Mechanical Comparison of Arrangement Strategies for Topological Interlocking Assemblies

Lukas Schnelle

Dec 2023

Topological Interlocking

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- planar topological interlocking assemblies, i.e. between two parallel planes in 3D-space,
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- only copies of the same block differently arranged.

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and triangular faces

$$\begin{aligned} B_2 := \{&\{v_1, v_2, v_3\}, \{v_1, v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_1, v_4, v_9\}, \{v_1, v_5, v_9\}, \\ &\{v_2, v_3, v_7\}, \{v_2, v_5, v_6\}, \{v_2, v_6, v_7\}, \{v_3, v_4, v_7\}, \{v_4, v_7, v_8\}, \\ &\{v_4, v_8, v_9\}, \{v_5, v_6, v_7\}, \{v_5, v_7, v_9\}, \{v_7, v_8, v_9\}\}. \end{aligned}$$

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$$v_1 = (0,0,0), v_2 = (1,1,0), v_3 = (2,0,0),$$
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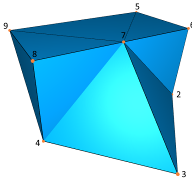

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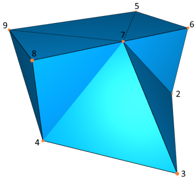

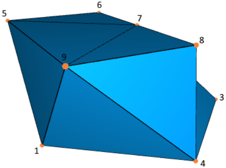

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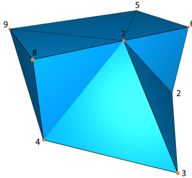

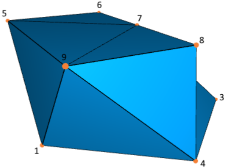

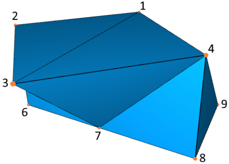

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Crystallograhpic/Wallpaper Groups Let $\Gamma \leq E(n)$ a subgroup of the group of isometries of dimension n.
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Proposition ([1, Prop. 1.9])

Let $\Gamma \leq E(n)$.
Then $E(n)/\Gamma$ is compact iff the orbit space $\mathbb{R}^n/\Gamma$ is compact.

Examples of wallpaper groups

$p1 :=$

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$$p4 :=$$

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Planar Assemblies of the Versatile Block

Finite Element Method

Problem formulation

Simulation Setup

Stresses

Combinatorial Method

Combinatorial Results

[1] A. Szczepanski and W. Scientific. *Geometry of Crystallographic Groups*. Algebra and discrete mathematics. World Scientific, 2012. URL: https://books.google.de/books?id=wX26CgAAQBAJ.