Resolutions for Unit Groups of Orders

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Overview

1. From densest sphere packings to perfect forms
   - The sphere packing problem
   - Perfect lattices
2. From perfect forms to the homology of infinite groups
   - Some definitions from homological algebra
   - Quadratic forms and unit groups of orders
   - Perturbations
   - The well-rounded retract
3. Computational Results
   - Linear groups over imaginary quadratic number fields
   - Maximal orders in quaternion algebras
   - Further applications of the well-rounded complex
4. References

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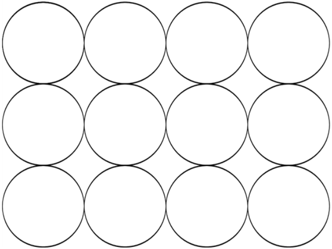

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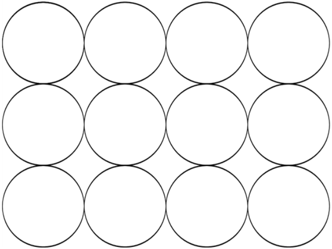

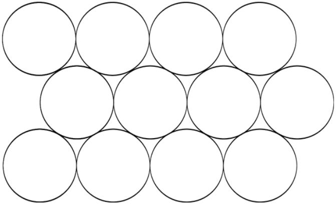

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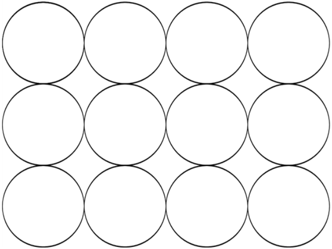

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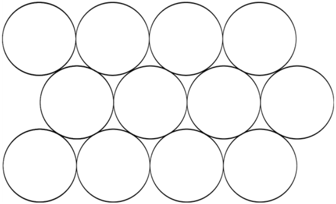

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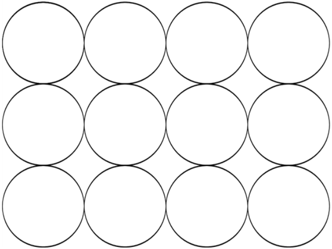

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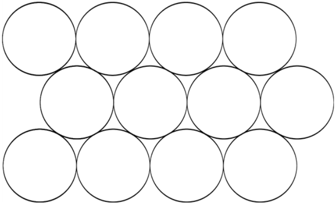

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Let $B = (b_1, ..., b_n)$ be a basis of $\mathbb{R}^n$. The *($\mathbb{Z}$-)lattice* generated by B is the set

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In dimension 2: The square packing belongs to the lattice with basis (e_1, e_2), the hexagonal packing to the lattice with basis $(e_1, 1/2(e_1 + \sqrt{3}e_2))$.

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- Let Λ be the 24-dimensional Leech-lattice. Then Λ has density equal to $V_{24} \approx 0.0019$.

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Densest known lattice sphere packings in some low dimensions

Dim.	2	3	4	5	6	7	8	24
Lattice	A_2^{**}	A_3^{**}	D_4^*	D_5^*	E_6^*	E_7^*	E_8^*	Λ
Density	0.907	0.740	0.617	0.465	0.373	0.295	0.254	0.002

*: Provably best lattice sphere packing in the given dimension.

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Basic idea: Use the action of certain groups on the space of quadratic forms (with the perfect forms as distinguished points) to obtain structural information about these groups.

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Definition: Chain Complexes

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- C is called *projective (free)*, if C_n is projective (free) for all $n \in \mathbb{Z}$.

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A a left R-module. An acyclic and projective (free) chain complex of the form

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- Every R-module has a free resolution.

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Group Homology

G a group, $R = \mathbb{Z}G$, B an R-module.
$\mathrm{H}_n(G, B) := \mathrm{Tor}_n^R(B, \mathbb{Z})$ is called the n-th homology group of G with coefficients in B.

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- No known generally applicable methods.

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The Cell Structure

$F \in \mathrm{P}$. Define $\mathcal{C}\ell_L(F) := \{F' \in \mathrm{P} \mid \mathrm{S}_L(F) = \mathrm{S}_L(F')\}$ the *minimal class* of F.

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The cellular chain complex

The decomposition yields an acyclic chain complex C, where C_n is the free Abelian group on the minimal classes in dimension n. C_n becomes a G-module by means of the G-action.

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Perturbations (C. T. C. Wall 1961, [7])

Assume we are given a free $\mathbb{Z}G$-resolution $A_{p,*}$ (boundary d_0) of C_p for all p. Then there are homomorphisms $d_k : A_{p,q} \to A_{p-k,q+k-1}$, such that

$$d := d_0 + d_1 + d_2 + \ldots : R_n := \bigoplus_{p+q=n} A_{p,q} \to R_{n-1} := \bigoplus_{p+q=n-1} A_{p+q}$$

is the boundary of an acyclic chain complex of free $\mathbb{Z}G$-modules, where $\mathrm{H}_0(R) \cong \mathbb{Z}$.

A diagram:

$$\cdots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \longrightarrow \mathrm{H}_0(C)$$

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\begin{tikzcd}
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The situation in the space of positive definite forms:

Let $(C_{k,i})_i$ be a system of representatives of the G-orbits of minimal classes of dimension k and $S_{k,i} := \mathrm{Stab}_G(C_{k,i})$.

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- Use these resolutions and C. T. C. Wall's lemma to construct a resolution of $\mathbb{Z}$ over G.

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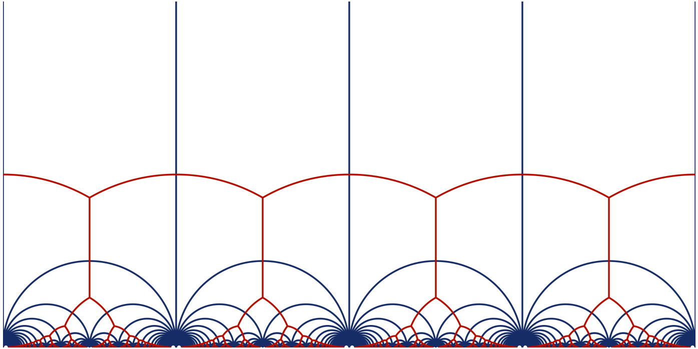
- The topological closure of each cell is a polytope.
- $\mathrm{P}^{\mathrm{wr}}_{=1}$ is a retract of P, especially we have that the cellular chain complex is again acyclic and $\mathrm{H}_0 \cong \mathbb{Z}$ (A. Ash, 1984 [1]).

The Well-Rounded-Retract for $\mathrm{SL}_2(\mathbb{Z})$



Quelle: `http://www.uncg.edu/mat/numbertheory/quadratic_form.html`

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- There is a subspace such that each cell in it is a polytope and has finite stabiliser in G.
- We may use this cellular decomposition and the *finite* stabilisers to construct a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

Overview

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- There are essentially $|\mathcal{C}\ell_K / \mathcal{C}\ell_K^n|$ different maximal orders in $K^{n\times n}$.

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Especially: $G_1 \ncong G_2$.

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$$\mathrm{H}_n(S_1, \mathbb{Z}) = \begin{cases} C_2 \times C_6 \times \mathbb{Z}^2 & n = 1 \\ C_2^2 \times C_{12} \times \mathbb{Z} & n = 2 \\ C_2^2 \times C_{24} & n = 3 \\ C_4 \times C_{12} & n = 4 \end{cases}$$

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2. $S_2 := \mathrm{SL}(L_2)$:

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$$\mathrm{H}_n(\mathcal{O}^*, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0 \\ C_{24}, & n \equiv 1 \pmod 2 \\ C_2, & n \equiv 0 \pmod 2, n > 0 \end{cases}$$

Further information available from the well-rounded complex

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- G has periodic homology if and only if any finite Abelian subgroup is cyclic.

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Theorem (R. Coulangeon, G. Nebe, 2013, [2])

The set $\{\mathrm{Stab}_G(C) \mid C \text{ cell in the well-rounded complex}\}$ contains a system of representatives of the conjugacy classes of maximal finite subgroups of G.

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- Use the recently introduced technique of torsion subcomplexes (A. Rahm) to acquire further information.
- Apply these methods to non-congruence subgroups of the Bianchi-Groups (i.e. SL_2 over imaginary quadratic integers).

Overview

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Thank you for your attention.