Mechanical Comparison of Arrangement Strategies for Topological Interlocking Assemblies

Lukas Schnelle

Dec 2023

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and triangular faces

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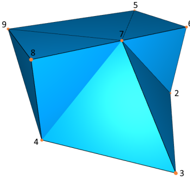

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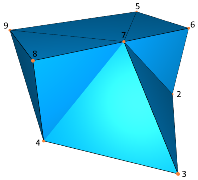

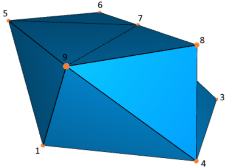

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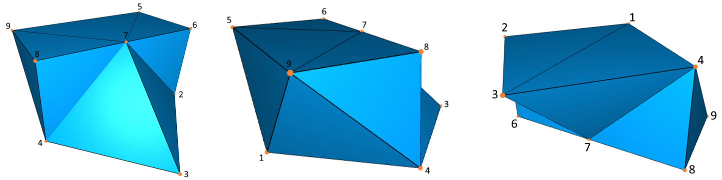

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Crystallograhpic/Wallpaper Groups Let $\Gamma \leq E(n)$ a subgroup of the group of isometries of dimension n.
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Proposition ([1, Prop. 1.9])

Let $\Gamma \leq E(n)$.
Then $E(n)/\Gamma$ is compact iff the orbit space $\mathbb{R}^n/\Gamma$ is compact.

Examples of wallpaper groups

$p1 :=$

$$\left\langle \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right), \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right\rangle$$

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$$p4 :=$$

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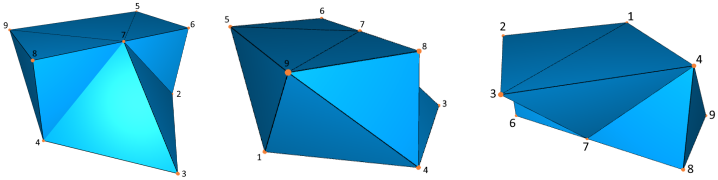
crystallographic groups on the Versatile Block



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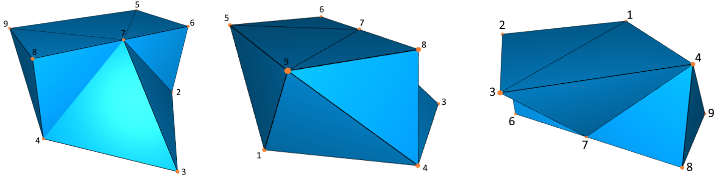


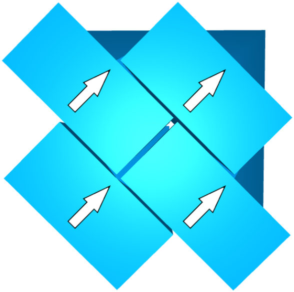
Notice: Versatile Block has only vertices with $z = 0$ and $z = 1$. Now consider the an isometry φ from before lifted to $\mathbb{R}^3$ by

$$\hat{\varphi} : \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \varphi \begin{pmatrix} x \\ y \end{pmatrix} \\ z \end{pmatrix}$$

Applied to all coordinates of the vertices of the Versatile Block we get a Versatile Block between the same planes.

Planar Assemblies of the Versatile Block

With the groups $p1$, pg and $p4$ from before we get three interlocking assemblies consisting only of copies of the Versatile Block:



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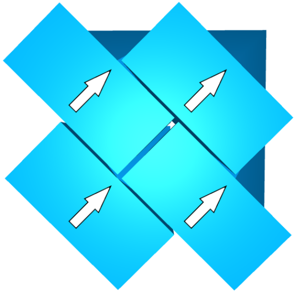

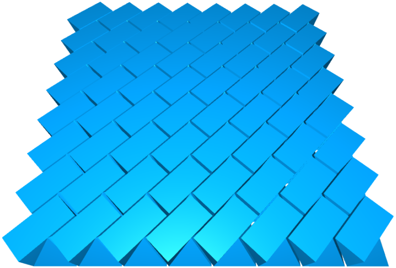

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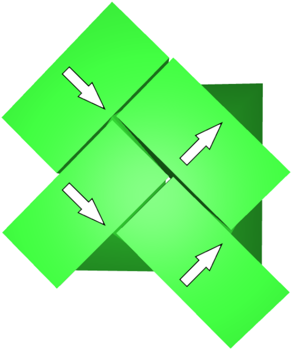

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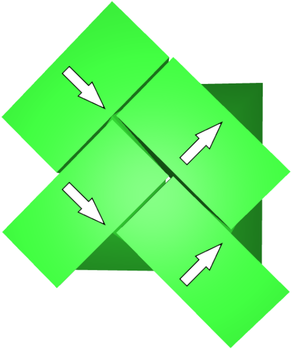

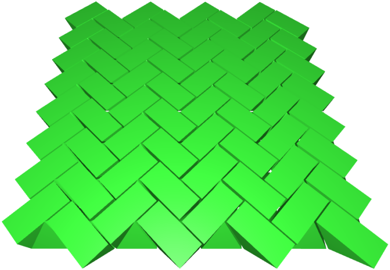

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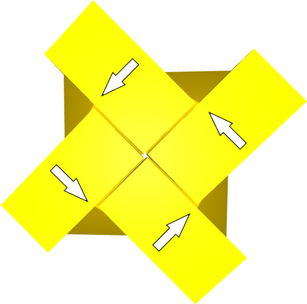

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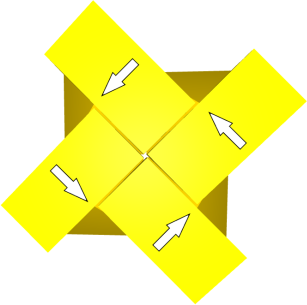

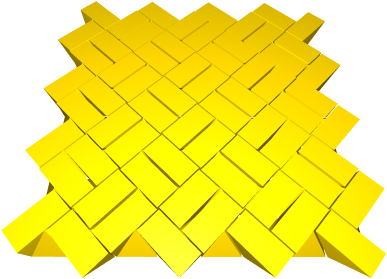

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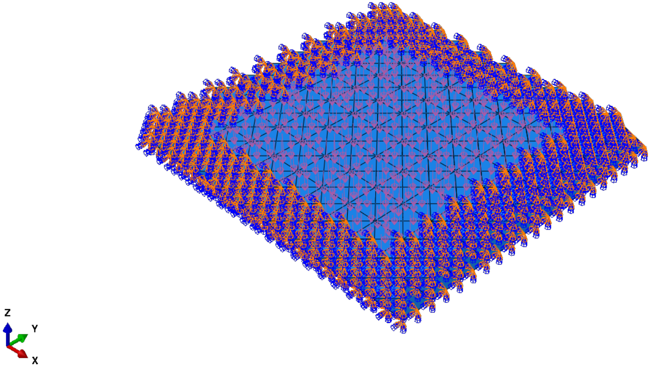

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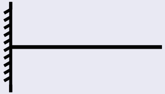

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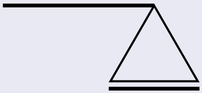

Only movement in direction of line below triangle and rotation.

Problem formulation

Simulation Setup

Stresses

Combinatorial Method

Combinatorial Results

[1] A. Szczepanski and W. Scientific. *Geometry of Crystallographic Groups*. Algebra and discrete mathematics. World Scientific, 2012. URL: https://books.google.de/books?id=wX26CgAAQBAJ.