Computing fundamental domains of crystallographic groups

With connections to topological interlocking

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Lemma

Let $E(n)$ be the set of all isometries of a dimension $n \in \mathbb{N}$.
Then $E(n)$ is a group with the composition of homomorphisms as the group operation.

Proposition ([1, Exa. 1.1, Prop. 1.6])

There is an isometry

$$E(n) \cong O(n) \ltimes \mathbb{R}^n.$$

We denote with φ_o the orthogonal part of φ and with φ_t the vector/translation part of φ.

Then the group operation of $\varphi, \psi \in E(n)$ is as follows:

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$$(\varphi_o, \varphi_t) \circ (\psi_o, \psi_t) := (\underbrace{\varphi_o \circ \psi_o}_{\substack{\text{op. in } O(n) \\ \text{i.e. comp. of maps}}}, \psi_t^{\varphi_o} + \varphi_t),$$

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$$\mathbb{R}^n \times (O(n) \ltimes \mathbb{R}^n) \to \mathbb{R}^n : (v, (\varphi_o, \varphi_t)) \mapsto v^{(\varphi_o, \varphi_t)} = v^{\varphi_o} + \varphi_t.$$

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(i) $\bigcup_{\gamma \in \Gamma} F^{\langle \gamma \rangle} = \mathbb{R}^n$,

(ii) there is a system of representatives $V \subseteq \mathbb{R}^n$ w.r.t. the partition given by the orbits of Γ acting on $\mathbb{R}^n$ such that

$$F^{\circ} \subseteq V \subseteq F.$$

the problem of Computing Dirichlet Cells current approach to computations - theorem that word length corresponds to distance - algo that incorporates that knowledge connections to TIA -> deformations

Thank you for your attention

References:

[1] A. Szczepanski. *Geometry of Crystallographic Groups*. Algebra and discrete mathematics. World Scientific, 2012. ISBN: 9789814412261. URL: https://books.google.de/books?id=wX26CgAAQBAJ.