university of groningen / faculty of science and engineering

Principles of Measurement Systems Assignment

Non-linear Oscillators

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Introduction

Part of the joy of having computational tools at your disposal is that you can easily solve almost every differential equation. While most analytic treatments of oscillations are limited to small displacements where the restoring forces are linear, numerically we can eliminate these restrictions and explore some interesting nonlinear physics. In this assignment we look at oscillators that may be harmonic for certain parameter values but then become anharmonic. We explore their behaviour when a driving force and/or damping are included and investigate non-linear resonances and beating.

A file implementing a basic ordinary differential equation solver is provided on Brightspace course page (more information on how this solver works can be found here). You can use this file as a basis for your simulation. After downloading this file, you can upload and run it on Google Collab.

For this assignment you are expected to hand in a report with your results as well as the code you used to produce them (see here for more information).

Theory

Nonlinear oscillators are a problem in classical mechanics in which the Equation Of Motion (hereafter: EoM) is given by Newton's second law:

$$m\frac{d^2x}{dt^2} = F_k(x) + F_{ext}(x,t) \tag{1}$$

where $F_k(x)$ is the restoring force exerted from the spring with spring constant, k, and $F_{ext}(x,t)$ is the external force. As a first model we explore a potential, $V_k(x)$, that is a harmonic oscillator for small displacements, but also contains a perturbation parameter, α, that introduces non-linearity for large displacements, x:

$$V_k(x) = \frac{1}{2}kx^2(1 - \frac{2}{3}\alpha x) \tag{2}$$

With this potential and assuming that there is no external force, the EoM becomes (remember $F_k = -dV_k/dx$):

$$m\frac{d^2x}{dt^2} = -kx(1 - \alpha x) \tag{3}$$

If $\alpha x << 1$ the motion is essentially harmonic, but as $x \to 1/\alpha$ the anharmonic effects become significant.

As a second model for our non-linear oscillator, assume that the spring's potential function is proportional to some arbitrary *even* power p of the displacement:

$$V_k(x) = \frac{1}{p} k x^p \tag{4}$$

which gives rise to the following restoring force:

$$F_k(x) = -\frac{dV_k}{dx} = -kx^{p-1} \tag{5}$$

We require an *even* p to ensure that the restoring force contains an odd power p, which guarantees that it is a *restoring* force for both negative and positive x. The motion is harmonic only if $p = 2$.

More information can be found here [1,2].

Questions

When answering the following questions, make sure you report the values of all the important parameters used.

1. The code for model 1 is provided on Brightspace. Use this to produce harmonic and anharmonic motion. Show and compare your plots for the position, $x(t)$, and velocity, $x'(t)$. [**10 points**]

2. Demonstrate that anharmonic oscillators are non-synchronous (that is, vibrations with different amplitudes have different periods). [**5 points**]

3. Use the provided plots of the harmonic and anharmonic potentials to describe the physics at play. Consider what happens to the force at small and large displacements. How does the force change for each potential? [**10 points**]

4. Write your own code to investigate model 2 (where the restoring force is given by equation 5). Keep $F_{ext}(x, t) = 0$. Demonstrate that this model also produces non-synchronous oscillations. [**5 points**]

5. Keep the initial condition, $x(0)$, comparable to the amplitude of F_k. Plot $x(t)$ for different powers, $p = 2, 4, 6, 8, ...$ Plot the potentials $V_k(x)$ for the same powers. Explain what happens to the motion as p increases. [**10 points**]

Before answering the following questions, add a time-dependent external force to your model, $F_{ext} = F_o sin(wt)$, where w is the driving frequency. The EoM becomes:

$$m\frac{d^2x}{dt^2} = F_o sin(wt) - kx^{p-1} \tag{6}$$

1. Keep $p = 2$ for this question. Try different magnitudes for the external force, F_o. Demonstrate that for large magnitudes, F_o, the system is overwhelmed by the driving force and, after the transients die out, the system oscillates in phase with the driver regardless of the driver's frequency (a.k.a. 500-pound gorilla effect, see figure 1). Show the resulting plots for three different driving frequencies, w. [**7 points**]

2. Do the same as above but now with $p = 4$. Estimate what is the *minimum* value of F_o for which the 500-pound gorilla effect occurs when $p = 4$? What about when $p = 2$? Explain why these values are different. [**8 points**]

3. Keep $p = 2$ for this question. Demonstrate the beating effect (see figure 2). For beating to occur you need $F_k \approx F_{ext}$ and $w \approx w_o$, where w_o is the natural frequency. Show the resulting plots for three different driving frequencies, w. [**5 points**]

4. Keep $F_k \approx F_{ext}$. Consider multiple driving frequencies in the range $w_o/10 < w < 10w_o$ (make sure to include $w = w_o$). Do this for both $p = 2$ and $p = 4$. Describe the resulting plots for $x(t)$. Explain the physics when $w = w_o$ (i.e. resonance) and when $w \approx w_o$ (i.e. beating). Remember to do this for both $p = 2$ and $p = 4$. [**15 points**]

Before answering the remaining questions, add a friction term to your model, such that the EoM becomes:

$$m\frac{d^2x}{dt^2} = F_o sin(wt) - kx^{p-1} - b\frac{dx}{dt} \tag{7}$$

where b is the coefficient of friction.

1. Keep $p = 2$. Create a plot of maximum amplitude against driving frequency in the range $w_o/10 < w < 10w_o$. Describe how this curve is changing when b is very large and very low. [**10 points**]

2. Take $F_o << F_k$. Consider different coefficients of friction, b, to investigate underdamping, overdamping and critical damping. Produce plots for all three cases. Do this for both $p = 2$ and $p = 4$. Describe the results. [**5 points**]

3. Repeat the above question for $F_o \approx F_k$ and $F_o = 10F_k$. Report your observations. [**10 points**]

BONUS: As a bonus part of this assignment you can experiment further with your model. You are free to investigate your own ideas (make sure to explicitly state what it is you are investigating and all the parameters used in your model). Below are some suggestions, you can use these as an inspiration.

1. Extend your harmonic oscillator for different types of friction:

$$F_f^{(static)} \leq -\mu N$$
$$F_f^{(kinetic)} = -\mu N \frac{v}{|v|}$$
$$F_f^{(viscous)} = -bv$$

where N is the normal force, μ and b are the coefficient of friction (you can set these values yourself), and v is the velocity. How does the motion differ for each type of friction?

2. Consider a simple model without external force and without friction. Plot the potential energy, the kinetic energy and the total energy (kinetic plus potential) for 50 periods. Comment on the relation between potential and kinetic energy and how it depends on the potential's parameters (α in equation 2 and/or p in equation 4).

3. Consider a simple model without external force, without friction and a potential given by equation 4. Verify the Virial theorem:

$$< KE >= \frac{p}{2} < PE > \tag{8}$$

where $< KE >$ and $< PE >$ are the time-averaged values for the kinetic and potential energy, respectively. Test this for different values of p and report your findings.

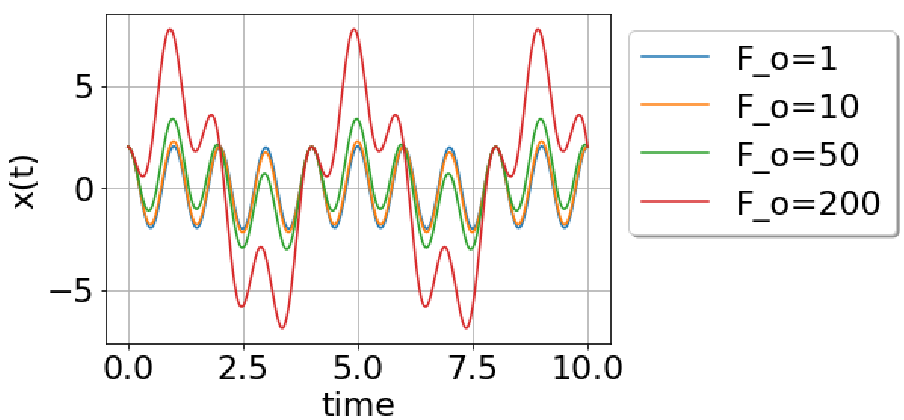

Figure 1: The 500 pound gorilla effect (red line). The period of the harmonic oscillator is $T = 1$, and for the driving force the period is $T = 4$.

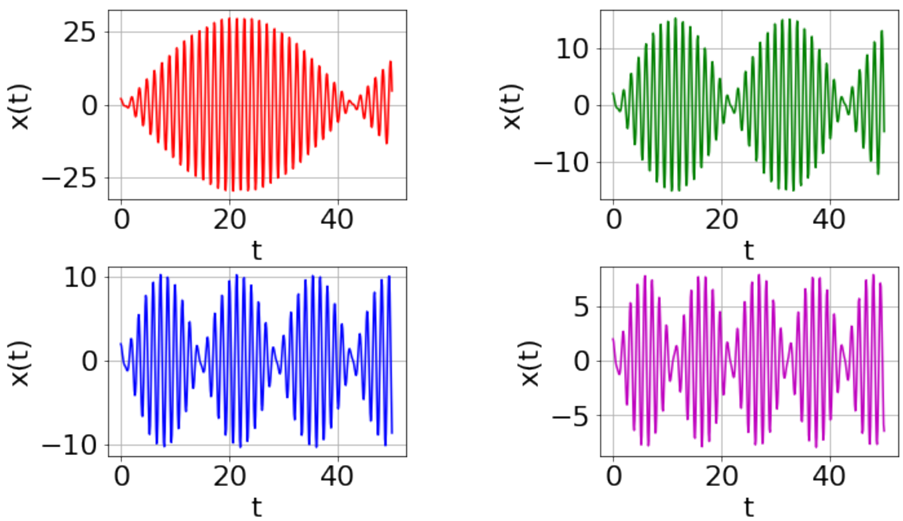

Figure 2: Beating effect for different driving frequencies

Report

Write a report about your work explaining your method and results. Make sure to be clear about your way of thinking and the steps you followed. When answering the above questions make sure to report the values of all the important parameters. The report should not exceed 5 pages (figures excluded) unless a bonus part is also considered in which case the report should not exceed 10 pages.

Note 1: You should also submit your code that reproduces your results.
Note 2: The points from answering the above questions counts as 80% of your final grade for this assignment. The remaining 20% is based on your report-writing skills (inclusion of relevant introduction and theory, clarity in presenting results and methods, conciseness etc.)

Bibliography

[1] Landau Rubin H. , Páez Manuel J., Bordeianu Cristian C. , "Computational Physics Problem Solving with Python", 3rd edition, Published in 2015 by WILEY-VCH Verlag GmbH & Co.KGaA. Relevant chapter: 8.

[2] Figliola Richard S. , Beasley Donald E. "Theory and Design for Mechanical Measurements" , 6th Edition, Publised in 2015 by John Wiley & Sons, Inc. Relevant chapters: 2, 3