Computing fundamental domains of crystallographic groups

With connections to topological interlocking

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Lemma

Let $E(n)$ be the set of all isometries of a dimension $n \in \mathbb{N}$. Then $E(n)$ is a group with the composition of homomorphisms as the group operation.

Proposition ([4, Exa. 1.1, Prop. 1.6])

There is an isometry

$$E(n) \cong O(n) \ltimes \mathbb{R}^n.$$

We denote with φ_o the orthogonal part of φ and with φ_t the vector/translation part of φ.

Then the group operation of $\varphi, \psi \in E(n)$ is as follows:

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$$(\varphi_o, \varphi_t) \circ (\psi_o, \psi_t) := (\underbrace{\varphi_o \circ \psi_o}_{\substack{\text{op. in } O(n) \\ \text{i.e. comp. of maps}}}, \psi_t^{\varphi_o} + \varphi_t),$$

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$$\mathbb{R}^n \times (O(n) \ltimes \mathbb{R}^n) \to \mathbb{R}^n : (v, (\varphi_o, \varphi_t)) \mapsto v^{(\varphi_o, \varphi_t)} = v^{\varphi_o} + \varphi_t.$$

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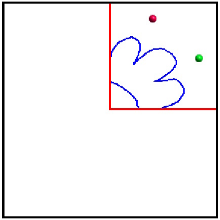
(i) $\bigcup_{\gamma \in \Gamma} F^{\langle \gamma \rangle} = \mathbb{R}^n$,

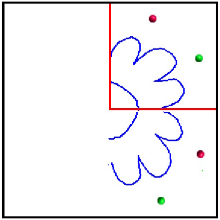
(ii) there is a system of representatives $V \subseteq \mathbb{R}^n$ w.r.t. the partition given by the orbits of Γ acting on $\mathbb{R}^n$ such that

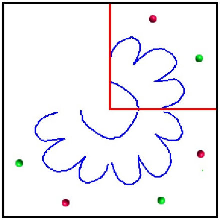
$$F^\circ \subseteq V \subseteq F.$$

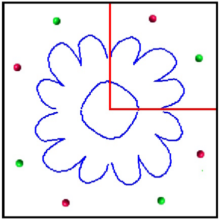
Example

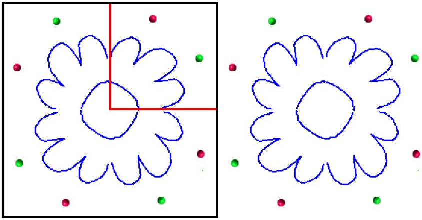
$$p4 := \left\langle \rho := \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}^T \right), \right.$$
$$\tau_1 := \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \right),$$
$$\left. \tau_2 := \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \right) \right\rangle$$

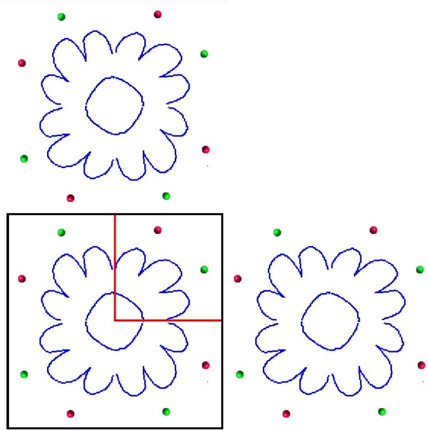


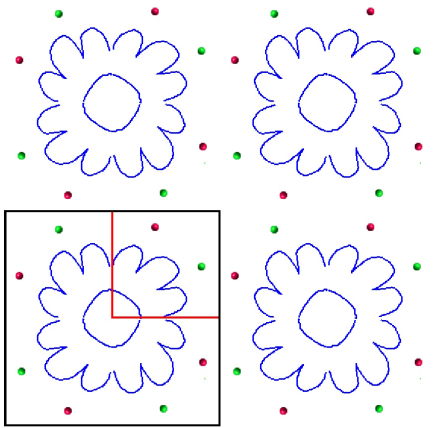


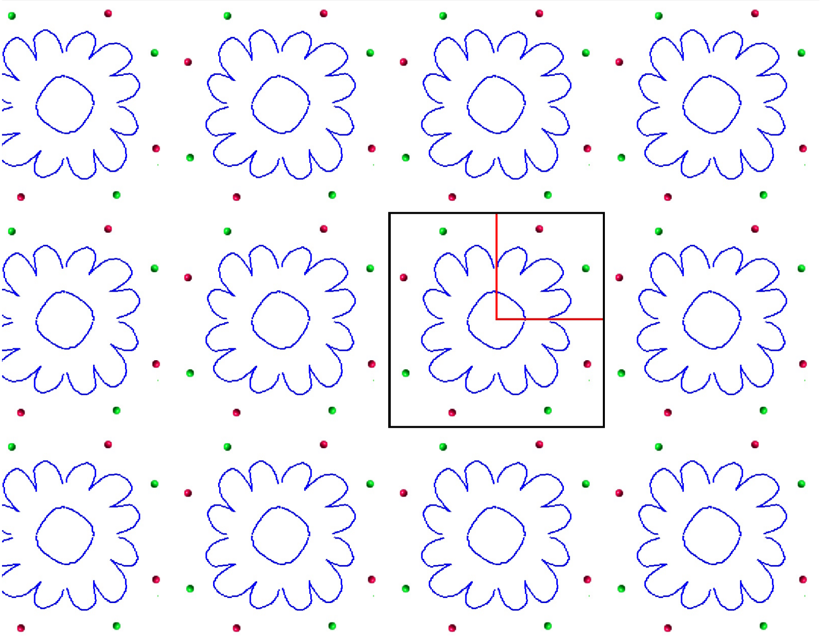












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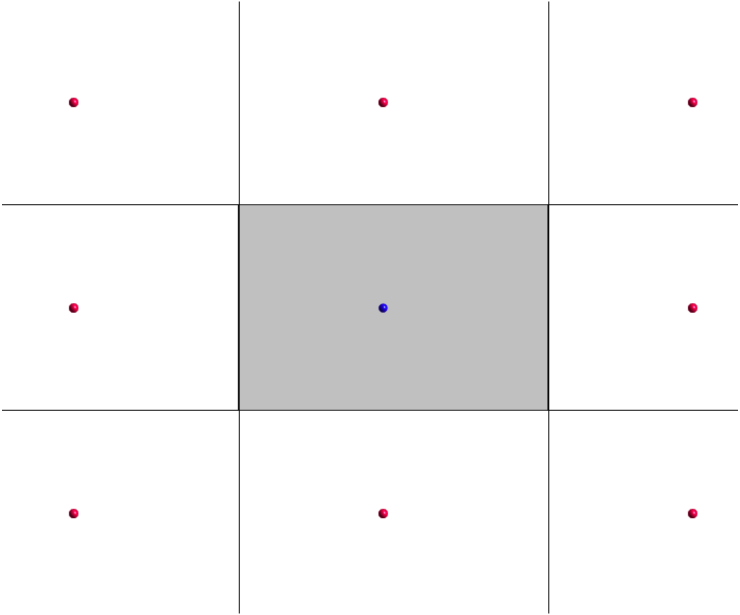
Definition ([3, Def. III.1])

Let $O \subseteq \mathbb{R}^n$ be a discrete set and $u \in O$ be a point. We call

$$D(u, O) = \bigcap_{w \in O, w \neq u} H^+(u, w).$$

the *Dirichlet cell of* u.



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Theorem ([2, Thm. III.11 (ii)])

Let $\Gamma \leq E(n)$ be a crystallographic group and $u \in \mathbb{R}^n$ a point in general position.

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Theorem ([2, Thm. III.11 (ii)])

Let $\Gamma \leq E(n)$ be a crystallographic group and $u \in \mathbb{R}^n$ a point in general position. Then the Dirichlet cell $D(u, u^\Gamma)$ is a fundamental domain for Γ.

Definition ([1, §3, Thm. 7, with remark after])

Let $B \subset \mathbb{R}^n$ be a closed subset. We define the *volume* of B as the Lebesgue measure of B, so $\mathrm{vol}(B) := \lambda(B)$ in the notation of [1].

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Let $B \subset \mathbb{R}^3$ a closed subset, $\varphi \in E(3)$.

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Then $vol(B^\varphi) = vol(B)$.

It can be shown that all fundamental domains of crystallographic groups have the same volume.

current approach to computations - theorem that word length corresponds to distance - algo that incorporates that knowledge connections to TIA -> deformations

Thank you for your attention

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