for each product type pt :

- The currently set lead time $LT_g \in \{1, 2, \dots, dds\}$ (Recall: Due Date Period – Lead Time = Release Period). Note that we bound the maximum lead time with due date slack dds , which is 7 in our setup.
- Counters $OP_{g,d} \in \mathbb{N}$ for the number of orders in the order pool divided in time buckets with $d \in \{1, 2, \dots, dds\}$, which stands for the number of periods until the due date.
- Counters $Q_{g,i,d} \in \mathbb{N}$ for the number of orders in the machine and queue $i \in \{M_1, M_2, \dots, M_6\}$ divided in time buckets with $d \in \{1, 2, \dots, dds\}$, which stands for the number of periods until the due date.
- Counters $FGI_{g,d} \in \mathbb{N}$ of orders in the finished goods inventory divided in time buckets with $d \in \{1, 2, \dots, dds\}$, which stands for the number of periods until the due date. Orders with a due date with more than 3 periods ago are listed in the counter FGI_{-3} .
- Counters $S_{g,d} \in \mathbb{N}$ of shipped orders from the last period divided in time buckets with $d \in \{-3, -2, \dots, dds\}$, which stands for the number of periods until the due date. Orders with a due date with more than 3 periods ago are listed in the counter S_{-3} .

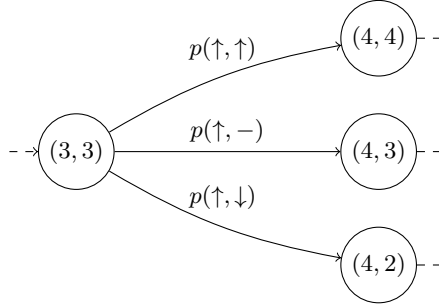


Fig. 4: Illustration of independent agents as agent one increases the lead time

The algorithm implicitly learns a function which maps the current state of the production system to a lead time update and thus a release decision. In an optimal situation it does so by multiply exploring every action for each state and assessing its economic viability. Orders are released once the due date is within the interval $[t, t + \text{lead time}]$, whereas t is the current period. Clearly, this yields bulk releases, meaning that either all or no orders with the same due date and of the same product type are released.

The *action space* is composed of two decisions, the relative change of the lead times to the currently set lead times $LT_g \in \{1, 2, \dots, \text{dds}\}$ for each product type **pt**. We restrict the lead time update for state s_{t+1} according to the lead time LT_g from state s_t by a maximum change of 1. Thus, if LT_g is the current lead time of product type **pt** the agent can choose a lead time in $\{1, 2, \dots, \text{dds}\} \cap \{LT_g - 1, LT_g, LT_g + 1\}$. A naive approach is to define one action for each combination of actions over all product type. This however, due to the exponential growth, results in only being able to solve very small production systems with only few product types. A common solution in production planning is to aggregate the products into product groups, cf. hierarchical production planning (Schneeweiß, 1995). Although applicable in our setup, if we group the products into bottleneck and non-bottleneck products as done by Schneckeneither et al. (2020), also this approach simply does not scale.

Therefore, in the seek of a broader solution we establish the idea of *independent agents*. Each independent agent operates on the same system state, but chooses only a part of the actual executed action. In our case each independent agent is responsible for the lead time update of a specific product type. To the best of the author’s knowledge this is the first work to reduce the action space in this way. Figure 4 illustrates the idea, where wlog. we use two agents and for simplicity omit the reward and truncate the state to the lead times. Assume that the first independent agent chooses to increase the lead time from 3 to 4. Thus, the future state, seen from this agent, depends on the probability of the other independent agent choosing to increase (\uparrow), decrease (\downarrow) or not change ($-$) the lead time. The Markov property, that is $p(s_{t+1}, r_t \mid s_t, a_t) = p((4, \cdot), r_t \mid (3, 3), (\uparrow, \cdot))$, is not touched by this adaption. While for the first agent the reward and future

state only depends on the current state and action taken, the corresponding probabilities for future state and reward change during the learning process as the the agent starts to behave differently. This allows us to investigate the algorithms behaviour for even larger solution spaces. In particular, this adaption enables us to present the algorithm the above defined state space not only on the basis of product groups, but directly for each product type pt . Note that without this extension the problem is computationally infeasible due to $3^6 = 729$ possible actions. Removing the dependency between the lead times of different product types reduces the set of possible actions, and thus the number of ANN output layer nodes, to $3 \cdot 6 = 18$. In preliminary experiments on small examples we only observed a very slight increase of required steps during training. Therefore, this is a vital adaption to ensure scalability.

Reward. In each period the agents choose actions, which when executed generate a reward while traversing to the next period by simulating the production system. The rewards are the accumulated costs at the end of the period consisting of the backorders, the WIP orders and the number of orders in the FGI. The objective is to minimise the returned reward.

Optimised Algorithm. Our implementation includes several optimisations, that help the algorithm in better exploring the solution space and stabilise the learning process. These include the above discussed independent agents that operate on the same state, n -step learning (Mnih et al., 2016), and ensuring to constantly reduce the average reward as searching for policies with higher average costs is unnecessary. Algorithm 2 depicts a detailed view on the algorithm including these extensions and with the objective to minimise the returned rewards. The idea is the same as in Algorithm 1, except that the action selection procedure is a loop over all independent agents and the implementation of the n -step logic. The algorithm is based around The agent first choose an action (steps 4 – 5), executes the selected action and stores the experience in the replay memory (step 6) and then samples a mini-batch of experiences each of length $nstep$ to improve the policy (steps 7 – 21). Step 17 uses an exponentially smoothed overestimation (by 2.5%) of the average reward and rate 2^{-5} . All average reward updates above this overestimation are rejected.

As in (Mnih et al., 2015, 2016) the algorithm uses a *Replay Memory* to save experiences and then randomly selects experiences from there for learning. This ensures that the independent and identically distributed (i.i.d.) assumption of the samples for the ANN backpropagation algorithms are satisfied. We extend this idea by splitting the available size of the replay memory into subsets, where each subset represents one action and sample uniformly over these subsets first. This prevents overfitting when one action is overrepresented in the policy, e.g. no lead time change occurs much more often than the other actions.

N-step learning collects a set of consecutive experiences, in our case retrieved from the replay memory, and connects the state values starting from the latest observed experience, such that the newly computed state value of the corresponding experience propagates to the experience of the previous period. This

Algorithm 2 Optimised near-Blackwell-optimal deep RL for unichain MDPs

```
1: Initialise state  $s_0$  and network parameters  $\theta_X^T, \theta_X^W$  randomly, set an exploration
   rate  $0 \leq p_{\text{learn}} \leq p_{\text{expl}} \leq 1$ , exponential smoothing learning rates  $0 < \alpha_{\text{max}}, \alpha, \gamma < 1$ ,
   a sufficiently large default value for  $\rho_{\text{max}}^\pi$  and discount factors  $0 < \gamma_0 < \gamma_1 \leq 1$ 
2: while the stopping criterion is not fulfilled do
3:    $a_t^{\text{rd}} \leftarrow \text{False}$ 
4:   for each independent agent  $i \in \text{ag}_1, \text{ag}_2, \dots, \text{ag}_n$  do
5:     With probability  $\frac{p_{\text{expl}}}{n}$  choose action  $a_{t,i}$  randomly or otherwise one of the
       set defined by  $\min_{a \in \mathcal{A}(i)} \leq_\epsilon (X_{\gamma_1}^\pi(s_t, a; \theta_X^W), X_{\gamma_0}^\pi(s_t, a; \theta_X^W))$ . Set  $a_t^{\text{rd}} \leftarrow \text{True}$ 
       if the action was chosen randomly.
6:   Carry out action  $a_t = (a_{t,\text{ag}_1}, \dots, a_{t,\text{ag}_n})$ , observe reward  $r_t$  and resulting state
        $s_{t+1}$ . Store experience  $(s_t, a_t, a_t^{\text{rd}}, r_t, s_{t+1})$  in the experience replay memory  $M$ .
7:   if  $t \bmod n_{\text{step}} \equiv 0$  then
8:     Reset gradient:  $d\theta_X^W \leftarrow 0$ 
9:     Sample  $m$  random mini-batches of  $n_{\text{step}}$  consecutive experiences from  $M$ 
       numbered by  $1 \leq i \leq n_{\text{step}}$ , each with  $(s_i, a_i^{\text{rd}}, a_i, r_i, s_{i+1})$ .
10:    for each mini-batch do
11:      for each experience, starting with the latest seen experience  $n_{\text{step}}$  do
12:        if experience  $n_{\text{step}}$  or  $\neg a_i^{\text{rd}}$  or  $p_{\text{expl}} > p_{\text{learn}}$  then
13:           $\text{target}_{\gamma_0} \leftarrow \min_a X_{\gamma_0}^\pi(s_{i+1}, a; \theta_X^T)$ 
14:           $\text{target}_{\gamma_1} \leftarrow \min_a X_{\gamma_1}^\pi(s_{i+1}, a; \theta_X^T)$ 
15:        if  $\neg a_i^{\text{rd}}$  or  $p_{\text{expl}} > p_{\text{learn}}$  then
16:           $\rho^\pi \leftarrow (1 - \alpha)\rho^\pi + \alpha[r_i + \min_a X_{\gamma_1}^\pi(s_{i+1}, a; \theta_X^W) - X_{\gamma_1}^\pi(s_i, a_i; \theta_X^W)]$ 
17:          Ensure improvement of  $\rho^\pi$ , otherwise reset to old value
18:           $\text{target}_{\gamma_0} \leftarrow r_i + \gamma_0 \text{target}_{\gamma_0} - \rho^\pi$ 
19:           $\text{target}_{\gamma_1} \leftarrow r_i + \gamma_1 \text{target}_{\gamma_1} - \rho^\pi$ 
20:           $d\theta_X^W \leftarrow d\theta_X^W + \partial(\text{target}_{\gamma_0} - X_{\gamma_0}^\pi(s_{i+1}, a; \theta_X^W))^2 / \partial \theta_X^W$ 
21:           $d\theta_X^W \leftarrow d\theta_X^W + \partial(\text{target}_{\gamma_1} - X_{\gamma_1}^\pi(s_{i+1}, a; \theta_X^W))^2 / \partial \theta_X^W$ 
22:        Perform update of  $\theta_X^W$  using  $d\theta_X^W / (m \cdot n_{\text{step}})$ 
23:      Every  $C$  steps exponentially set target network:  $\theta_X^T \leftarrow (1 - \gamma)\theta_X^T + \gamma\theta_X^W$ 
24:      Set  $s \leftarrow s', t \leftarrow t + 1$  and decay parameters
```

allows a faster convergence, as values bootstrap over multiple periods in contrast to just one step. The experiences are stored according to the first taken action in the n -step series. We uniformly sample over the actions to ensure each action is constantly trained. This adaption reduces catastrophic forgetting of already accumulated knowledge. Furthermore, we use 8 additional **worker threads** (parallel RL) that operate on the same environment to collect a broader range of experiences (as e.g. in Mnih et al., 2016). The average reward value is shared and computed as average over the main agent and all workers.

The algorithm initiation strategy is to first fill the replay memory buffer by skipping any learning (steps 7 – 21 in Algorithm 2) until the period, that corresponds to the replay memory size, is reached. Then for 500 steps the average reward is set to the last computed exponentially smoothed value to prevent divergence due to the untrained network weight initialisation values.

Neural Network Setup. Due to the vast possibilities the neural network architecture was manually optimised, starting with less stochastic systems and aiming for a coffin shape that can be used in all setups. During this process we constantly adapted and reevaluated the best network on various production system configurations. The resulting architecture depends on the number of features n_{inp} and the number of output nodes n_{out} , which itself depends on the number of actions $n_{\text{act}} = |\mathcal{S}|$, the independent agent configuration, and the number of functions that are approximated. For the later, in case of ARAL this is 2, as we represent both functions $X_{\gamma_0}^\pi$ and $X_{\gamma_1}^\pi$ with the same neural network, while deep Q-Learning only approximates the function Q_γ^π . The resulting feedforward deep ANN architecture is given as $n_{\text{inp}} \rightarrow 1.75n_{\text{inp}} \rightarrow 1.5n_{\text{inp}} \rightarrow n_{\text{out}}$, where each arrow symbolises a fully-connected layer with a leaky rectified linear unit (ReLU) as activation function, which leaks values smaller than 0 with rate 0.02. Note that we explicitly do not use an output activation function. The idea is that we do not want to restrict the approximations to values inside this range. This is especially important for Q-Learning, as the range of the state values are difficult to forecast. The actual values n_{inp} and n_{out} depend on the used algorithm. In our case case using 6 independent agents we gain $n_{\text{inp}} = 324$ and $n_{\text{out}} = 36$. For the Q-Learning algorithm, which only approximates one function, the output nodes n_{out} are halved.

The features representing order counts are scaled using min-max scaling with minimum 0 and maximum 5, the lead times with minimum 0 and maximum 7 respectively. To further ensure smaller ANN parameter values the output is transformed using min-max scaling to the interval $(-400, 800)$, s.t. an ANN output of 1 represents a value of 800. On each training step, i.e. every n -step steps, randomly m sets of experiences the training mini-batch size each consisting of n -step experiences are selected, there gradients computed and the sum of gradients backpropagated through the worker network using Adam (Kingma and Ba, 2014), where the weight decay with parameter λ is implemented as presented by Loshchilov and Hutter (2017). Every C steps the target network is updated by exponentially smoothing the parameters with rate γ . As neural network initialisation method we use the commonly known Xavier initialisation (Glorot and Bengio, 2010), i.e. the initial value for each weight i is sampled from a uniform distribution $W_{l,i} \sim \mathcal{U}(-1/\sqrt{n_l}, 1/\sqrt{n_l})$, where n_l is the number of nodes in layer l .

3.4 Deep Q-Learning

[MS: explain here, alg in appendix?]

4 Computational Experiments

This section presents the experimental results and gives an overview of the performance of the established algorithm compared to the conventional static lead time setting algorithms.

Parameter	α	γ_0	γ_1	γ	ϵ	p_{exp}	p_{learn}
Start value	0.01	0.8	0.99/1.0	0.01	0.25	1.0	0.5
Exp. Smth. Rate	0.25			0.25		0.25	
Exp. Smth. Steps in 10^3	50			100		100	
Minimum value	5^{-5}			0.001		0.005	

Parameter	Value
Workers Min. Exploration	0.01, 0.02, 0.03, 0.04, 0.05, 0.10, 0.15, 0.20
Training Batch Size	4
Target Network Update	Exp. smooth. with rate γ every $C = 100$ steps
N-Step	5
Training Mini-Batch Size	$m = 4$
Adam ANN Backpropagation	$\alpha = 0.005, \beta_1 = 0.9, \beta_2 = 0.999, \epsilon = 10^{-8}, \lambda = 0.001$
Replay Memory Size	10800

Table 1: Overview of Parameter Setup.

Table 1 presents an overview of the parameter setup. The parameters α , γ and p_{exp} are exponentially decayed as indicated. [MS: TODO] As further comparison we implement the deep Q-Learning algorithm proposed by Mnih et al. (2015), where for comparison reasons we also integrate n-Step learning, the experience replay memory adaption and independent agents. [MS: Discount factor in table?]

[MS: Exp - Exp in simulation model?]

5 Conclusion

[MS: It remains to be shown that independent agents find the same (or better) policies.]

[MS: Multiple agents on different tasks (hierarchical PP): e.g. Order acceptance + release, scheduling + release, etc.]

[MS: todo]

70- \mathcal{U}	WIPC	FGIC	BOC	FGIC+BOC	SUM	SFTT	FGIT	SL(%)	Cost.p.P	ρ^π
QL _{0.99}	38.56	134.70	81.70	216.41	254.97	1.34	1.22	90.6	36.42	
ARAL _{0.99}	37.20	113.10	100.40	213.50	250.70	1.31	1.11	88.9	35.81	35.04
ARAL _{1.00}	38.66	156.66	59.74	216.40	255.06	1.34	1.34	93.2	36.44	36.50
BIL1	32.06	0.00	512.90	512.90	544.96	1.20	0.49	44.9	77.85	
BIL2	32.06	83.46	105.01	188.47	220.53	1.20	0.94	88.6	31.50	
BIL3	32.16	247.73	21.68	269.41	301.56	1.20	1.83	97.8	43.08	
BIL4	32.14	429.10	5.08	434.17	466.32	1.20	2.81	99.5	66.62	
80- \mathcal{U}	WIPC	FGIC	BOC	FGIC+BOC	SUM	SFTT	FGIT	SL(%)	CostpP	ρ^π
QL _{0.99}	63.10	180.40	121.53	301.93	365.03	1.68	1.34	89.7	52.15	
ARAL _{0.99}	65.31	168.49	148.59	317.08	382.39	1.72	1.28	86.4	54.63	52.92
ARAL _{1.00}	60.78	234.05	76.17	310.21	370.99	1.64	1.59	93.4	53.00	52.29
BIL1	51.89	0.00	830.22	830.22	882.11	1.47	0.49	32.8	126.02	
BIL2	52.00	70.11	255.17	325.28	377.27	1.48	0.82	79.4	53.90	
BIL3	51.98	240.18	77.82	318.00	369.97	1.48	1.62	94.0	52.85	
BIL4	51.83	442.31	26.71	469.02	520.85	1.47	2.56	98.1	74.41	
90- \mathcal{U}	WIPC	FGIC	BOC	FGIC+BOC	SUM	SFTT	FGIT	SL(%)	CostpP	ρ^π
QL _{0.99}	98.91	190.68	297.33	488.01	586.92	2.13	1.28	82.7	83.85	
ARAL _{0.99}	105.35	263.95	207.18	471.13	576.48	2.24	1.59	87.9	82.35	79.83
ARAL _{1.00}	107.87	240.80	253.87	494.66	602.54	2.28	1.49	84.7	86.08	80.43
BIL1	88.66	0.00	1,418.53	1,418.53	1,507.19	1.96	0.50	21.0	215.31	
BIL2	88.32	50.85	646.00	696.86	785.17	1.96	0.71	64.5	112.17	
BIL3	88.25	207.21	299.89	507.10	595.35	1.96	1.35	85.2	85.05	
BIL4	87.77	415.18	153.61	568.79	656.57	1.95	2.21	92.9	93.80	

Table 2: Results with moderate variability in the system due to uniformly distributed demand interarrival times.

70-Exp	WIPC	FGIC	BOC	FGIC+BOC	SUM	SFTT	FGIT	SL(%)	Cost.p.P	ρ^π
QL _{0.99}	53.87	142.40	164.43	306.83	360.70	1.59	1.21	83.8	51.53	
ARAL _{0.99}	61.42	181.70	115.29	296.99	358.41	1.73	1.41	89.3	51.20	50.79
ARAL _{1.00}	56.58	199.31	115.71	315.02	371.60	1.64	1.50	88.7	53.09	50.58
BIL1	48.55	0.00	776.75	776.75	825.30	1.48	0.50	33.7	117.90	
BIL2	48.62	66.93	248.64	315.57	364.18	1.48	0.83	78.0	52.03	
BIL3	48.63	222.60	74.57	297.17	345.80	1.48	1.61	93.7	49.40	
BIL4	48.51	409.53	23.41	432.94	481.45	1.48	2.55	98.1	68.78	
80-Exp	WIPC	FGIC	BOC	FGIC+BOC	SUM	SFTT	FGIT	SL(%)	Cost.p.P	ρ^π
QL _{0.99}	87.71	227.49	225.15	452.64	540.34	2.04	1.50	83.4	77.19	
ARAL _{0.99}	89.84	253.48	185.69	439.16	529.00	2.08	1.61	87.6	75.57	70.53
ARAL _{1.00}	87.17	242.92	198.46	441.37	528.54	2.03	1.57	86.6	75.51	72.94
BIL1	78.73	0.00	1,259.71	1,259.71	1,338.44	1.88	0.50	22.9	191.21	
BIL2	78.20	52.30	548.74	601.04	679.24	1.88	0.73	65.0	97.03	
BIL3	79.03	200.27	242.10	442.37	521.40	1.89	1.38	85.8	74.49	
BIL4	79.38	395.49	116.94	512.44	591.82	1.90	2.23	93.8	84.55	
90-Exp	WIPC	FGIC	BOC	FGIC+BOC	SUM	SFTT	FGIT	SL(%)	Cost.p.P	ρ^π
QL _{0.99}	168.66	222.15	788.66	1,010.81	1,179.46	3.13	1.37	71.5	168.49	
ARAL _{0.99}	158.03	212.73	701.30	914.03	1,072.06	2.97	1.33	71.9	153.15	115.80
ARAL _{1.00}	160.45	286.07	539.80	825.87	986.32	3.01	1.62	79.6	140.90	109.91
BIL1	146.84	0.00	2,349.37	2,349.37	2,496.21	2.80	0.50	13.3	356.60	
BIL2	143.76	34.11	1,412.75	1,446.86	1,590.62	2.75	0.63	47.1	227.23	
BIL3	144.28	154.15	877.45	1,031.60	1,175.88	2.76	1.10	70.6	167.98	
BIL4	147.08	335.07	622.14	957.21	1,104.29	2.80	1.81	82.5	157.76	

Table 3: Results with high variability in the system due to exponentially distributed interarrival times of the demand.

Bibliography

- S.S. Ackerman. Even-flow a scheduling method for reducing lateness in job shops. *Management Technology*, 3:20–32, 1963.
- M Emin Aydin and Ercan Öztemel. Dynamic job-shop scheduling using reinforcement learning agents. *Robotics and Autonomous Systems*, 33(2-3):169–178, 2000.
- Marc G Bellemare, Joel Veness, and Michael Bowling. Investigating contingency awareness using atari 2600 games. In *Twenty-Sixth AAAI Conference on Artificial Intelligence*, 2012.
- J. W. M. Bertrand. The effect of workload dependent due-dates on job shop performance. *Management Science*, 29(7):799–816, 1983.
- Dimitri P Bertsekas, Dimitri P Bertsekas, Dimitri P Bertsekas, and Dimitri P Bertsekas. *Dynamic programming and optimal control*, volume 1. Athena scientific Belmont, MA, 1995.
- David Blackwell. Discrete dynamic programming. *The Annals of Mathematical Statistics*, 34:719–726, 1962. doi: 016/j.cam.2018.05.030.
- S.-H. Chung and H.-W. Huang. Cycle time estimation for wafer fab with engineering lots. *IIE Transactions*, 34(2):105–118, 2002. doi: 10.1080/07408170208928854. URL <https://doi.org/10.1080/07408170208928854>.
- Tapas K. Das, Abhijit Gosavi, Sridhar Mahadevan, and Nicholas Marchallick. Solving semi-markov decision problems using average reward reinforcement learning. *Management Science*, 45(4):560–574, 1999.
- Stefanos Doltsinis, Pedro Ferreira, and Niels Lohse. Reinforcement learning for production ramp-up: A q-batch learning approach. In *2012 11th International Conference on Machine Learning and Applications*, volume 1, pages 610–615. IEEE, 2012.
- S. T. Enns and P. Suwanruji. Work load responsive adjustment of planned lead times. *Journal of Manufacturing Technology Management*, 15(1):90–100, 2004.
- L. F. Gelders and L. N. Van Wassenhove. Hierarchical integration in production planning: Theory and practice. *Journal of Operations Management*, 3(1):27–35, 1982.
- Joren Gijsbrechts, Robert N Boute, Jan A Van Mieghem, and Dennis Zhang. Can deep reinforcement learning improve inventory management? performance and implementation of dual sourcing-mode problems. *Performance and Implementation of Dual Sourcing-Mode Problems (December 17, 2018)*, 2018.
- Xavier Glorot and Yoshua Bengio. Understanding the difficulty of training deep feedforward neural networks. In *Proceedings of the thirteenth international conference on artificial intelligence and statistics*, pages 249–256, 2010.
- Ronald A Howard. Dynamic programming and markov processes. 1964.
- J. Hoyt. Dynamic lead times that fit today’s dynamic planning (quoat lead times). *Production and Inventory Management*, 19(1):63–71, 1978.
- S.-C. Kim and P. M. Bobrowski. Evaluating order release mechanisms in a job shop with sequence-dependent setup times. *Production and Op-*

- erations Management*, 4(2):163–180, 1995. doi: 10.1111/j.1937-5956.1995.tb00048.x. URL <https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1937-5956.1995.tb00048.x>.
- Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014.
- M. Knollmann and K. Windt. Control-theoretic analysis of the lead time syndrome and its impact on the logistic target achievement. *Procedia CIRP*, 7: 97–102, 2013.
- Ilya Loshchilov and Frank Hutter. Decoupled weight decay regularization. *arXiv preprint arXiv:1711.05101*, 2017.
- Sridhar Mahadevan. An average-reward reinforcement learning algorithm for computing bias-optimal policies. In *AAAI/IAAI, Vol. 1*, pages 875–880, 1996a.
- Sridhar Mahadevan. Average reward reinforcement learning: Foundations, algorithms, and empirical results. *Machine Learning*, 22:159–195, 1996b.
- Sridhar Mahadevan. Optimality criteria in reinforcement learning. In *Proceedings of the AAAI Fall Symposium on Learning Complex Behaviors in Adaptive Intelligent Systems*, 1996c.
- Sridhar Mahadevan. Sensitive discount optimality: Unifying discounted and average reward reinforcement learning. In *ICML*, pages 328–336, 1996d.
- Sridhar Mahadevan and Georgios Theodorou. Optimizing production manufacturing using reinforcement learning. In *FLAIRS Conference*, pages 372–377, 1998.
- Sridhar Mahadevan, Nicholas Marchallick, Tapas K. Das, and Abhijit Gosavi. Self-improving factory simulation using continuous-time average-reward reinforcement learning. In *MACHINE LEARNING-INTERNATIONAL WORKSHOP THEN CONFERENCE*, pages 202–210, 1997.
- H. Mather and G. W. Plossl. Priority fixation versus throughput planning. *Production and Inventory Management*, 19:27–51, 1978.
- B. L. Miller and A. F. Veinott. Discrete dynamic programming with a small interest rate. *The Annals of Mathematical Statistics*, 40(2):366–370, 1969. ISSN 00034851. URL <http://www.jstor.org/stable/2239451>.
- Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Andrei A Rusu, Joel Veness, Marc G Bellemare, Alex Graves, Martin Riedmiller, Andreas K Fidjeland, Georg Ostrovski, et al. Human-level control through deep reinforcement learning. *Nature*, 518(7540):529, 2015.
- Volodymyr Mnih, Adria Puigdomenech Badia, Mehdi Mirza, Alex Graves, Timothy Lillicrap, Tim Harley, David Silver, and Koray Kavukcuoglu. Asynchronous methods for deep reinforcement learning. In *International conference on machine learning*, pages 1928–1937, 2016.
- Julia Pahl, Stefan Voss, and David L. Woodruff. Production planning with load dependent lead times: an update of research. *Annals of Operations Research*, 153(1):297–345, 2007. ISSN 0254-5330. doi: 10.1007/s10479-007-0173-5. URL <GotoISI>://WOS:000247203300013.
- Carlos D Paternina-Arboleda and Tapas K Das. Intelligent dynamic control policies for serial production lines. *Iie Transactions*, 33(1):65–77, 2001.
- Martin L Puterman. Markov decision processes. j. *Wiley and Sons*, 1994.

- G. J. Ragatz and V. A. Mabert. An evaluation of order release mechanisms in a job-shop environment. *Decision Sciences*, 19:167–189, 1988.
- M. Schneckenreither. Blackwell-optimal reinforcement learning. Working paper., 2019.
- M. Schneckenreither and S. Haeussler. *Reinforcement Learning Methods for Operations Research Applications: The Order Release Problem*, page 46. Springer International Publishing, 2019. ISBN 978-3-030-13708-3.
- Manuel Schneckenreither. Average reward adjusted discounted reinforcement learning: Near-blackwell-optimal policies for real-world applications. *arXiv preprint arXiv:2004.00857*, 2020.
- Manuel Schneckenreither, Stefan Haeussler, and Christoph Gerhold. Order release planning with predictive lead times: a machine learning approach. *International Journal of Production Research*, 2020. doi: 10.1080/00207543.2020.1859634.
- Christoph Schneeweiß. Hierarchical structures in organisations: A conceptual framework. *European Journal of Operational Research*, 86(1):4–31, 1995.
- B. Selcuk, J. C. Fransoo, and T. G. De Kok. The effect of updating lead times on the performance of hierarchical planning systems. *International Journal of Production Economics*, 104(2):427–440, 2006.
- David Silver, Aja Huang, Chris J Maddison, Arthur Guez, Laurent Sifre, George Van Den Driessche, Julian Schrittwieser, Ioannis Antonoglou, Veda Panneershelvam, Marc Lanctot, et al. Mastering the game of go with deep neural networks and tree search. *nature*, 529(7587):484, 2016.
- David Silver, Thomas Hubert, Julian Schrittwieser, Ioannis Antonoglou, Matthew Lai, Arthur Guez, Marc Lanctot, Laurent Sifre, Dhharshan Kumaran, Thore Graepel, et al. Mastering chess and shogi by self-play with a general reinforcement learning algorithm. *arXiv preprint arXiv:1712.01815*, 2017.
- Richard S Sutton, Andrew G Barto, et al. *Introduction to reinforcement learning*, volume 2. MIT press Cambridge, 1998.
- Prasad Tadepalli and DoKyeong Ok. Model-based average reward reinforcement learning. *Artificial intelligence*, 100(1-2):177–224, 1998.
- I. P. Tatsiopoulos and B. G. Kingsman. Lead time management. *European Journal of Operational Research*, 14(4):351–358, 1983. ISSN 0377-2217.
- Arthur F. Veinott. Discrete dynamic programming with sensitive discount optimality criteria. *The Annals of Mathematical Statistics*, 40(5):1635–1660, 1969. doi: 10.1214/aoms/1177697379.
- Yi-Chi Wang and John M Usher. Application of reinforcement learning for agent-based production scheduling. *Engineering Applications of Artificial Intelligence*, 18(1):73–82, 2005.
- Bernd Waschneck, André Reichstaller, Lenz Belzner, Thomas Altenmüller, Thomas Bauernhansl, Alexander Knapp, and Andreas Kyek. Optimization of global production scheduling with deep reinforcement learning. *Procedia CIRP*, 72(1):1264–1269, 2018.
- H. P. Wiendahl. *Load-Oriented Manufacturing Control*. Springer, Berlin, 1st edition, 1995. ISBN 9783642633430.

- Günther Zäpfel. *Produktionswirtschaft: Operatives Produktions-Management*. de Gruyter, 1982.
- Wei Zhang and Thomas G Dietterich. A reinforcement learning approach to job-shop scheduling. In *IJCAI*, volume 95, pages 1114–1120. Citeseer, 1995.

A Deep Q-Learning Algorithm

Algorithm 3 presents the detailed deep Q-Learning algorithm. It is very similar to the established ARAL algorithm that is given in Algorithm 2. However, it does not compute the average reward. Furthermore, it also operates on a single estimation of the overall future discounted reward.

Algorithm 3 Deep Q-Learning

- 1: Initialise state s_0 and network parameters θ_X^T, θ_X^W randomly, set an exploration rate $0 < p_{\text{expl}} \leq 1$, exponential smoothing learning rates $0 < \alpha_{\text{max}}, \alpha, \gamma < 1$ and discount factor $0 < \gamma_1 < 1$
 - 2: **while** the stopping criterion is not fulfilled **do**
 - 3: **for each** independent agent $i \in \text{ag}_1, \text{ag}_2, \dots, \text{ag}_n$ **do**
 - 4: With probability $\frac{p_{\text{expl}}}{n}$ choose action $a_{t,i}$ randomly or otherwise one of the set defined by $\min_{a \in \mathcal{A}(i)} X_{\gamma_1}^\pi(s_t, a; \theta_X^W)$.
 - 5: Carry out action $a_t = (a_{t,\text{ag}_1}, \dots, a_{t,\text{ag}_n})$, observe reward r_t and resulting state s_{t+1} . Store experience $(s_t, a_t, a_t^{\text{rd}}, r_t, s_{t+1})$ in the experience replay memory M .
 - 6: **if** $t \bmod n_{\text{step}} \equiv 0$ **then**
 - 7: Reset gradient: $d\theta_X^W \leftarrow 0$
 - 8: Sample m random mini-batches of n_{step} consecutive experiences from M numbered by $1 \leq i \leq n_{\text{step}}$, each with $(s_i, a_i^{\text{rd}}, a_i, r_i, s_{i+1})$.
 - 9: **for each** mini-batch **do**
 - 10: **for each** experience, starting with the latest seen experience n_{step} **do**
 - 11: **if** experience n_{step} or $\neg a_i^{\text{rd}}$ or $p_{\text{expl}} > p_{\text{learn}}$ **then**
 - 12: $\text{target}_{\gamma_1} \leftarrow \min_a X_{\gamma_1}^\pi(s_{i+1}, a; \theta_X^T)$
 - 13: $\text{target}_{\gamma_1} \leftarrow r_i + \gamma_1 \text{target}_{\gamma_1}$
 - 14: $d\theta_X^W \leftarrow d\theta_X^W + \partial(\text{target}_{\gamma_1} - X_{\gamma_1}^\pi(s_{i+1}, a; \theta_X^W))^2 / \partial \theta_X^W$
 - 15: Perform update of θ_X^W using $d\theta_X^W / (m \cdot n_{\text{step}})$
 - 16: Every C steps exponentially set target network: $\theta_X^T \leftarrow (1 - \gamma)\theta_X^T + \gamma\theta_X^W$
 - 17: Set $s \leftarrow s', t \leftarrow t + 1$ and decay parameters
-

B

70- \mathcal{U}	QL _{0.99}	ARAL _{0.99}	ARAL _{1.00}	BIL1	BIL2	BIL3
ARAL _{0.99}	3.37e-04					
ARAL _{1.00}	3.55e-01	1.04e-05				
BIL1	1.94e-45	2.48e-51	6.90e-44			
BIL2	1.09e-24	3.35e-16	9.57e-27	3.66e-64		
BIL3	1.78e-20	9.27e-29	2.44e-18	1.35e-22	6.26e-47	
BIL4	6.30e-34	3.85e-41	4.83e-32	1.16e-08	3.00e-56	1.16e-08
80- \mathcal{U}	QL _{0.99}	ARAL _{0.99}	ARAL _{1.00}	BIL1	BIL2	BIL3
ARAL _{0.99}	4.78e-16					
ARAL _{1.00}	5.26e-04	3.96e-08				
BIL1	1.02e-40	1.98e-21	3.92e-34			
BIL2	6.98e-10	7.12e-03	2.06e-03	1.31e-27		
BIL3	2.16e-02	8.93e-11	2.03e-01	1.57e-36	2.46e-05	
BIL4	6.16e-33	1.10e-11	2.74e-25	5.33e-05	5.38e-18	5.30e-28
90- \mathcal{U}	QL _{0.99}	ARAL _{0.99}	ARAL _{1.00}	BIL1	BIL2	BIL3
ARAL _{0.99}	5.18e-01					
ARAL _{1.00}	1.94e-04	1.56e-05				
BIL1	8.40e-42	9.61e-43	1.10e-34			
BIL2	2.36e-33	1.10e-34	5.21e-25	1.56e-05		
BIL3	4.16e-04	3.64e-05	8.19e-01	6.27e-35	1.81e-25	
BIL4	2.33e-20	6.57e-22	1.79e-11	2.93e-18	4.39e-08	5.97e-12

Table 4: Results from the statistical analysis for the results of moderate variability in the system.

70-Exp	QL _{0.99}	ARAL _{0.99}	ARAL _{1.00}	BIL1	BIL2	BIL3
ARAL _{0.99}	1.75e-02					
ARAL _{1.00}	6.15e-07	5.50e-12				
BIL1	7.62e-35	1.74e-39	1.32e-23			
BIL2	3.33e-01	1.02e-03	3.17e-05	6.06e-33		
BIL3	5.96e-17	1.79e-11	6.09e-29	7.00e-52	3.47e-19	
BIL4	1.16e-24	6.25e-30	1.68e-12	4.68e-06	1.56e-22	3.43e-44
80-Exp	QL _{0.99}	ARAL _{0.99}	ARAL _{1.00}	BIL1	BIL2	BIL3
ARAL _{0.99}	7.07e-02					
ARAL _{1.00}	1.74e-03	1.72e-01				
BIL1	2.39e-35	6.77e-39	2.38e-41			
BIL2	1.22e-25	1.00e-29	1.21e-32	1.13e-05		
BIL3	2.72e-05	1.33e-02	2.44e-01	3.65e-43	5.68e-35	
BIL4	2.07e-13	1.12e-17	6.55e-21	3.62e-17	6.09e-07	1.45e-23
90-Exp	QL _{0.99}	ARAL _{0.99}	ARAL _{1.00}	BIL1	BIL2	BIL3
ARAL _{0.99}	2.61e-06					
ARAL _{1.00}	5.03e-13	1.74e-03				
BIL1	9.64e-23	2.73e-33	2.22e-39			
BIL2	3.46e-12	1.32e-23	1.30e-30	1.23e-05		
BIL3	6.86e-01	1.23e-05	3.46e-12	1.32e-23	5.03e-13	
BIL4	2.45e-04	2.38e-01	2.60e-05	6.39e-31	6.30e-21	9.46e-4

Table 5: Results from the statistical analysis for the results of of moderate variability in the system.