# Hecke Operators for Algebraic Modular Forms

Von der Fakultät für Mathematik, Informatik und
Naturwissenschaften der RWTH Aachen University zur Erlangung
des akademischen Grades eines Doktors der Naturwissenschaften
genehmigte Dissertation vorgelegt von

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Tag der mündlichen Prüfung: 8. September 2016

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# 1. Introduction

## 1.1. Algebraic Modular Forms

The interplay between the theory of modular forms on the one hand and algebraic number theory on the other hand has proven to be very fruitful. Perhaps the best-known example of this is the proof of Fermat's last theorem via Wiles's proof of the modularity theorem which was formerly known as the Taniyama-Shimura-Weil conjecture. In this instance the deep connection between modular forms and elliptic curves allows to answer the innocuous looking question whether the equation $a^n + b^n = c^n$ with $a, b, c \in \mathbb{Q}$ and $n \in \mathbb{Z}_{\geq 3}$ has non-trivial solutions.
The following two examples - which have a stronger computational flavour and are therefore closer to the heart of this thesis - also illustrate this remarkable link between the analytic and the algebraic world.

Beginning in the 1950s Heinrich Brandt and Martin Eichler studied the problem of finding a "nice" basis of the space of elliptic modular forms of weight 2 and (square-free) level $N$ (cf. [Eic55]), where "nice" means that the Fourier coefficients of its elements are already known or easily computable. To that end they studied the theta series of lattices which arise from (right) ideals of quaternion orders of discriminant $N$ endowed with the positive definite norm form. It turns out that certain linear combinations of these theta series - where the coefficients arise from the combinatorial structure of the ideals in the quaternion algebra - always constitute modular forms of weight 2 and level $N$. Moreover these linear combinations actually generate the whole space of weight 2 modular forms and one obtains a basis with the desired properties in this way. In addition the so-called Brandt matrices which capture the combinatorial structure of the ideals are closely related to the action of Hecke operators on the space of weight 2 modular forms via Eichler's trace formula (cf. [Eic73]).

A different (but closely related) connection between the two theories can be found in the work of Martin Eichler and Anatolij Andrianov. Consider an even, integral lattice $L$ in the rational vector space $V$ of even dimension $n$, endowed with a positive definite form $Q$. The theta series of $L$ is the generating function of the values of $Q$ on the lattice vectors,

$$\theta_L(q) = \sum_{v \in L} q^{Q(v)}.$$

It turns out that by setting $q = \exp(\pi i z)$ we obtain an elliptic modular form of weight $n/2$ and level $N$, where $N$ is the smallest integer such that $NL^{\#}$ is again even integral.
We call the set of lattices in $V$ that are locally isometric to $L$ at all primes the *genus* of $L$. While two lattices in the same genus are not necessarily isometric (i.e. in the same orbit under the orthogonal group with respect to $Q$), the genus does decompose into only finitely many isometry classes. Following Eichler's and Martin Kneser's work (cf. [Eic52a, Kne57]) we call two lattices in the same genus $p$-neighbours (for a fixed prime $p$) if their intersection has index $p$ in both of them. It is algorithmically possible to enumerate all $p$-neighbours of a given lattice $L$ and Kneser showed that under certain assumptions one can obtain a system of representatives of the isometry classes in its genus by simply computing iterated $p$-neighbours and continuously checking for isometry. In this situation let $L_1, ..., L_h$ be such a system of representatives and define the $p$-neighbouring graph of the genus as the (weighted and directed) graph whose vertices are the $L_i$ and whose arcs $L_i \to L_j$ have weight the number of $p$-neighbours of $L_i$ isomorphic to $L_j$. Since all lattices in the same genus have identical level, the theta series of $L_1, ..., L_h$ span a subspace of the weight $n/2$ and level $N$ modular forms. Eichler showed (cf. [Eic52b]) that the adjacency matrix of the $p$-neighbouring graph acts as a Hecke operator on this subspace, whenever $p \nmid N$. This result was generalized by Andrianov (cf. [And80]) to degree $k$ theta series (in which case one obtains Siegel modular forms) and later Yoshida - working in an adelic setting - obtained results including vector valued modular forms (cf. [Yos86]).

This thesis is concerned with the explicit computation of Hecke operators acting on spaces of so-called algebraic modular forms. Let $\mathbb{G}$ be a connected semisimple linear algebraic group over the rationals[1] and assume that the real Lie group $\mathbb{G}(\mathbb{R})$ is compact (for example let $\mathbb{G}$ be the special orthogonal group of a positive definite quadratic form). In 1999 Benedict Gross noticed that under these conditions one can set up a theory of automorphic forms for $\mathbb{G}$ in an entirely algebraic way which led to his definition of *algebraic modular forms* (cf. [Gro99]).

Denote the finite adeles of $\mathbb{Q}$ by $\hat{\mathbb{Q}}$, let $K \leq \mathbb{G}(\hat{\mathbb{Q}})$ be an open and compact subgroup (a so-called *level*), and let $V$ be an irreducible representation of $\mathbb{G}$ (the so-called *weight*). Furthermore let $\Sigma := K\backslash\mathbb{G}(\hat{\mathbb{Q}})/\mathbb{G}(\mathbb{Q})$ and define

$$M(V, K) := \left\{ f : \mathbb{G}(\hat{\mathbb{Q}}) \to V \mid {\scriptstyle f(\kappa\gamma g) = f(\gamma)g \text{ for } \gamma \in \mathbb{G}(\hat{\mathbb{Q}}), \atop g \in \mathbb{G}(\mathbb{Q}), \kappa \in K} \right\}. \tag{1.1}$$

It turns out that the set $\Sigma$ is always finite, and since every function in $M(V, K)$ is determined by its finitely many values on a system of representatives of $\Sigma$, the vector space $M(V, K)$ is finite-dimensional[2]. The space $M(V, K)$ comes equipped with an action of the Hecke operators $T(\gamma)$, $\gamma \in \mathbb{G}(\hat{\mathbb{Q}})$, one for each

[1] Note that we only restrict ourselves to the rational field for the purpose of this introduction and later work with arbitrary (totally real) number fields.
[2] In the case $V = \mathbb{Q}$ we can even identify $M(V, K)$ with the set of $\mathbb{Q}$-valued functions on $\Sigma$.

double coset $K\gamma K$, which form the basis of an algebra - the so-called Hecke algebra - under convolution. From an algorithmic standpoint the main task is now to explicitly compute this action. That means for given $\mathbb{G}, K, V$ and $\gamma$ we would like to be able to compute a basis for $M(V, K)$ and the action of $T(\gamma)$ on $M(V, K)$ with respect to this basis.

Interesting examples of open compact subgroups in $\mathbb{G}(\hat{\mathbb{Q}})$ often arise from integral forms $\underline{G}$ of $\mathbb{G}$ i.e. stabilizers of lattices $L$ in some faithful representation of $\mathbb{G}$. In this case the problem of finding a basis for $M(V, K)$ (or, essentially equivalently, finding representatives for $\Sigma$) amounts to the same as computing a system of representatives of the $\mathbb{G}(\mathbb{Q})$-isomorphism classes in the $\mathbb{G}$-genus of the given lattice $L$ (or equivalently the genus of $\underline{G}$). The cardinality of $\Sigma$ is then known as the *class number* of $L$ and in some sense measures the obstruction to $\mathbb{G}$-isomorphism being a local property. For $\mathbb{G} = \mathrm{SO}_n$, the special orthogonal group of a positive definite form, this is exactly the problem we already considered above in the context of Kneser's $p$-neighbours. In this situation we saw that one can list a system of representatives by the Kneser method of forming iterative $p$-neighbours at a prime $p$ by virtue of the *Strong Approximation Property.* For other algebraic groups this property does not necessarily hold; however, in many examples we get a slightly weaker result, the so-called *Almost Strong Approximation Property* which still leads to an algorithm for enumerating the desired representatives (cf. [HJ97, CH02, Cha04]). Moreover the adjacency matrix of the arising graph is again a Hecke operator which in this case acts on the space of algebraic modular forms of trivial weight.

The second task of computing the action of $T(\gamma)$ is usually significantly harder to achieve (in many cases one even obtains the enumeration of representatives of $\Sigma$ as a byproduct). The first algorithmic work formulated in the language of algebraic modular forms was published by Joshua Lansky and David Pollack (cf. [LP02, Lan01]) who achieved a certain double coset decomposition in a fairly general setup (a result which we will use extensively throughout this thesis). They then performed some explicit computations for compact forms of $G_2$ and $\mathrm{PGSp}_4$ over the rationals.
In 2009 Clifton Cunningham and Lassina Dembélé performed computations with compact forms of $\mathrm{GSp}_4$ over totally real fields of narrow class number one (cf. [CD09]).
Around the same time David Loeffler presented algorithms for working with algebraic modular forms for unitary groups over CM-fields which give rise to compact forms of $\mathrm{GL}_n$ (cf. [Loe08]). For compact forms of classical groups (in particular unitary groups over CM-fields and orthogonal groups over totally real number fields) Matthew Greenberg and John Voight implemented methods making use of lattice techniques (cf. [GV14]). Lastly Jeffery Hein, a student of John Voight's, worked out the cases of special orthogonal and spin groups in his dissertation thesis. These articles employ various methods (often depending on the specific structure of the group at hand) and do not necessarily make use of the standard approach by Joshua Lansky and David Pollack.

This thesis's contribution to this ongoing project is as follows. For compact forms of $G_2$ we present an implementation that works over arbitrary totally real number fields, generalizing the work of Lansky and Pollack and reproducing their results over the rationals. Furthermore we present an implementation to work with algebraic modular forms for unitary groups over definite quaternion algebras. These groups constitute compact forms of symplectic groups which have (to the best of our knowledge) not yet been worked with in this context. The implemented algorithms work over arbitrary totally real number fields and in arbitrary dimension (subject to the obvious constraints on memory and processing speed). They can be used to compute spaces of algebraic modular forms and the action of the Hecke algebra for arbitrary weight and parahoric level structure defined by a lattice.

In addition we present a novel method of computing certain Hecke operators based on an idea by Eichler. While this method is already more efficient than the standard approach in computing these operators, it comes with the added benefit of actually computing two Hecke operators at once as well as a system of representatives with respect to a second level $K'$. Using this approach one can in general not obtain the whole action of the Hecke algebra, and the answer to the question which operators are obtainable highly depends on the specific group at hand. We study this in detail in Chapter 5 and it turns out that the compact forms of symplectic groups are among the cases where one actually can obtain the whole action. In this situation the method yields a significant speed-up compared to the standard approach (in addition to the added benefit of computing several Hecke operators and systems of representatives at once).

Now let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two groups with the same properties as $\mathbb{G}$ above. We denote by $\widehat{\mathbb{G}_i}$, $i = 1, 2$, the complex dual of $\mathbb{G}_i$, i.e. the group over $\mathbb{C}$ whose root datum is dual to that of $\mathbb{G}_i/\mathbb{C}$. If $\rho : \widehat{\mathbb{G}_1} \to \widehat{\mathbb{G}_2}$ is a homomorphism one might then hope to be able to *lift* algebraic modular forms for $\mathbb{G}_1$ to modular forms for $\mathbb{G}_2$ via $\rho$ by employing the so-called *Satake homomorphism*. The Langlands functoriality conjecture suggests that (under certain circumstances) a lifting procedure should exist and while this remains unproven so far, one can still use an algorithmic approach to find numerical evidence for the existence of such a procedure. In their article [LP02] Joshua Lansky and David Pollack in particular discuss lifts from $\mathrm{PGL}_2$ to $\mathrm{PGSp}_4$ via the symmetric cube map $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C}) \cong \mathrm{Spin}_5(\mathbb{C})$. In this case they find quite a few forms on $\mathrm{PGSp}_4$ which appear to be lifts from $\mathrm{PGL}_2$ and the existence of a lifting procedure was later proven by Dinakar Ramakrishnan and Freydoon Shahidi (cf. [RS07]). In this thesis we present some numerical evidence for lifts from $G_2$ to $\mathrm{Sp}_6$ via the embedding $G_2(\mathbb{C}) \hookrightarrow \mathrm{SO}_7(\mathbb{C})$.

Finally we present some alternative usage of our algorithms. If $\underline{G}$ is an integral form of $\mathbb{G}$ and $S$ is a finite set of primes one may consider the $S$-arithmetic group $\underline{G}(\mathbb{Z}_S)$ where $\mathbb{Z}_S$ is the ring of $S$-integers (i.e. the rational numbers that are integral "away from $S$"). The group $\underline{G}(\mathbb{Z}_S)$ acts on the direct product of the Euclidean buildings of $\mathbb{G}$ at the primes in $S$ and this action can be used to study the structure of $\underline{G}(\mathbb{Z}_S)$ (e.g. see [BS76]). We employ this action together with an algorithm of Graham Ellis's to explicitly construct a free resolution of the trivial $\underline{G}(\mathbb{Z}_S)$-module and use this to compute the integral homology up to a certain degree.

## 1.2. From Classical to Algebraic Modular Forms

We want to take the time to briefly motivate the term "algebraic modular form" by outlining the passage from elliptic modular forms to adelic automorphic forms and representations and then describing the connection between automorphic representations and algebraic modular forms.

Let us denote by $\mathbb{H}$ the upper half plane in $\mathbb{C}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ via

$$gz = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \tag{1.2}$$

and we set $j(g,z) = (cz+d)$. Then an *elliptic modular form* of weight $k$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that

1. $f$ is holomorphic.

2. $j(g,z)^{-k} f(gz) = f(z)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$.

3. $f$ is holomorphic at $i\infty$ (i.e. $f$ has a Fourier expansion of the form $f(z) = \sum_{n\in\mathbb{Z}} a_n \exp(2\pi i z)$ with $a_n = 0$ for $n < 0$).

Since $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ and the stabilizer of $i \in \mathbb{H}$ is $\mathrm{SO}_2(\mathbb{R})$, we can identify $\mathbb{H}$ with $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. Thus every function $f$ on $\mathbb{H}$ can also be considered as a ($\mathrm{SO}_2(\mathbb{R})$-invariant) function on $\mathrm{SL}_2(\mathbb{R})$ (by evaluating at $gi$). For a modular form $f$ of weight $k$ we now set $\hat{f} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C},\ g \mapsto j(g,i)^{-k} f(gi)$. By construction $\hat{f}$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant, so we have turned an $\mathrm{SO}_2(\mathbb{R})$-invariant form with a suitable transformation under $\mathrm{SL}_2(\mathbb{Z})$ into an $\mathrm{SL}_2(\mathbb{Z})$-invariant function with a suitable transformation under $\mathrm{SO}_2(\mathbb{R})$. In addition the other conditions imposed on $f$ for being a modular form translate to certain well-behavedness properties for $\hat{f}$. Functions satisfying these conditions are called *classical automorphic forms* on $\mathrm{SL}_2(\mathbb{R})$.

On the other hand let us now consider the adele ring $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \hat{\mathbb{Q}}$ of $\mathbb{Q}$ and let $K$ be a maximal compact subgroup of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, for instance we could take $K = \mathrm{O}_2(\mathbb{R}) \times K_{fin}$, where $K_{fin} = \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$. We call a smooth function $f : \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ a ($K$-finite) adelic $L^2$-automorphic form if it satisfies the following conditions:

1. $f \in L^2(\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}))$.

2. $f(gx) = f(x)$ for all $g \in \mathrm{GL}_2(\mathbb{Q})$.

3. $f$ is right $K$-finite, i.e. the space $\langle f \circ g : x \mapsto f(xg) \mid g \in K \rangle$ is finite-dimensional.

4. $f$ satisfies a certain differential equation.

5. $f$ is of moderate growth.

Let us denote by $Z \cong \mathrm{GL}_1$ the center of $\mathrm{GL}_2$, then we have the following identification:

$$\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}) \simeq Z(\mathbb{A}_\mathbb{Q})\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_{fin}. \tag{1.3}$$

Thus a classical automorphic form of $\mathrm{SL}_2(\mathbb{R})$ (or starting one step earlier, an elliptic modular form) lifts to a function on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. This lift is $\mathrm{GL}_2(\mathbb{Q})$-invariant by construction and it turns out that it also satisfies the other conditions for being an adelic automorphic form, which yields the desired passage from elliptical modular forms to adelic automorphic forms.

The definition of adelic automorphic forms naturally generalizes in a few ways. One may look at an algebraic number field $k$ instead of $\mathbb{Q}$, consider non-maximal compact subgroups in place of $K$ and, most importantly, one may exchange $\mathrm{GL}_2$ for any connected reductive group $\mathbb{G}$ over $k$. If we keep the ground field $\mathbb{Q}$, the group $\mathrm{GL}_2$ and exchange $K_f$ with the group of matrices whose $p$-part is upper triangular modulo $p^{n_p}$ (where $n_p = 0$ for all but finitely many $p$) we obtain an analogous passage from elliptical modular forms of level $N = \prod_p p^{n_p}$ to the so defined adelic automorphic forms. If we look at $\mathrm{GL}_2(\mathbb{A}_k)$ where $k$ is a totally real number field we obtain the adelic setting of Hilbert modular forms and finally, if we consider $\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q})$ we are in the adelic setting of Siegel modular forms.

Now let $\mathbb{G}$ be a semisimple connected group over $\mathbb{Q}$ such that $\mathbb{G}(\mathbb{R})$ is compact, $V$ an irreducible $\mathbb{G}$-module and $K$ an open and compact subgroup of $\mathbb{G}(\hat{\mathbb{Q}})$ (as in the definition of algebraic modular forms). The space $L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_\mathbb{Q}))$ comes equipped with an action of $\mathbb{G}(\mathbb{A}_\mathbb{Q})$ by right translation (i.e. $gf(x) = f(xg)$) and thus constitutes an infinite-dimensional $\mathbb{G}(\mathbb{A}_\mathbb{Q})$-module. A (complex) representation of $\mathbb{G}(\mathbb{A}_\mathbb{Q})$ is now called *automorphic* if it appears as a subrepresentation of $L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_\mathbb{Q}))$.

Let $M = M(V, K)$ be the space of algebraic modular forms of weight $V$ and level $K$. Then we can identify $M \otimes \mathbb{R}$ with the (real) vector space of all functions

$$f : \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_\mathbb{Q})/K \to V \otimes \mathbb{R} \tag{1.4}$$

subject to $f(gg_\infty) = g_\infty^{-1} f(g)$ (cf. [Gro99, Prop. (8.3)]). Additionally this identification is compatible with the action of $H_K \otimes \mathbb{R}$, the scalar extension of the Hecke algebra $H_K$. We choose a $\mathbb{G}(\mathbb{R})$-invariant inner product on $V \otimes \mathbb{R}$ (which is possible since $\mathbb{G}(\mathbb{R})$ is compact) and associate to each function $f$ as above a map $l_f : V \otimes \mathbb{R} \to L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_\mathbb{Q}))$ via the formula $l_f(v)(g) = \langle f(g), v\rangle$. By another result of Gross's (cf. [Gro99, Prop. (8.5)]) this gives us an $H_K\otimes\mathbb{R}$-linear isomorphism between $M \otimes \mathbb{R}$ and $\mathrm{Hom}_{\mathbb{G}(\mathbb{R})\times K}(V \otimes \mathbb{R}, L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_\mathbb{Q})))$.

In particular in the case where $V$ is absolutely irreducible we obtain a bijection between the simple $H_K$-submodules of $M \otimes \mathbb{C}$ and the irreducible automorphic representations $\pi = \pi_\infty \otimes \hat{\pi}$ of $\mathbb{G}(\mathbb{A}_\mathbb{Q})$ with the properties $\pi_\infty \cong V \otimes \mathbb{C}$ and $\hat{\pi}^K \neq \{0\}$ (i.e. $\hat{\pi}$ admits $K$-fixed points).

This constitutes the relation between algebraic modular forms and automorphic representations.

## 1.3. Outline

This thesis is organized as follows. Chapter 2 gives an introduction to the theory of integral forms of algebraic groups. In the third chapter we introduce the space of algebraic modular forms, the Hecke algebra, and its action on the space of modular forms. We also provide some important facts about this action and the structure of the space of modular forms and the Hecke algebra (in particular in the split local case which is of greatest interest for our applications).

An important step in computing the action of Hecke operators is the decomposition of double cosets into right cosets. For this reason we describe how to perform decompositions of double cosets with respect to parahoric subgroups into right cosets in groups over local fields in Chapter 4. In addition we present some new results on certain double cosets with respect to the intersection of two parahoric subgroups.

Eichler's method usually provides a way to compare the mass of two genera of integral forms. Chapter 5 contains a novel application of this idea to the computation of Hecke operators. We introduce intertwining operators and Eichler elements and work out explicitly what these operators look like. Furthermore we describe the subalgebra that is generated by the Eichler elements for all (almost simple) simply connected and some adjoint groups. For instance we show that these operators (which are often less expensive to compute than the operators one would usually consider) generate the whole Hecke algebra in the case of symplectic groups.

In Chapter 6 we present the necessary results on the structure of groups of type $G_2$ and $C_n$ to the extent that is needed for our computations. We describe their structure over local fields and how one constructs integral forms in these settings. Chapter 7 introduces the concept of lifting of algebraic modular forms. We present the necessary basics on the Satake homomorphism, work it out explicitly for $\mathrm{Sp}_6$, and determine what lifts from $G_2$ to $\mathrm{Sp}_6$ should look like.

Chapter 8 contains some alternative applications of our implemented algorithms. For instance we explain how to obtain a free resolution of the trivial module over $S$-arithmetic subgroups and compute the integral homology of some $S$-arithmetic subgroups of $\mathrm{Sp}_4$ for $S = \{2\}$. In Chapter 9 we give an overview of the results which were obtained using our implementation of the aforementioned algorithms. In particular we present the decomposition of certain genera into isometry classes and the decomposition of spaces of algebraic modular forms into simple submodules for the action of the Hecke algebra. In addition we compare the performance of the algorithm which we developed in Chapter 5 with the standard algorithm, showing that it exhibits a significant improvement.

Lastly the appendix contains a short introduction to the theory of algebraic affine group schemes, to the extent needed in this thesis. Furthermore we present the structure theory of split semisimple algebraic groups over local fields and provide the potential user with a short handbook for the implemented algorithms.

## 1.4. Preliminaries and Notation

At various points throughout this thesis we will need certain results from the theory of linear algebraic groups over fields (in most cases of characteristic zero). We do not present a self-contained introduction into the theory here but instead refer the reader to Tonny A. Springer's book [Spr98], which covers all that is necessary on this topic for our purposes except for the structure of reductive groups over local fields. The basic definitions and some results on the structure (at least in the split semisimple case) of groups over local fields are presented in Appendix B. In particular we introduce the affine building (following Jacques Tits's exposition in [Tit79]) and the generalized Tits system (following Nagayoshi Iwahori's exposition in [Iwa66]).

Moreover we will, at some points, need to look at algebraic groups not over fields but over rings (which are, in our cases, always integral domains). For that purpose we present a short introduction into the theory of algebraic affine group schemes in Appendix A.

Let us agree on some notational conventions for all that follows: The letter $k$ will always denote an algebraic number field, the symbol $\mathcal{O}_k$ its maximal order (i.e. the integral closure of $\mathbb{Z}$ in $k$) and ideals in $\mathcal{O}_k$ will be denoted by lowercase German type letters, i.e. $\mathfrak{a}, \mathfrak{p}, \mathfrak{q}, ...$, the letters $\mathfrak{p}$ and $\mathfrak{q}$ being exclusively used for prime ideals. If $\mathfrak{p} < \mathcal{O}$ is a prime we will denote the completion of $k$ at $\mathfrak{p}$ by $k_{\mathfrak{p}}$ and its ring of integers by $\mathcal{O}_{\mathfrak{p}}$. The letter $F$ will always denote a local field,

$\mathcal{O}_F$ its ring of integers and $\pi$ a generator for the unique maximal ideal in $\mathcal{O}_F$ (sometime called a uniformizer).

Linear algebraic groups over $k$ (or $F$) will be denoted either by their standard symbol if it exists (e.g. $\mathrm{GL}_n$ (or alternatively $\mathrm{GL}_V$), $\mathrm{SL}_n, \mathrm{Sp}_{2n}, ...$) or by an uppercase blackboard bold letter (e.g. $\mathbb{G}$). If $\mathbb{G}$ is such a linear algebraic group we will denote a group scheme over $\mathcal{O}_k$ (or $\mathcal{O}_F$), from which we may obtain $\mathbb{G}$ by restricting to $k$-algebras (or $F$-algebras), by $\underline{G}$, so that the base ring becomes clear from the choice of symbol. Finally we use standard uppercase letters for abstract groups, for instance we might want to use $G = \mathbb{G}(F)$ for ease of notation.

## Acknowledgements

During the process of writing this thesis numerous people have supported me, and I would like to take the opportunity to express my gratitude for their help. First and foremost I want to thank my advisor Prof. Dr. Gabriele Nebe who suggested this topic, and without whose invaluable advice and support this work would not have been possible. I am also grateful to my coadvisor Prof. Dr. Julia Hartmann for welcoming me in Philadelphia for two weeks and some interesting discussions on this occasion.

In addition I would also like to thank my friend, colleague and office mate Oliver Braun for his encouragement, a multitude of discussions (mathematical and otherwise), and carefully proofreading large parts of this this thesis (needless to say all remaining errors are entirely on me). Furthermore I wish to thank Dr. Markus Kirschmer for many ad-hoc Magma support sessions and providing me with some of his programs, and my brother Christoph Schönnenbeck for also proofreading parts of this thesis.

Finally I am grateful to the German research foundation (DFG) who funded my studies through the scholarship I held in the research training group "Experimental and constructive algebra".

# 2. Integral Forms of Algebraic Groups

In this chapter we want to study ways of defining given algebraic groups over the integers (or more generally over maximal orders in algebraic number fields). To that end we will first give some generalities on lattices, then present the central definition of integral forms, and later gather some important results on their structure.

## 2.1. Lattices

In this section $k$ always denotes an algebraic number field and $\mathcal{O}_k$ its maximal order (i.e. the integral closure of $\mathbb{Z}$ in $k$).

**Definition 2.1.1** *Let $V$ be a finite-dimensional $k$-vector space. A (full) lattice in $V$ is an $\mathcal{O}_k$-submodule $L \subset V$ that is finitely generated and contains a $k$-basis of $V$. The dimension of $V$ is then called the rank of $L$.*

Equivalently we could define a lattice of rank $n$ to be a finitely generated projective $\mathcal{O}_k$-module $L$ such that $L \otimes_{\mathcal{O}_k} k \cong k^n$.

Since $\mathcal{O}_k$ is a Dedekind domain we have the following structure theorem attributed to Steinitz.

**Theorem 2.1.2** *Let $V$ be a finite-dimensional $k$-vector space of dimension $n$ and $L \subset V$ a full lattice. Then there is a basis $e_1, ..., e_n$ of $V$ and fractional ideals $\mathfrak{a}_1, ..., \mathfrak{a}_n \subset k$ such that*

$$L = \bigoplus_{i=1}^{n} \mathfrak{a}_i e_i. \tag{2.1}$$

Obviously, the ideals in the above theorem are not uniquely defined. For example one can always rescale the given basis and in turn rescale the corresponding ideals. The ideal class of their product, however, does not depend on any choice.

**Definition 2.1.3** *In the notation of the last theorem we call the class $\mathrm{St}(L) := [\mathfrak{a}_1 \cdot ... \cdot \mathfrak{a}_n]$ in the class group of $\mathcal{O}_k$ the Steinitz class of $L$.*

**Theorem 2.1.4** *Two $\mathcal{O}_k$-lattices are isomorphic (as $\mathcal{O}_k$-modules) if and only if they are of equal rank and have the same Steinitz class.*

For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ (or equivalently for a finite place of $k$) we set $k_\mathfrak{p}$ the completion of $k$ at $\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ its ring of integers. The ring $\mathcal{O}_\mathfrak{p}$ is no longer only a Dedekind domain but a principal ideal domain whence all finitely generated torsion-free modules are free; in particular this holds for completions of lattices. For such a lattice $L$ we set $L_\mathfrak{p} := L \otimes_{\mathcal{O}_k} \mathcal{O}_\mathfrak{p}$ which is naturally a submodule of $V_\mathfrak{p} := V \otimes_k k_\mathfrak{p}$ when $L \subset V$. In this way we can associate to every lattice $L$ the collection $(L_\mathfrak{p})_\mathfrak{p}$ where $\mathfrak{p}$ runs through all prime ideals of $\mathcal{O}_k$.

Now let $L, L' \subset V$ be two lattices in the finite-dimensional $k$-vector space $V$. Since the intersection $L \cap L'$ has finite index in both $L$ and $L'$ we have $L_\mathfrak{p} = L'_\mathfrak{p}$ for almost all prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ (in particular $L_\mathfrak{p} = L'_\mathfrak{p}$ for all $\mathfrak{p}$ that do not divide the product $[L : (L \cap L')] \cdot [L' : (L \cap L')]$). In other words the sequences $(L_\mathfrak{p})_\mathfrak{p}$ and $(L'_\mathfrak{p})_\mathfrak{p}$ coincide in all but finitely many places. The following theorem reverses this relation and can be seen as a local-global-principle for equality of lattices.

**Theorem 2.1.5** *Let $V$ be a finite-dimensional $k$-vector space and $L$ a lattice in $V$. Then there is a bijection*

$$\left\{L' \mid L' \text{ lattice in } V\right\} \leftrightarrow \left\{(M_\mathfrak{p})_\mathfrak{p} \mid {}_{M_\mathfrak{p}=L_\mathfrak{p}}\ {}^{M_\mathfrak{p} \text{ lattice in } V_\mathfrak{p},}_{\text{for all but finitely many } \mathfrak{p}}\right\}. \tag{2.2}$$

*In particular we have $L = L'$ (for a second lattice $L' \subset V$) if and only if $L_\mathfrak{p} = L'_\mathfrak{p}$ for all $\mathfrak{p}$.*

Obviously the group $\mathrm{GL}(V)$ acts on the set of lattices in $V$ but the above theorem now also allows us to define an action of the adelic general linear group as follows. Denote by $\hat{k}$ the set of finite adeles of $k$ and identify $V$ with $k^n$ via the choice of some basis (this naturally also identifies $\mathrm{GL}(V)$ and $\mathrm{GL}_n(k)$). The group $\mathrm{GL}_n(\hat{k}) \cong \mathrm{GL}(V \otimes_k \hat{k})$ can be identified with the set of matrices

$$\mathrm{GL}_n(\hat{k}) = \{(g_\mathfrak{p})_\mathfrak{p} \mid g_\mathfrak{p} \in \mathrm{GL}_n(k_\mathfrak{p})\ \forall\ \mathfrak{p}, g_\mathfrak{p} \in \mathrm{GL}_n(\mathcal{O}_\mathfrak{p}) \text{ for almost all } \mathfrak{p}\}. \tag{2.3}$$

In particular for $L \subset k^n$ an $\mathcal{O}_k$-lattice and $g = (g_\mathfrak{p})_\mathfrak{p} \in \mathrm{GL}_n(\hat{k})$ the collections $(L_\mathfrak{p})_\mathfrak{p}$ and $(L_\mathfrak{p} g_\mathfrak{p})_\mathfrak{p}$ coincide at all but finitely many primes. Hence the collection $(L_\mathfrak{p} g_\mathfrak{p})_\mathfrak{p}$ defines a new lattice $L'$ in $k^n$ and setting $Lg := L'$ thus yields an action of $\mathrm{GL}_n(\hat{k})$ on the set of lattices in $k^n$. Note that our identification above is independent of the chosen basis since any two bases define the same sets $\mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$ at all but finitely many primes.

We give an algorithm to make this action computationally feasible. Obviously we only need to concern ourselves with constructing a lattice which differs from a given one at one prime $\mathfrak{p}$ only.

**Algorithm 2.1.6** pVARIATION$(L, \mathfrak{p}, B)$

***Input:*** *An $\mathcal{O}_k$-lattice $L$ in $k^n$, a prime ideal $\mathfrak{p} \subset \mathcal{O}_k$ and a $k_\mathfrak{p}$-basis $B$ of $V_\mathfrak{p}$.*

***Output:*** *A lattice $L'$ such that $L'_\mathfrak{q} = L_\mathfrak{q}$ for all $\mathfrak{q} \neq \mathfrak{p}$ and $L'_\mathfrak{p} = \langle B \rangle_{\mathcal{O}_\mathfrak{p}}$.*

1: *Find $m \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{p}^m L_\mathfrak{p} \subset \langle B \rangle_{\mathcal{O}_\mathfrak{p}} \subset \mathfrak{p}^{-m} L_\mathfrak{p}$.*

2: *Choose arbitrary preimages $B'$ of $B + \mathfrak{p}^m L_\mathfrak{p}$ in $k^n$.*

3: ***return*** $L' := (\langle B' \rangle_{\mathcal{O}_k} + \mathfrak{p}^m L) \cap \mathfrak{p}^{-m} L$

*Proof.* The preimages in row 2 exist since we have

$$(\mathfrak{p}^{-m} L_\mathfrak{p})/(\mathfrak{p}^m L_\mathfrak{p}) \cong (\mathfrak{p}^{-m} L)/(\mathfrak{p}^m L) \tag{2.4}$$

as $\mathcal{O}_k/\mathfrak{p}^m \cong \mathcal{O}_\mathfrak{p}/\mathfrak{p}^m$-modules. Thus we could already choose the preimages in $\mathfrak{p}^{-m} L_\mathfrak{p}$ in which case we can skip taking the intersection in the end. However, in applications it might sometimes be easier to find some preimage lying in $k^n$ rather than searching in $\mathfrak{p}^{-m} L_\mathfrak{p}$. By construction we have $L'_\mathfrak{p} = \langle B \rangle$ and since $\mathfrak{p}^m L \subset L' \subset \mathfrak{p}^{-m} L$, the lattices $L$ and $L'$ coincide away from $\mathfrak{p}$. □

## 2.2. Integral Forms

The idea of studying certain algebraic groups by looking at lattices in finite-dimensional vector spaces on which they act is hardly new. For instance, lattices feature prominently in Eichler's work on quadratic forms. However, the definitions and notations we want to introduce now are only about 20 years old and first appeared in [CNP98].

As in the last section $k$ denotes an algebraic number field with ring of integers $\mathcal{O}_k$ and ring of finite adeles $\hat{k}$. In addition let $\mathbb{G}$ be a reductive algebraic matrix group over $k$.

**Definition 2.2.1** *An integral form $\underline{G} = \mathbb{G}_L$ of $\mathbb{G}$ is an affine group scheme over $\mathcal{O}_k$ which we obtain in the following way: For $\mathbb{G} \hookrightarrow \mathrm{GL}_n$ a faithful representation and $L$ an $\mathcal{O}_k$-lattice in $k^n$ we set*

$$\underline{G}(\mathcal{O}_l) := \mathrm{Stab}_{\mathbb{G}(l)}(L \otimes_{\mathcal{O}_k} \mathcal{O}_l) \tag{2.5}$$

*for any finite extension $l$ of $k$ with ring of integers $\mathcal{O}_l$ and analogously*

$$\underline{G}(\mathcal{O}_\mathfrak{p}) := \mathrm{Stab}_{\mathbb{G}(k_\mathfrak{p})}(L \otimes_{\mathcal{O}_k} \mathcal{O}_\mathfrak{p}) \tag{2.6}$$

*for any finite prime $\mathfrak{p}$ of $\mathcal{O}_k$.*

Note that two distinct lattices can define the same integral form. For instance each lattice defines the same integral form as all its $k^\times$-multiples. As a more involved example consider the group $\mathrm{SO}(V, q)$ where $V$ is some $k$-vector space with a non-degenerate quadratic form $q$. In this scenario each lattice $L \leq V$

defines the same integral form (of $\mathrm{SO}(V,q)$) as its dual $L^{\#} = \{v \in V \mid b_q(v,w) \in \mathcal{O}_k \ \forall\ w \in L\}$.

Since a finite matrix group $F \leq \mathbb{G}(k)$ always fixes a lattice (for instance the $\mathcal{O}_k$-module generated by the $F$-orbits of a basis) there is necessarily an integral form $\underline{G}$ of $\mathbb{G}$ such that $F \leq \underline{G}(\mathcal{O}_k)$.

The $k$-rational points $\mathbb{G}(k)$ act on the set of integral forms by conjugation (with $g \in \mathbb{G}(k)$ mapping the integral form corresponding to the lattice $L$ to the integral form corresponding to $Lg$). However, we also defined an action of the adelic general linear group on lattices which enables us in this situation to get an action of the adelic points of $\mathbb{G}$ on integral forms as follows: Let $\underline{G} = \mathbb{G}_L$ be an integral form of $\mathbb{G}$ defined by the lattice $L$ and $g = (g_{\mathfrak{p}})_{\mathfrak{p}} \in \mathbb{G}(\hat{k})$. Then $\underline{G}g$ is defined as the integral form corresponding to $Lg$; in particular at each prime $\mathfrak{p}$ we have $\mathbb{G}_{Lg}(\mathcal{O}_{\mathfrak{p}}) = g_{\mathfrak{p}}^{-1}\mathbb{G}_L(\mathcal{O}_{\mathfrak{p}})g_{\mathfrak{p}}$.

**Definition 2.2.2** 1. *Two lattices $L, L'$ in the same faithful $\mathbb{G}$-module $V$ are called $\mathbb{G}$-isomorphic (we write $L \cong L'$ assuming the underlying group is clear) if they are in the same orbit under $\mathbb{G}(k)$; we say they are in the same $\mathbb{G}$-genus (or simply in the same genus if $\mathbb{G}$ is clear from context) if they are in the same orbit under $\mathbb{G}(\hat{k})$. The set of all lattices isomorphic to $L$ is called the class of $L$, denoted* $\mathrm{class}(L)$*, and we denote the genus of $L$ by* $\mathrm{genus}(L)$*.*

2. *We say two integral forms of $\mathbb{G}$ are in the same genus if they are in the same orbit under $\mathbb{G}(\hat{k})$; we call them isomorphic if they are already in the same orbit under $\mathbb{G}(k)$. The notation for the isomorphism class and the genus of an integral form are analogous to those of lattices. In particular we write $\underline{G} \cong \underline{G}'$ if two integral forms are isomorphic (again assuming the underlying group is clear from context).*

The concept of the genus naturally arises when studying why $\mathbb{G}$-isomorphism fails to be a local property. For general groups $\mathbb{G}$, lying in the same genus might be a far weaker property than being isomorphic; however, at least for semisimple groups the extent of the failure of the local-global property is limited in the following sense.

**Theorem 2.2.3 ([BHC62, Thm. 7.8])** *Assume $\mathbb{G}$ to be semisimple and let $\underline{G}$ be an integral form of $\mathbb{G}$. Then* $\mathrm{genus}(\underline{G})$ *decomposes into a finite number of isomorphism classes and this number is called the class number of $\underline{G}$, denoted $h(\underline{G})$.*

**Example 2.2.4** *Consider the group* $\mathrm{GL}_n$*. Any two lattices in $k^n$ are locally isomorphic at every prime $\mathfrak{p}$ since $\mathcal{O}_{\mathfrak{p}}$ is a principal ideal domain. However, two lattices are only globally isomorphic if their Steinitz classes coincide. Hence we can conclude that the class number (of an arbitrary lattice) is just the class number of $\mathcal{O}_k$.*

**Proposition 2.2.5 ([CNP98, Prop. 3.3])** *Let $\underline{G}$ be an integral form of $\mathbb{G}$. Then $\underline{G}(\mathcal{O}_\mathfrak{p})$ is an open and compact subgroup of $\mathbb{G}(k_\mathfrak{p})$ (and hence of finite index in a maximal compact subgroup) for all $\mathfrak{p}$. For all but finitely many $\mathfrak{p}$ the group $\underline{G}(\mathcal{O}_\mathfrak{p})$ is a hyperspecial maximal compact subgroup.*

For notational ease let us denote by $\mathfrak{P}$ the set of finite primes of $\mathcal{O}_k$ and by $\mathfrak{V}$ the set of infinite places of $k$.

In light of the last proposition it is natural to look at the following:

**Definition 2.2.6** *Let $\underline{G}$ be an integral form of $\mathbb{G}$. We call $\underline{G}$ maximal (cf. [CNP98]) if $\underline{G}(\mathcal{O}_\mathfrak{p})$ is maximal compact for all primes $\mathfrak{p}$. We call $\underline{G}$ a model for $\mathbb{G}$ (cf. [Gro96]) if $\underline{G}(\mathcal{O}_\mathfrak{p})$ is hyperspecial maximal compact for all primes $\mathfrak{p} \in \mathfrak{P}$.*

## 2.3. Mass Formulas

Under certain circumstances it is possible to assign to a genus of integral forms a so-called mass which measures its complexity in some sense and yields a useful termination condition for explicit computations.

As before, let $k$ denote an algebraic number field with ring of integers $\mathcal{O}_k$ and adele ring $\hat{k}$, and $\mathbb{G}$ a (in contrast to before) semisimple group over $k$. Additionally we now assume that $k$ is totally real and that $\mathbb{G}(k_\nu)$ is compact for all $\nu \in \mathfrak{V}$ (in fact the second condition already implies the first).

Fix an integral form $\underline{G} = \underline{G}_1$ of $\mathbb{G}$ and representatives $\underline{G}_j = \underline{G}\gamma_j$ with $\gamma_j \in \mathbb{G}(\hat{k}), 1 \leq j \leq h$, for the isomorphism classes in the genus of $\underline{G}$. Denote by $S_j \leq \mathbb{G}(\hat{k})$ the stabilizer of $\underline{G}_j$; then we have

$$\mathbb{G}(\hat{k}) = \bigsqcup_{j=1}^{h} S_1 \gamma_j \mathbb{G}(k) \tag{2.7}$$

and furthermore

$$S_j = \gamma_j^{-1} S_1 \gamma_j = \prod_{\mathfrak{p} \in \mathfrak{P}} \underline{G}_j(\mathcal{O}_\mathfrak{p}) \tag{2.8}$$

by construction. Due to the compactness assumption the groups $S_j \cap \mathbb{G}(k) = \underline{G}_j(\mathcal{O}_k)$ are finite. In this situation a mass formula is simply a function making the sum

$$\text{mass}(\text{genus}(\underline{G})) := \sum_{j=1}^{h} \frac{1}{|\underline{G}_j(\mathcal{O}_k)|} \tag{2.9}$$

computable from only local information on $\underline{G}$ (that is without knowing a system of representatives). Obviously, such formulas are highly useful when trying to compute a system of representatives for a given genus both as a plausibility check and a stopping condition for the computation. We will see some examples of mass formulas in explicit cases in later chapters and for now only give two general results. First of all we need a method of comparing the mass of two integral forms of the same group. This idea is sometimes attributed to Venkov but actually appeared first in Eichler's work on quadratic forms (cf. [Eic52a, Satz 8])[1]. For this reason we will refer to the following theorem as Eichler's method or Eichler's trick.

**Theorem 2.3.1** *Let $\underline{G}'$ be another integral form of $\mathbb{G}$ (not necessarily with respect to the same faithful representation) and $S_1'$ its stabilizer in $\mathbb{G}(\hat{k})$. Since $\underline{G}$ and $\underline{G}'$ coincide at all but finitely many primes in $\mathfrak{P}$, the intersection $S_1 \cap S_1'$ has finite index in both $S_1$ and $S_1'$. The following holds:*

$$\operatorname{mass}(\operatorname{genus}(\underline{G})) \cdot [S_1 : (S_1 \cap S_1')] = \operatorname{mass}(\operatorname{genus}(\underline{G}')) \cdot [S_1' : (S_1 \cap S_1')]. \quad (2.10)$$

*Proof.* Since there is no proof of this theorem (in this generality) to be found in the literature, we give one here. However, we can follow closely along the lines of [BN97, Prop. 2.4]. Let $\underline{G}_i' = \underline{G}'\gamma_i', 1 \le i \le h'$, be a system of representatives in the genus of $\underline{G}'$.

We define

$$\begin{aligned} a_{j,i} &:= \left|\left\{\underline{H} \in \underline{G}'\gamma_j S_j \mid \underline{H} \cong \underline{G}_i'\right\}\right| \\ b_{i,j} &:= \left|\left\{\underline{H} \in \underline{G}\gamma_i' S_i' \mid \underline{H} \cong \underline{G}_j\right\}\right| \end{aligned} \quad (2.11)$$

for $1 \le j \le h$ and $1 \le i \le h'$. Here $\underline{G}\gamma_i' S_i'$ simply denotes the orbit of the integral form $\underline{G}$ under the set $\gamma_i' S_i' \subset \mathbb{G}(\hat{k})$. By construction we have

$$\sum_{i=1}^{h'} a_{j,i} = |\underline{G}'\gamma_j S_j| = |S_j : (S_j \cap \gamma_j^{-1} S_1' \gamma_j)| = |S_1 : (S_1 \cap S_1')| \quad (2.12)$$

and analogously

$$\sum_{j=1}^{h} b_{i,j} = |S_1' : (S_1 \cap S_1')|. \quad (2.13)$$

Now for $g, g' \in \mathbb{G}(k)$ we have $\underline{G}'\gamma_j = \underline{G}_i' g = \underline{G}_i' g'$ if and only if $gg^{-1} \in S_i' \cap \mathbb{G}(k) = \underline{G}_i'(\mathcal{O}_k)$ and an analogous equality holds in the genus of $\underline{G}$. Thus

$$\begin{aligned} |\underline{G}_i'(\mathcal{O}_k)| \cdot a_{j,i} &= \left|\left\{g \in \mathbb{G}(k) \mid \underline{G}_i' g \in \underline{G}'\gamma_j S_j\right\}\right| \\ &= \left|\left\{g \in \mathbb{G}(k) \mid \underline{G}'\gamma_i' g \in \underline{G}' S_1 \gamma_j\right\}\right| \text{ and} \\ |\underline{G}_j(\mathcal{O}_k)| \cdot b_{i,j} &= \left|\left\{g \in \mathbb{G}(k) \mid \underline{G}_j g \in \underline{G}\gamma_i' S_i'\right\}\right| \\ &= \left|\left\{g \in \mathbb{G}(k) \mid \underline{G}\gamma_j g \in \underline{G} S_1' \gamma_i\right\}\right|. \end{aligned} \quad (2.14)$$

[1] I would like to thank Professor Schulze-Pillot for pointing this out to me.

But for $g \in \mathbb{G}(k)$ the following holds:

$$\begin{aligned} & \underline{G}\gamma_j g^{-1} \in \underline{G}S_1'\gamma_i \\ \Leftrightarrow\ & \underline{G} \in \underline{G}S_1'\gamma_i g \gamma_j^{-1} \\ \Leftrightarrow\ & S_1 \cap S_1'\gamma_i g\gamma_j^{-1} \neq \emptyset \\ \Leftrightarrow\ & S_1\gamma_j g^{-1}\gamma_i^{-1} \cap S_1' \neq \emptyset \\ \Leftrightarrow\ & \underline{G}'\gamma_i' g \in \underline{G}' S_1 \gamma_j, \end{aligned} \tag{2.15}$$

whence $|\underline{G}_i'(\mathcal{O}_k)| \cdot a_{j,i} = |\underline{G}_j(\mathcal{O}_k)| \cdot b_{i,j}$.

We put all of this together:

$$\begin{aligned} \operatorname{mass}(\operatorname{genus}(\underline{G})) \cdot [S_1 : (S_1 \cap S_1')] &= \left(\sum_{j=1}^{h} \frac{1}{|\underline{G}_j(\mathcal{O}_k)|}\right)\left(\sum_{i=1}^{h'} a_{j,i}\right) \\ &= \sum_{i,j} \frac{a_{j,i}}{|\underline{G}_j(\mathcal{O}_k)|} \\ &= \sum_{i,j} \frac{b_{i,j}}{|\underline{G}_i'(\mathcal{O}_k)|} \\ &= \left(\sum_{i=1}^{h'} \frac{1}{|\underline{G}_i'(\mathcal{O}_k)|}\right)\left(\sum_{j=1}^{h} b_{i,j}\right) \\ &= \operatorname{mass}(\operatorname{genus}(\underline{G}')) \cdot [S_1' : (S_1 \cap S_1')]. \end{aligned} \tag{2.16}$$

This was exactly our initial assertion. $\square$

Having this method available we are now prepared to give a general mass formula at least in a particularly nice situation. Let us assume that $\mathbb{G}(k_\mathfrak{p})$ is split for all $\mathfrak{p} \in \mathfrak{P}$ and choose a hyperspecial maximal compact subgroup $H_\mathfrak{p}$ of $\mathbb{G}(k_\mathfrak{p})$ for each of the finitely many $\mathfrak{p}$ where $\underline{G}$ is not already hyperspecial. Set

$$c_\mathfrak{p} := \frac{[H_\mathfrak{p} : (H_\mathfrak{p} \cap \underline{G}(\mathcal{O}_\mathfrak{p}))]}{[\underline{G}(\mathcal{O}_\mathfrak{p}) : (H_\mathfrak{p} \cap \underline{G}(\mathcal{O}_\mathfrak{p}))]} \tag{2.17}$$

and denote by $c$ the product of all these $c_\mathfrak{p}$. Note that this depends neither on the choice of the hyperspecial subgroups nor on the faithful representation with respect to which the integral form is defined (cf. [CNP98, La. 3.8]).

**Theorem 2.3.2 ([CNP98, Thm. 3.7])** *Let $\mathbb{G}$ be simply connected and almost simple over the totally real number field $k$. Assume $\mathbb{G}(k_\nu)$ to be compact for all infinite places $\nu$ of $k$ and $\mathbb{G}(k_\mathfrak{p})$ split for all $\mathfrak{p} \in \mathfrak{P}$. Let $\underline{G}_j, 1 \leq j \leq h$, as before be a system of representatives of the genus of $\underline{G}$ and $c$ defined as above.*

*Furthermore let $N := [k : \mathbb{Q}]$ and $d_1, ..., d_r$ be the degrees of the primary polynomial invariants of the Weyl group of $\mathbb{G}$ over $\overline{\mathbb{Q}}$ (sometimes referred to as the degrees of $\mathbb{G}$). Then we have the following mass formula:*

$$\mathrm{mass}(\mathrm{genus}(\underline{G})) = \sum_{j=1}^{h} \frac{1}{|\underline{G}_j(\mathcal{O}_k)|} = c \prod_{i=1}^{r} \frac{1}{2^N} \zeta_k(1 - d_i), \tag{2.18}$$

*where $\zeta_k$ denotes the Dedekind zeta function of $k$.*

## 2.4. Almost Strong Approximation

Having mass formulas at our disposal we are equipped with a method of checking whether a given set of integral forms in a genus actually constitutes a system of representatives for the isomorphism classes. However we still do not know where to search for such a system. In the case of genera for the spin group of a quadratic form it is possible to find a system of representatives by looking at iterated $\mathfrak{p}$-neighbours of a lattice (with mild restrictions on the prime $\mathfrak{p}$) by virtue of the strong approximation theorem. Here two lattices (in the same genus) are called $\mathfrak{p}$-neighbours if the quotient modulo their intersection is a one-dimensional $\mathcal{O}_k/\mathfrak{p}$-vector space for both of them. One may use the affine building of $\mathbb{G}(k_\mathfrak{p})$ to generalize this notion to other algebraic groups and we will see that this is the right idea to enumerate a system of representatives for the isomorphism classes in a given genus.

The notion of the almost strong approximation property and the results we want to present here are due to the three articles [HJ97], [CH02], and [Cha04].

For this section let $\mathbb{G}$ be a simply connected, semisimple, linear algebraic group defined over the totally real number field $k$. The set of finite places of $k$ is denoted by $\mathfrak{P}$ and the set of infinite places by $\mathfrak{V}$. We assume that $\mathbb{G}(k_\nu)$ is compact for every $\nu \in \mathfrak{V}$.

By $\mathfrak{B}(\mathbb{G}, \mathfrak{p})$ we denote the affine building of $\mathbb{G}(k_\mathfrak{p})$ (for $\mathfrak{p} \in \mathfrak{P}$). Remember that the hyperspecial maximal compact subgroups of $\mathbb{G}(k_\mathfrak{p})$ are precisely the stabilizers of hyperspecial vertices in $\mathfrak{B}(\mathbb{G}, \mathfrak{p})$. Thus it makes sense to call two hyperspecial maximal compact subgroups of $\mathbb{G}(k_\mathfrak{p})$ adjacent if they stabilize hyperspecial vertices of adjacent chambers.

**Definition 2.4.1** *Let $\underline{G}$ and $\underline{G}'$ be two integral forms of $\mathbb{G}$ and $\mathfrak{p} \in \mathfrak{P}$. We call $\underline{G}$ and $\underline{G}'$ $\mathfrak{p}$-neighbours if they coincide away from $\mathfrak{p}$ while $\underline{G}(\mathcal{O}_\mathfrak{p})$ and $\underline{G}(\mathcal{O}'_\mathfrak{p})$ are adjacent hyperspecial maximal compact subgroups of $\mathbb{G}(k_\mathfrak{p})$.*

Now let $\underline{G}$ be an arbitrary integral form of $\mathbb{G}$.

**Definition 2.4.2 ([CH02])** *1. Let $\emptyset \neq S \subset \mathfrak{P}$ be a finite set of finite primes of $k$. We say $\mathbb{G}$ has the almost strong approximation property with respect to $S$ if the following holds:*

*Given a family $F = \{\sigma_\mathfrak{p} \in \mathbb{G}(k_\mathfrak{p}) \mid \mathfrak{p} \in S\}$ and an integer $s$ there is a finite set of primes $\Omega = \Omega(F, s)$ such that for each $\mathfrak{q} \notin \Omega \cup S$ there is an element $\sigma \in \mathbb{G}(k)$ satisfying:*

*a) $\sigma \equiv \sigma_\mathfrak{p} \pmod{\mathfrak{p}^s}$ for all $\mathfrak{p} \in S$.*

*b) $\sigma \in \underline{G}(\mathcal{O}_\mathfrak{p})$ for all $\mathfrak{p} \notin S \cup \{\mathfrak{q}\}$.*

*c) $\underline{G}(\mathcal{O}_\mathfrak{q})$ and $\sigma^{-1}\underline{G}(\mathcal{O}_\mathfrak{q})\sigma$ are adjacent hyperspecial maximal compact subgroups of $\mathbb{G}(k_\mathfrak{q})$.*

*2. $\mathbb{G}$ is said to have the almost strong approximation property (ASAP) if (ASAP) holds for all finite subsets of $\mathfrak{P}$.*

Note that this definition is independent of the chosen integral form $\underline{G}$ since two integral forms coincide at all but finitely many primes.

**Lemma 2.4.3 ([CH02, Prop. 5.1])** *If (ASAP) holds for $\mathbb{G}$ and $\underline{G}$ is an integral form of $\mathbb{G}$, there is a finite set $\Omega \subset \mathfrak{P}$ such that for each $\mathfrak{q} \notin \Omega$ and each integral form $\underline{H}$ in the genus of $\underline{G}$ there is an integral form $\underline{H}'$ isomorphic to $\underline{H}$ which is a $\mathfrak{q}$-neighbour of $\underline{G}$.*

*Proof.* Since the genus of $\underline{G}$ contains only finitely many isomorphism classes, it suffices to prove this for one arbitrarily chosen integral form $\underline{H}$. Let $\mathbb{G} \hookrightarrow \mathrm{GL}_m$ be a closed embedding and let $L, M \leq k^m$ be two lattices corresponding to the integral forms $\underline{G}$ and $\underline{H}$ respectively. Set $F \subset \mathfrak{P}$ the finite set of primes $\mathfrak{p}$ with $L_\mathfrak{p} \neq M_\mathfrak{p}$. For $\mathfrak{p} \in F$ choose some $\sigma_\mathfrak{p}$ such that $L_\mathfrak{p}\sigma_\mathfrak{p} = M_\mathfrak{p}$ and set $s$ to some integer such that $\mathfrak{p}^s L_\mathfrak{p} \subset M_\mathfrak{p}$ for all $\mathfrak{p} \in F$.

Since (ASAP) holds for $\mathbb{G}$ there is some finite set $\Omega$ such that for $\mathfrak{q} \notin \Omega$ there is a $\sigma \in \mathbb{G}(k)$ which approximates $\sigma_\mathfrak{p}$ modulo $\mathfrak{p}^{2s}$ for $\mathfrak{p} \in F$, fixes $L_\mathfrak{p} = M_\mathfrak{p}$ for $\mathfrak{p} \notin F \cup \{\mathfrak{q}\}$, and $\sigma^{-1}\underline{G}(\mathcal{O}_\mathfrak{p})\sigma$ is a neighbour of $\underline{G}(\mathcal{O}_\mathfrak{p})$.

By the choice of $s$ we know that $L_\mathfrak{p}\sigma \subset M_\mathfrak{p}$ and hence $L_\mathfrak{p}\sigma = M_\mathfrak{p}$. The integral form $\underline{H}' := \underline{H}\sigma^{-1}$ is now isomorphic to $\underline{H}$ and a $\mathfrak{q}$-neighbour of $\underline{G}$ by construction. □

The question remains which groups actually exhibit the almost strong approximation property. We summarize the known results:

**Theorem 2.4.4 ([CH02],[Cha04])** *The following groups exhibit the almost strong approximation property:*

1. a) *The special unitary group of a (totally) positive definite Hermitian form on a space of dimension at least* $2$ *over a CM-extension of* $k$.

   b) *The unitary group of a (totally) positive definite Hermitian form on a space of dimension at least* $2$ *over a (totally) definite quaternion algebra over* $k$.

   c) *The special unitary group of a (totally) positive definite Hermitian form on a space of dimension at least* $2$ *over a (totally) definite quaternion algebra over* $k$ *with involution of the second kind.*

   d) *The spin group of a (totally) positive definite quadratic form on a space of dimension at least* $3$.

   e) *The spin group of a skew-Hermitian form on a space of dimension at least* $4$ *over a quaternion division algebra over* $k$ *where the form is anisotropic at all finite primes.*

2. a) *The groups of type* ${}^3D_4$ *and* ${}^6D_4$.

   b) *The groups of type* $G_2$ *and* $F_4$.

   c) *The groups of type* $E_7$ *and* $E_8$.

Now if $\mathbb{G}$ is one of the groups in the last theorem, the results of this section together with the mass formula from the previous one suggest the following algorithm for enumerating the $\mathbb{G}$-isomorphism classes of lattices in a given genus.

**Algorithm 2.4.5** ComputeGenus($L$)

***Input:*** *An $\mathcal{O}_k$-lattice $L$ in the faithful $\mathbb{G}(k)$-module $V$.*

***Output:*** *A system $(L = L_1, ..., L_h)$, representing the isomorphism classes in* genus$(L)$ *together with elements $\sigma_i \in \mathbb{G}(\hat{k}), 1 \leq i \leq h$, such that $L_i = L\sigma_i$.*

*Initialize* $M \leftarrow \left(\mathrm{mass}(\mathrm{genus}(L)) - |\,\mathrm{Stab}_{\mathbb{G}}(k)(L)|^{-1}\right)$, $S \leftarrow \emptyset$ *and* $\mathcal{G} \leftarrow [(L, 1_{\mathbb{G}(\hat{k})})]$.

***while*** $M \neq 0$ ***do***

  *Choose a prime ideal* $\mathfrak{p} \notin S$ *such that* $\mathbb{G}$ *is split at* $\mathfrak{p}$ *and* $K_\mathfrak{p} := \mathbb{G}_L(\mathcal{O}_\mathfrak{p})$ *is hyperspecial.*

  $S \leftarrow S \cup \{\mathfrak{p}\}$.

  *Fix a system* $R$ *of representatives of the* $K_\mathfrak{p} \backslash K_\mathfrak{p} s_0 K_\mathfrak{p}$.[2]

  ***for*** $(L', \mu) \in \mathcal{G}$ ***do***

    ***for*** $\sigma_\mathfrak{p} \in R$ ***do***

      *Let* $\sigma \in \mathbb{G}(\hat{k})$ *be the element with component* $\sigma_\mathfrak{p}$ *at* $\mathfrak{p}$ *and* 1 *else.*

      *Compute* $L'' := L\sigma\mu$ *and set* new $\leftarrow$ true.

      ***for*** $(N, \nu) \in \mathcal{G}$ ***do***

        ***if*** $N \cong_{\mathbb{G}(k)} L''$ ***then***

          new $\leftarrow$ false

          *Break* $(N, \nu)$.

        ***end if***

      ***end for***

      ***if*** new ***then***

        *Append* $(L'', \sigma)$ *to* $\mathcal{G}$.

        $M \leftarrow \left(M - |\,\mathrm{Stab}_{\mathbb{G}}(k)(L'')|^{-1}\right)$.

        ***if*** $M = 0$ ***then***

          ***return*** $\mathcal{G}$

        ***end if***

      ***end if***

    ***end for***

  ***end for***

***end while***

[2]Here $s_0$ is the unique generator of the affine Weyl group not fixing $L_\mathfrak{p}$ for some suitably chosen apartment. For details see Chapter 4.

# 3. Algebraic Modular Forms and Hecke Algebras

## 3.1. Algebraic Modular Forms

In this section we want to introduce the basic definitions and notations in the theory of algebraic modular forms which we will use extensively in the remainder of this work. In particular the exposition in the beginning follows along the lines of Benedict Gross's aforementioned article [Gro99]. Since we have certain examples in mind, we will only consider semisimple groups and work over an algebraic number field instead of the rationals. This was (for example) also done in [GV14].

Let $k$ be a totally real algebraic number field with ring of integers $\mathcal{O}_k$ and let

$$k_\infty := \mathbb{R} \otimes_{\mathbb{Q}} k \cong \mathbb{R}^{[k:\mathbb{Q}]}.$$

Considering the finite places we set $\hat{\mathbb{Q}}$ the finite adeles of $\mathbb{Q}$ (i.e. the elements of $\prod_{p \text{ prime}} \mathbb{Q}_p$ which are integral at all but finitely many places) and $\hat{k} := k \otimes_{\mathbb{Q}} \hat{\mathbb{Q}}$ the finite adeles of $k$. We identify $\hat{k}$ with the set

$$\hat{k} = \left\{ (x_\mathfrak{p})_\mathfrak{p} \in \prod_{\mathfrak{p} \subset \mathcal{O}_k \text{ prime}} k_\mathfrak{p} \mid x_\mathfrak{p} \in \mathcal{O}_\mathfrak{p} \text{ f.a.a. } \mathfrak{p} \right\} \tag{3.1}$$

and we denote the (full) ring of adeles of $k$ by $\mathbb{A}_k := k_\infty \times \hat{k}$.

Let $\mathbb{G}$ be a connected semisimple linear algebraic group over $k$ such that $\mathbb{G}(k_\infty)$ is compact (had we not already stipulated $k$ to be totally real we would get it as a consequence here) and let $\rho : \mathbb{G} \to \mathrm{GL}_V$ be an irreducible finite-dimensional rational representation of $\mathbb{G}$ defined over some extension of $k$.

The fact that the Lie group $\mathbb{G}(k_\infty)$ is compact has the following computationally very useful consequences:

**Proposition 3.1.1 ([Gro99, Prop. (1.4),(2.5)])** *The following holds under the general assumptions of this section:*

1. *Every arithmetic subgroup $\Gamma \leq \mathbb{G}(k)$ is finite.*

*2. The rational points $\mathbb{G}(k)$ form a discrete subgroup of the adelic group $\mathbb{G}(\hat{k})$ and the quotient space $\mathbb{G}(\hat{k})/\mathbb{G}(k)$ is compact.*

*3. For every irreducible representation $V'$ of $\mathbb{G}$ there is a character $\mu \in$ $\mathrm{Hom}(\mathbb{G}, \mathbb{G}_m)$ and a (totally) positive definite bilinear form $\langle\ ,\ \rangle : V' \times V' \to$ $k$ such that*

$$\langle vg, v'g\rangle = \mu(g)\langle v, v'\rangle \tag{3.2}$$

*for all $g \in \mathbb{G}(k)$ and $v, v' \in V'$. This character takes totally positive values on $\mathbb{G}(k_\infty)$.*

### 3.1.1. The Space of Modular Forms and its Inner Product

In addition to our notation from before we set $D := \mathrm{End}_{\mathbb{G}}(V)$ and let $\mu \in$ $\mathrm{Hom}(\mathbb{G}, \mathbb{G}_m)$ be the character defined by $V$ whose existence was guaranteed in Proposition 3.1.1. Then $D$ is a (finite-dimensional) division algebra over $k$ which coincides with $k$ if $V$ is absolutely irreducible.

Gross's original aim was to study the space of modular forms on $\mathbb{G}$ with coefficients in $V$, i.e. the space

$$M(V) = \left\{ f : \mathbb{G}(\hat{k}) \to V \mid {}_{f(\gamma g) = f(\gamma)g \text{ for } \gamma \in \mathbb{G}(\hat{k}),\ g \in \mathbb{G}(k)}^{f \text{ locally constant,}} \right\}. \tag{3.3}$$

This left $D$-vector space is far too large to be suitable for computations, but since each $f \in M(V)$ is locally constant we can conclude that it is constant on the cosets of an open and compact subgroup $K$ in $\mathbb{G}(\hat{k})$. This leads to the following definition:

$$\begin{aligned} M(V,K) &= \left\{ f : K\backslash\mathbb{G}(\hat{k}) \to V \mid {}^{f(\gamma g) = f(\gamma)g \text{ for}}_{\gamma \in \mathbb{G}(\hat{k}),\ g \in \mathbb{G}(k)} \right\} \\ &\cong \left\{ f : \mathbb{G}(\hat{k}) \to V \mid {}^{f(\kappa\gamma g) = f(\gamma)g \text{ for } \gamma \in \mathbb{G}(\hat{k}),}_{g \in \mathbb{G}(k), \kappa \in K} \right\}. \end{aligned} \tag{3.4}$$

The space $M(V,K)$ is also known as the space of modular forms of weight $V$ and level $K$ (for the group $\mathbb{G}$). Under this definition $M(V)$ is the direct limit of the spaces $M(V,K)$ which we want to study in greater detail. The following proposition establishes the fact that $M(V,K)$ is indeed a suitable candidate for explicit computations:

**Proposition 3.1.2 ([Gro99, Prop. (4.3),(4.5)])** *Set $\Sigma_K := K\backslash\mathbb{G}(\hat{k})/\mathbb{G}(k)$. The following holds:*

*1. The set $\Sigma_K$ is finite.*

*2. If $\alpha_i, 1 \le i \le r$, is a system of representatives for $\Sigma_K$ and*

$$\Gamma_i := \mathbb{G}(k) \cap \alpha_i^{-1} K \alpha_i, \tag{3.5}$$

*then*

$$M(V,K) \to \bigoplus_{i=1}^{r} V^{\Gamma_i}, \ f \mapsto (f(\alpha_1), ..., f(\alpha_r)) \tag{3.6}$$

*is an isomorphism of $D$-vector spaces. In particular $M(V,K)$ is finite-dimensional.*

*Proof.* The set $\Sigma_K$ is finite since $\mathbb{G}(k)$ is discrete and cocompact in $\mathbb{G}(\hat{k})$. The groups $\Gamma_i, \ 1 \le i \le r$, are arithmetic subgroups of $\mathbb{G}(k)$ and hence finite. Furthermore for $g \in \Gamma_i$ we have $\alpha_i g = \kappa\alpha_i$ for some suitable $\kappa \in K$ and so

$$f(\alpha_i)g = f(\alpha_i g) = f(\kappa\alpha_i) = f(\alpha_i) \tag{3.7}$$

which implies $f(\alpha_i) \in V^{\Gamma_i}$. Hence every choice of $f(\alpha_i)$ for all $1 \le i \le r$ gives rise to a unique element in $M(V,K)$ which proves the second assertion. □

Remember that, by Proposition 3.1.1, the space $V$ admits a totally positive definite bilinear form $\langle -,- \rangle$ which is connected to the character $\mu$ via

$$\langle vg, v'g \rangle = \mu(g)\langle v, v' \rangle. \tag{3.8}$$

This bilinear form gives rise to an inner product - that is frequently called the *Peterson inner product* - on the space $M(V,K)$ in the following way:

Once more, fix representatives $\alpha_i, \ 1 \le i \le r$, for the set $\Sigma_K$ and define $\Gamma_i$ as before. Then we can define the inner product of $f, f' \in M(V,K)$ as

$$\langle f, f' \rangle_M := \sum_{i=1}^{r} \frac{1}{|\Gamma_i|\mu_{\hat{k}}(\alpha_i)} \langle f(\alpha_i), f'(\alpha_i) \rangle. \tag{3.9}$$

Here $\mu_{\hat{k}}$ denotes the composite $\mathbb{G}(\hat{k}) \xrightarrow{\mu} \hat{k}^\times \xrightarrow{N} \mathbb{Q}^\times$ where $N$ is the norm. Note that this implies that $\mu_{\hat{k}}$ is trivial on all compact subgroups of $\mathbb{G}(\hat{k})$.

The so defined map $\langle -,- \rangle_M$ is obviously a totally positive definite symmetric bilinear from on $M(V,K)$. Furthermore it does not depend on our choice of representatives $\alpha_i$, for if $\beta_i = \kappa\alpha_i g$ then

$$\mathbb{G}(k) \cap \beta_i^{-1} K \beta_i = g^{-1}\Gamma_i g \cong \Gamma_i \tag{3.10}$$

and

$$\langle f(\beta_i), f'(\beta_i) \rangle = \mu(g)\langle f(\alpha_i), f'(\alpha_i) \rangle, \ \mu_{\hat{k}}(\beta_i) = \mu(g)\mu_{\hat{k}}(\alpha_i). \tag{3.11}$$

As usual the inner product on $M(V,K)$ allows us to define an anti-involution on $\mathrm{End}_k(M(V,K))$ via the adjoint:

$$\langle fT, f' \rangle_M = \langle f, f'T' \rangle_M. \tag{3.12}$$

## 3.2. Hecke Algebras

Having defined algebraic modular forms as our objects of interest in the previous section, we now want to introduce an algebra which acts on the spaces $M(V, K)$ in a particularly nice way. First though let us take a step back and consider a wider viewpoint which does not depend on our previous setup but will be of use for all that follows.

### 3.2.1. An Abstract Introduction to Hecke Algebras

For this subsection only, let $G$ be a group (of arbitrary cardinality) and $K \leq G$ a subgroup of $G$. By $G/\!\!/K$ we will denote the set of double cosets of $G$ with respect to $K$ and we will assume that the commensurator of $K$ in $G$ is just $G$ itself (i.e. each double coset decomposes into a finite number of left- (or right-) cosets).

**Definition 3.2.1** *The ($\mathbb{Q}$-)algebra $H_K := H(G, K)$ is the set of $K$-bi-invariant functions $G \to \mathbb{Q}$ which are non-zero only on a finite number of double cosets with multiplication given by convolution, i.e.*

$$\mathbb{1}_{KgK} \cdot \mathbb{1}_{KhK}(x) = \sum_{i,j} \mathbb{1}_{Kg_ih_j}(x)$$

*for all $x \in G$, where $KgK = \bigsqcup_i Kg_i$, $KhK = \bigsqcup_j Kh_j$, and $\mathbb{1}_{KgK}$ is the characteristic function of the set $KgK$.*

This definition completely determines the multiplication in $H_K$ since the functions $\mathbb{1}_{KgK}$ form a basis for $H_K$; the multiplication is well-defined by our fundamental assumption. In this sense we will think of the double-, right-, or left-cosets (as well as sums thereof) as elements of $H_K$ whenever there is no risk of confusion.

**Lemma 3.2.2** *Let $M = 1_K^G = \langle Kg \mid g \in G \rangle_{\mathbb{Q}}$ be the right permutation representation of $G$ on the right cosets of $H$ in $G$ or equivalently the induction of the trivial $\mathbb{Q}K$-module to $\mathbb{Q}G$. Then $H_K \cong \mathrm{End}_{\mathbb{Q}G}(M)$.*

*Proof.* Define $\phi : H_K \to \mathrm{End}_{\mathbb{Q}G}(M)$ via $\phi(\mathbb{1}_{KgK})(Kh) = \sum_i Kg_ih$ for all $h \in G$, where again $KgK = \bigsqcup_i Kg_i$. This is obviously a well-defined ring homomorphism. We will show that it is surjective and injective: To that end consider an arbitrary endomorphism $\psi \in \mathrm{End}_{\mathbb{Q}G}(M)$. We have $\psi(K) = \sum_j \lambda_j Kh_j$ for suitable $\lambda_j \in \mathbb{Q}$ and $h_j \in G$ and thus $\psi(Kg) = \psi(K)g = \sum_j \lambda_j Kh_jg$ for all $g \in G$. Hence we have $\psi = \phi(\sum_j \lambda_j \mathbb{1}_{Kh_j})$ pending the right $K$-invariance of $\sum_j \lambda_j \mathbb{1}_{Kh_j}$. But we see

$$\sum_j \lambda_j Kh_jk = \psi(Kk) = \psi(K) = \sum_j \lambda_j Kh_j \tag{3.13}$$

for all $k \in K$. On the subject of injectivity we note that if $\phi(\sum_i \mathbb{1}_{Kg_i}) = 0$ then $0 = \phi(\sum_i \mathbb{1}_{Kg_i})(K1) = \sum_i Kg_i$. $\square$

Now let $B \leq K \leq G$ be another subgroup of $G$ which we assume to have finite index $n$ in $K$. Then $B$ inherits the property of $K$ that each double coset of $G$ with respect to $B$ is a disjoint union of finitely many left- (or right-) cosets; so it makes sense to define the Hecke algebra $H_B$ of $G$ with respect to $B$.

**Lemma 3.2.3** *The function $\mathbb{1}_K$, considered as an element of $H_B$, fulfills $\mathbb{1}_K^2 = n \cdot \mathbb{1}_K$ where $n = [K : B]$, so $e := \frac{1}{n}\mathbb{1}_K \in H_B$ is an idempotent and $H_K \cong eH_Be$ as algebras.*

*Proof.* Let $K = \bigsqcup_{i=1}^n Bk_i$, then in $H_B$ we have

$$\mathbb{1}_K^2 = \sum_{i,j} \mathbb{1}_{Bk_ik_j} = \sum_j \mathbb{1}_{Kk_j} = \sum_j \mathbb{1}_K = n \cdot \mathbb{1}_K, \tag{3.14}$$

so $e := \frac{1}{n}\mathbb{1}_K$ is in fact an idempotent. Now consider the map $\phi = \phi_{K,B} : H_K \to eH_Be$ which is the additive extension of $\mathbb{1}_{KgK} \mapsto \frac{1}{n} \cdot \mathbb{1}_{KgK}$. Then $\mathbb{1}_K = 1 \in H_K$ maps to $e = 1 \in eH_Be$ and we only need to check that this map is well-defined and multiplicative. To that end let $KgK = \bigsqcup_r Kg_r$, $KhK = \bigsqcup_s Kh_s$ be two double cosets. Then

$$\begin{aligned} e\phi(\mathbb{1}_{KgK})e &= \frac{1}{n^3} \sum_{s,i,j,l} \mathbb{1}_{Bk_ik_jg_rk_l} = \frac{1}{n^3} \sum_{j,r,l} \mathbb{1}_{Kk_jg_rk_l} \\ &= \frac{1}{n^2} \sum_l \underbrace{\sum_r \mathbb{1}_{Kg_rk_l}}_{=KgK} = \frac{1}{n}\mathbb{1}_{KgK} \\ &= \phi(\mathbb{1}_{KgK}), \end{aligned} \tag{3.15}$$

hence $\phi$ is well-defined and furthermore

$$\begin{aligned} \phi(\mathbb{1}_{KgK})\phi(\mathbb{1}_{KhK}) &= \frac{1}{n^2} \sum_r \mathbb{1}_{Kg_r} \sum_s \mathbb{1}_{Kh_s} = \frac{1}{n^2} \sum_{r,i} \mathbb{1}_{Bk_ig_r} \sum_{s,j} \mathbb{1}_{Bk_jh_s} \\ &= \frac{1}{n^2} \sum_{r,s,i,j} \mathbb{1}_{Bk_ig_rk_jh_s} = \frac{1}{n^2} \sum_{r,s,j} \mathbb{1}_{Kg_rk_jh_s} \\ &= \frac{1}{n^2} \sum_s (\sum_j \underbrace{\sum_r \mathbb{1}_{Kg_rk_j}}_{=\mathbb{1}_{KgK}})h_s = \frac{1}{n} \sum_s \mathbb{1}_{KgK}h_s \\ &= \frac{1}{n} \sum_{s,r} \mathbb{1}_{Kg_rh_s} = \phi(\sum_{s,r} \mathbb{1}_{Kg_rh_s}) = \phi(\mathbb{1}_{KgK}\mathbb{1}_{KhK}), \end{aligned} \tag{3.16}$$

hence $\phi$ is multiplicative. $\square$

Let us denote by $|K\backslash KgK|$ (or $|B\backslash BgB|$) the number of distinct right $K$- (or $B$) cosets in $KgK$ (or $BgB$).

**Lemma 3.2.4** *Let $e$ be the idempotent from the previous lemma and $g \in G$. Then $\phi_{K,B}(\mathbb{1}_{KgK}) = \frac{|K\backslash KgK|}{|B\backslash BgB|} \cdot e\mathbb{1}_{BgB}e$.*

*Proof.* Obviously $e\mathbb{1}_{BgB}e$ is $K$-bi-invariant and supported only on $KgK$, hence a multiple of $\mathbb{1}_{KgK}$. Now let $K = \bigsqcup_{i=1}^{n} Bk_i$ and $BgB = \bigsqcup_{j=1}^{s} Bg_j$ ($s = |B\backslash BgB|$), then we compute $e\mathbb{1}_{BgB}e = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{s}\mathbb{1}_{Kg_jk_i}$ which equals $\eta \cdot \phi_{K,B}(\mathbb{1}_{KgK}) = \frac{\eta}{n}\mathbb{1}_{KgK}$ for some $\eta$.

Furthermore $\sum_{i=1}^{n}\sum_{j=1}^{s}\mathbb{1}_{Kg_jk_i}$ is a sum (with $n \cdot s$ summands) over $K$-right cosets in $KgK$ and each such right coset appears the same number of times in the sum. Thus each such coset appears exactly $\frac{ns}{s'}$ times, where $s' = |K\backslash KgK|$, which yields $\frac{\eta}{n} = \frac{1}{n^2}\frac{ns}{s'}$ and thus proves the assertion. □

### 3.2.2. The Hecke Algebra of an Open Compact Subgroup

After having taken a general look at double coset algebras in the last subsection we now return to the notations and assumptions of Section 3.1, so once more $K$ is an open and compact subgroup in the locally compact group $\mathbb{G}(\hat{k})$. We will denote the space of double cosets $K\backslash\mathbb{G}(\hat{k})/K$ as $\mathbb{G}(\hat{k})/\!\!/K$.

**Definition 3.2.5** *The* Hecke algebra $H_K = H(\mathbb{G}, K)$ *is the ($\mathbb{Q}$)-algebra of all locally constant, compactly supported functions $\mathbb{G}(\hat{k}) \to \mathbb{Q}$ which are $K$-bi-invariant. The multiplication in $H_K$ is given by convolution with respect to the (unique) Haar measure $d\lambda_K$ giving the compact group $K$ measure 1, i.e.*

$$(F \cdot F')(\gamma) = \int_{\mathbb{G}(\hat{k})} F(x)F'(x^{-1}\gamma)d\lambda_K(x) = \int_{\mathbb{G}(\hat{k})} F(\gamma y^{-1})F'(y)d\lambda_K(y) \quad (3.17)$$

*for $F, F' \in H_K$ and $\gamma \in \mathbb{G}(\hat{k})$.*

A basis for $H_K$ is given by the characteristic functions of the double cosets of $\mathbb{G}(\hat{k})$ with respect to $K$. These are obviously compactly supported as $K\gamma K$ is the image of the compact set $K \times K$ under the map $(k, k') \mapsto k\gamma k'$ and linearly independent by simply evaluating a linear combination of 0 at representatives of each double coset.

Furthermore since $K\gamma K$ is compact and $K\gamma'$ is open for all $\gamma, \gamma' \in \mathbb{G}(\hat{k})$, each double coset decomposes into finitely many right cosets. In particular $K \subset \mathbb{G}(\hat{k})$ fulfills the condition of the previous subsection. The next result shows that the definition of the Hecke algebra we have given here is equivalent to the one from the previous subsection:

**Proposition 3.2.6** *Let $\gamma_1, \gamma_2 \in \mathbb{G}(\hat{k})$, $K\gamma_iK = \bigsqcup_{j=1}^{s_i} K\gamma_{i,j}$, $i = 1, 2$, be the decomposition of the corresponding double cosets into right cosets and $F_i :=$*

$\mathbb{1}_{K\gamma_i K} \in H_K$, $i = 1, 2$, *the characteristic functions on the double cosets. Then the product of* $F_1$ *and* $F_2$ *in* $H_K$ *is given by:*

$$(F_1 \cdot F_2)(g) = \sum_{k=1}^{s_1} \sum_{j=1}^{s_2} \mathbb{1}_{K\gamma_{1,k}\gamma_{2,j}}(g) \tag{3.18}$$

*for all* $g \in \mathbb{G}(\hat{k})$.

*Proof.* Let additionally $K\gamma_1 K = \bigsqcup_{j=1}^{s_1'} \widetilde{\gamma_{1,j}} K$ be the decomposition of $K\gamma_1 K$ into left cosets. We evaluate the integral at $g \in \mathbb{G}(\hat{k})$:

$$\begin{aligned}
(F_1 \cdot F_2)(g) &= \int_{\mathbb{G}(\hat{k})} F_1(x) F_2(x^{-1}g) d\lambda_K(x) \\
&= \int_{K\gamma_1 K} \mathbb{1}_{K\gamma_2 K}(x^{-1}g) d\lambda_K(x) \\
&= \sum_{k=1}^{s_1'} \sum_{j=1}^{s_2} \int_{\widetilde{\gamma_{1,k}}K} \mathbb{1}_{K\gamma_{2,j}}(x^{-1}g) d\lambda_K(x) \\
&= \sum_{k=1}^{s_1'} \sum_{j=1}^{s_2} \int_{\widetilde{\gamma_{1,k}}K} \mathbb{1}_{K\gamma_{2,j}g^{-1}}(x^{-1}) d\lambda_K(x) \\
&= \sum_{k=1}^{s_1'} \sum_{j=1}^{s_2} \int_{\widetilde{\gamma_{1,k}}K} \mathbb{1}_{g\gamma_{2,j}^{-1}K}(x) d\lambda_K(x) \\
&\overset{(*)}{=} \sum_{k=1}^{s_1'} \sum_{j=1}^{s_2} \mathbb{1}_{\widetilde{\gamma_{1,k}}K\gamma_{2,j}}(g) \\
&= \sum_{k=1}^{s_1'} \sum_{j=1}^{s_2} \mathbb{1}_{\widetilde{\gamma_{1,k}}K}(g\gamma_{2,j}^{-1}) \\
&= \sum_{j=1}^{s_2} \mathbb{1}_{K\gamma_1 K}(g\gamma_{2,j}^{-1}) \\
&= \sum_{j=1}^{s_2} \sum_{k=1}^{s_1} \mathbb{1}_{K\gamma_{1,k}}(g\gamma_{2,j}^{-1}) \\
&= \sum_{k=1}^{s_1} \sum_{k=1}^{s_2} \mathbb{1}_{K\gamma_{1,k}\gamma_{2,j}}(g)
\end{aligned} \tag{3.19}$$

Here $(*)$ holds since

$$\int_{\widetilde{\gamma_{1,k}}K} \mathbb{1}_{g\gamma_{2,j}^{-1}K}(x) dx = \begin{cases} \int_{\widetilde{\gamma_{1,k}}K} 1 \; dx = 1, & \widetilde{\gamma_{1,k}}K = g\gamma_{2,j}^{-1}K \\ 0, & \text{else} \end{cases} \tag{3.20}$$

□

If $K$ is a product of local factors $K = \prod_{\mathfrak{p}} K_{\mathfrak{p}}$ where the product runs over the finite places of $k$ and each $K_{\mathfrak{p}}$ is an open and compact subgroup in the locally compact group $\mathbb{G}(k_\mathfrak{p})$, then the Hecke algebra $H_K$ is the restricted tensor product of the local Hecke algebras $H(\mathbb{G}(k_\mathfrak{p}), K_\mathfrak{p}) = H_{K_\mathfrak{p}}$. Hence it is advisable to first study these Hecke algebras for semisimple groups over local fields since most practical questions in the global situation can be reduced to their local counterparts.

### 3.2.3. Hecke Algebras of Semisimple Groups over Local Fields

Let $F$ be a local field of characteristic 0 with ring of integers $\mathcal{O}_F$, uniformizer $\pi$ and finite residue class field of order $q$ and characteristic $p$. For example we could consider (in the notation of the previous subsection) the field $F = k_\mathfrak{p}$ with ring of integers $\mathcal{O}_\mathfrak{p}$ and maximal ideal $\mathfrak{p}\mathcal{O}_\mathfrak{p}$. In fact this is the fundamental example we are primarily interested in and which we will always have in mind.

Furthermore let $\mathbb{G}$ now be a connected, semisimple, linear algebraic group defined and split over $F$. The theory of Hecke algebras for these groups was first considered by Iwahori and Matsumoto ([IM65]) who developed an analogue of the usual Bruhat decomposition and gave a complete set of generators and relations for the Hecke algebra with respect to an Iwahori subgroup. Here we want to review these results and give some additional descriptions of the structure of these algebras.

Let us agree upon the following notation which will be used once more in a later chapter and is, with notational compatibility in mind, essentially taken from the articles [LP02] and [Lan01]:

There is a Chevalley group scheme $\underline{G}$ over $\mathcal{O}_F$ such that $K := \underline{G}(\mathcal{O}_F) \leq \underline{G}(F) = \mathbb{G}(F)$ is a hyperspecial maximal compact subgroup and such that the special fiber $\underline{G}_{\mathcal{O}_F/\pi\mathcal{O}_F}$ is again semisimple of the same type as $\mathbb{G}$ (cf. Theorem B.2.11).

Let $\underline{T} \leq \underline{G}$ be a split maximal torus scheme (whose generic fiber $\underline{T}_F$ we call $\mathbb{T}$). We set $\mathbb{N}_\mathbb{T}$ the normalizer of $\mathbb{T}$ in $\mathbb{G}$, i.e. $\mathbb{N}_\mathbb{T}(A) := N_{\mathbb{G}(A)}(\mathbb{T}(A))$ for all $A \in F\text{-}\mathsf{Alg}$, and $X^*(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ and $X_*(\mathbb{G}_m, \mathbb{T})$ the character and cocharacter module of $\mathbb{T}$, respectively. Let $\Phi \subset X^*(\mathbb{T})$ be the (finite) set of roots (i.e. the non-trivial weights occuring in the adjoint representation). We choose some positive subset $\Phi^+ \subset \Phi$ (or equivalently a Borel subgroup $\mathbb{T} \subset \mathbb{B} \subset \mathbb{G}$) and let $\Delta$ be the corresponding system of simple (or indecomposable) roots. Dually to this, let $\Phi^\vee \subset X_*(\mathbb{T})$ be the set of coroots and $\alpha \mapsto \alpha^\vee$ the usual bijective correspondence.

Given $\alpha \in \Phi$, we denote by $x_\alpha : \mathbb{G}_a \to \underline{U_\alpha}$ the isomorphism between $\mathbb{G}_a$ and the one-dimensional unipotent subgroup scheme $\underline{U_\alpha} \leq \underline{G}$ (whose generic fiber $(\underline{U_\alpha})_F$ we will call $\mathbb{U}_\alpha$). The morphism $x_\alpha$, when considered as a map $F \to \mathbb{U}_\alpha(F)$, restricts to $\mathcal{O}_F$ with $\underline{U_\alpha}(\mathcal{O}_F) = \mathbb{U}_\alpha(F) \cap K$.

The (finite) Weyl group of $\mathbb{G}$, defined as $\mathbb{N}_{\mathbb{T}}(F)/\mathbb{T}(F) = (\mathbb{N}_T(F) \cap K)/\underline{T}(\mathcal{O}_F)$, will here be denoted by $W_0$ (to avoid ambiguity), while we use the symbol $\widetilde{W}$ for the extended affine Weyl group $\mathbb{N}_{\mathbb{T}}(F)/\underline{T}(\mathcal{O}_F)$. Then both $W_0$ and $\widetilde{W}$ act as groups of affine transformations on the vector space $X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $W_0$ is precisely the stabilizer of $0 \in X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ in $\widetilde{W}$. Furthermore there is an isomorphism

$$\widetilde{W} \cong X_*(\mathbb{T}) \rtimes W_0, \tag{3.21}$$

where we embed $X_*(\mathbb{T})$ into $\widetilde{W}$ as a normal subgroup of translations (acting in the obvious way on $X_*(\mathbb{T}) \otimes \mathbb{R}$). In this sense set $t_\lambda = t(\lambda) \in \widetilde{W}$ the translation corresponding to $\lambda \in X_*(\mathbb{T})$, which yields the identity

$$w^{-1} t_\lambda w = t_{\lambda w}, \tag{3.22}$$

for $w \in W_0$ and $\lambda \in X_*(\mathbb{T})$.

We follow the description in Appendix B. The Weyl group $W_0$ is a finite Coxeter group with set of involutive generators $S_0 = \{w_\alpha \mid \alpha \in \Delta\}$ where $w_\alpha$ simply denotes the reflection through the vanishing hyperplane of the root $\alpha \in \Phi$. Now decompose $\Phi = \Phi_1 \cup ... \cup \Phi_m$ into irreducible root systems with corresponding simple systems $\Delta_i$, $1 \leq i \leq m$ (such that each $\Phi_i$ is the root system of an almost simple constituent of $\mathbb{G}$). If we put $\alpha_{0,i}$ the (unique) highest root of $\Phi_i$ (with respect to the simple system $\Delta_i$) we can form a larger Coxeter group with generators $\widetilde{S} := S_0 \cup \{t_{\alpha_{0,i}^\vee} w_{\alpha_{0,i}}\}$ which is isomorphic to the affine Weyl group $W_{af}$ associated to $\Phi$. We will think of $W_{af}$ as a subgroup of $\widetilde{W}$ via this isomorphism.

We make the canonical choice for an Iwahori subgroup of $\mathbb{G}(F)$ by letting $I$ equal the subgroup generated by $\underline{T}(\mathcal{O}_F)$, the groups $x_\alpha(\mathcal{O}_F) = \underline{U_\alpha}(\mathcal{O}_F)$ for $\alpha \in \Phi^+$, and the groups $x_\alpha(\mathfrak{p})$ for $\alpha \in \Phi^- = -\Phi^+$. This is the canonical choice for $I$ since it is the inverse image of the Borel subgroup associated to $\Phi^+$ under the reduction modulo $\mathfrak{p}$ from $K$ to the special fiber of $\underline{G}$ (cf. [Tit79]). Under these definitions the triple $(\mathbb{G}(F), I, \mathbb{N}_{\mathbb{T}}(F))$ is a generalized Tits system.

The group $\widetilde{W}$ is an extension of $W_{af}$ by a group $\Omega$ which one can find as follows: Put $\widetilde{I}$ the normalizer of $I$ in $\mathbb{G}(F)$, then

$$\Omega = (\mathbb{N}_{\mathbb{T}}(F) \cap \widetilde{I})/\underline{T}(\mathcal{O}_F) \subset \widetilde{W}. \tag{3.23}$$

The group $\Omega$ is finite Abelian and canonically isomorphic to $X_*(\mathbb{T})/\Lambda$ where $\Lambda \leq_{\mathbb{Z}} X_*(\mathbb{T})$ is the lattice generated by the coroots $\Phi^\vee$ (cf. Section B.3) (in particular $\Omega$ is trivial if $\mathbb{G}$ is simply connected). It normalizes $W_{af}$ and we get a split extension

$$\widetilde{W} \cong W_{af} \rtimes \Omega. \tag{3.24}$$

As usual we can consider the length function $w \mapsto \ell(w)$ on the Coxeter group $W_{af}$ (with respect to the generating system $\widetilde{S}$). This length function extends to $\widetilde{W}$ by setting $\ell(\rho w) = \ell(w\rho) := \ell(w)$ for $w \in W_{af}$ and $\rho \in \Omega$. In this sense we will call an expression $w = w_1 ... w_r \rho$ with $w_i \in \widetilde{S},\ 1 \leq i \leq r$, and $\rho \in \Omega$ reduced if $\ell(w) = r$.

Having this notation at hand, we want to study the Hecke algebra of $\mathbb{G}(F)$ with respect to open compact subgroups which contain the Iwahori subgroup $I$. Since all of these arise as condensations of $H_I = H(\mathbb{G}(F), I)$ by Lemma 3.2.3 we will first study $H_I$ itself.

A natural basis for $H_I$ is given by the characteristic functions on the double cosets of $\mathbb{G}(F)$ with respect to $I$, so we are in need of a convenient indexing of these. This is achieved by the following lemma:

**Lemma 3.2.7 ([IM65, Prop. (2.1)])** $\mathbb{G}(F)$ *decomposes as:*

$$\mathbb{G}(F) = \bigsqcup_{w \in \widetilde{W}} IwI. \tag{3.25}$$

The following result summarizes the structure of $\mathbb{G}(F)/\!\!/I$ (and hence gives a first insight into the multiplication in the Hecke algebra with respect to $I$). It is also known as the *cell multiplication rule*:

**Proposition 3.2.8 ([IM65, Prop. 2.8, Thm. 3.3])** *Let* $w, w' \in \widetilde{W}$.

1. *For all* $s \in \widetilde{S}$ *the following holds:*

$$IwIsI = \begin{cases} IwsI & \text{if } \ell(ws) = \ell(w) + 1 \\ IwsI \cup IwI & \text{if } \ell(ws) < \ell(w). \end{cases} \tag{3.26}$$

2. *If* $\ell(ww') = \ell(w) + \ell(w')$ *then* $IwIw'I = Iww'I$. *If, in particular,* $w = s_1 ... s_{\ell(w)} \eta$ *is a reduced expression for* $w$ *as a word in* $\widetilde{S} \cup \Omega$, *we get* $IwI = Is_1I ... Is_{\ell(w)} I\eta I$.

As the support of the product of $\mathbb{1}_{IwI}$ and $\mathbb{1}_{IsI}$ in $H_I$ is contained in $IwIsI$, this product is (by the previous proposition) either a multiple of $\mathbb{1}_{IwsI}$ (if $\ell(ws) = \ell(w) + 1$) or a linear combination of $\mathbb{1}_{IwI}$ and $\mathbb{1}_{IwsI}$. Hence it remains to determine the structure constants in these cases. This was achieved by Iwahori and Matsumoto who described the Hecke algebra $H_I$ in [IM65].

**Lemma 3.2.9 ([IM65, (3.5)-(3.8)])** *For* $w \in \widetilde{W}$ *set* $T_w := \mathbb{1}_{IwI}$. *Then* $H_I$ *is the algebra with basis* $(T_w,\ w \in \widetilde{W})$ *subject (only) to the following relations:*

$$\begin{aligned} &T_w T_w' = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w'). \\ &(T_s + 1)(T_s - q) = 0 \text{ for } s \in \widetilde{S}. \end{aligned} \tag{3.27}$$

Note that via the first relation we get $T_w = T_{s_1}...T_{s_{\ell(w)}}T_\eta$ where $w = s_1...s_{\ell(w)}\eta$ is a reduced expression for $w$ and this is independent of the chosen reduced expression. In particular we see immediately that $T_w$ is invertible for every $w \in \widetilde{W}$ since

$$\begin{aligned} T_s^{-1} &= \frac{1}{q}(T_s - q + 1) \text{ for } s \in \widetilde{S} \\ T_\eta^{-1} &= T_{\eta^{-1}} \text{ for } \eta \in \Omega \end{aligned} \tag{3.28}$$

and each $T_w$ is a product of these.

The algebra $H_I$ has a variety of distinguished subalgebras which we want study in order to understand $H_I$ itself.

**Remark and Definition 3.2.10** *1. The subspace with basis $T_w,\ w \in W_{af}$, is a subalgebra of $H_I$ called the* affine Hecke algebra *and will be denoted by $H_I^a$.*

*2. The subspace with basis $T_w,\ w \in W_0$, is a (finite-dimensional) subalgebra of $H_I$ and will be denoted by $H_I^0$.*

*3. The subalgebra with basis $T_\eta,\ \eta \in \Omega$, is naturally isomorphic to $\mathbb{Q}[\Omega]$ (the group ring of the finite group $\Omega$) and we will identify the two via this isomorphism.*

The algebra $H_I^a$ is isomorphic to the one-parameter Iwahori-Hecke algebra of the Coxeter group $W_{af}$ (with parameter $q$), i.e. the algebra with generators $T_s,\ s \in \widetilde{S}$, subject to the relations

$$\begin{aligned} &(T_s + 1)(T_s - q) = 0 \text{ for } s \in \widetilde{S}, \\ &T_sT_{s'}T_s... = T_{s'}T_sT_{s'}... \text{ for } s \neq s' \in \widetilde{S}, \end{aligned} \tag{3.29}$$

where on the left hand side and the right hand side of the second relation there are exactly $m_{s,s'} = |ss'|$ factors. Analogously $H_I^0$ is the (finite-dimensional) Iwahori-Hecke algebra (with parameter $q$) of the finite Weyl group $W_0$. It is noteworthy that the Iwahori-Hecke algebra of a Coxeter group is a deformation of the standard group algebra which can be recovered as the Iwahori-Hecke algebra with parameter 1. In fact, for $q$ outside of a finite set (consisting of certain roots of unity) and finite Coxeter groups, the Iwahori-Hecke algebra with parameter $q$ over $\mathbb{C}$ is isomorphic to the group algebra (cf. [GP00, Thm. (8.1.7)]).

Having defined our subalgebras we now find two decompositions of $H_I$ which shine a brighter light on its structure. The first one stems from the semidirect product decomposition $\widetilde{W} = W_a \rtimes \Omega$ and is again due to Iwahori and Matsumoto:

**Lemma 3.2.11 ([IM65, Prop. (3.8)])** *The Hecke algebra $H_I$ has the following tensor product decomposition:*

$$H_I \cong H_I^a \otimes_{\mathbb{Q}} \mathbb{Q}[\Omega], \tag{3.30}$$

*with multiplication*

$$(T_w \otimes T_\eta)(T_{w'} \otimes T_{\eta'}) = (T_w T_{\eta w' \eta^{-1}}) \otimes T_{\eta\eta'} \tag{3.31}$$

*for $w, w' \in W_{af}$ and $\eta, \eta' \in \Omega$.*

Since we have a finite presentation for $H_I^a$ and $\Omega$ is a finite group (so the same is true for $\mathbb{Q}[\Omega]$), this description of $H_I$ is particularly useful for implementation purposes as it can be used with any computer algebra system capable of handling finitely presented algebras. However, this decomposition is of limited value when considering the condensation algebras we want to study, as hardly any of the relations are preserved when we descend to those. For this purpose it is far more advantageous to consider another decomposition of $H_I$ which is commonly attributed to Bernstein (though no article of Bernstein's appears to actually cover it). He realized that $H_I$ contains a large commutative subalgebra and was able to explicitly describe the center of $H_I$.

To attain this we start by setting

$$\begin{aligned} P^+ :&= \{\lambda \in X_*(T) \mid \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\} \\ &= \{\lambda \in X_*(T) \mid \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta\}. \end{aligned} \tag{3.32}$$

This is often called the set of dominant coweights. Additionally define $\rho \in \frac{1}{2}X^*(T)$ to be half the sum of the positive roots,

$$2\rho := \sum_{\alpha \in \Phi^+} \alpha. \tag{3.33}$$

If $\rho$ is not a character (i.e. an element of $\frac{1}{2}X^*(T) - X^*(T)$) we set $\mathbb{Q}' := \mathbb{Q}(q^{1/2})$ and consider from now on the scalar extension $H_I' := \mathbb{Q}' \otimes H_I$ (and accordingly the algebras $H_0'$ and $H_a'$ defined in the obvious way). The set $P^+$ is a fundamental domain for the action of $W_0$ on $X_*(T)$ (since $\alpha_i$ is the only positive root which turns into a negative one under $s_i$). Furthermore $\rho$ is connected to the length function in the following way:

$$\ell(t_\lambda) = \langle \lambda, 2\rho \rangle \tag{3.34}$$

for $\lambda \in P^+$. Hence the length function is additive on the set $T_\lambda,\ \lambda \in P^+$, (where we set $T_\lambda := T_{t_\lambda}$ for $\lambda \in X_*(T)$ for ease of notation) and so formula 3.27 yields

$$T_\lambda T_{\lambda'} = T_{\lambda+\lambda'} \text{ for all } \lambda, \lambda' \in P^+. \tag{3.35}$$

Now if $\lambda$ is an arbitrary element of $X_*(T)$ (so not necessarily in $P^+$) there are dominant coweights $\mu, \nu$ such that $\lambda = \mu - \nu$. Presuming this decomposition we define the following elements of $H'_I$:

$$\widetilde{T}_\lambda := q^{-\langle \lambda, \rho \rangle} T_\mu T_\nu^{-1}. \tag{3.36}$$

This is well-defined, for if $\lambda = \mu - \nu = \mu' - \nu'$ we get $q^{-\langle \lambda, \rho \rangle} = q^{-\langle \mu - \nu, \rho \rangle} = q^{-\langle \mu' - \nu', \rho \rangle}$ and

$$T_\mu T_{\nu'} = T_{\mu + \nu'} = T_{\mu' + \nu} = T_{\mu'} T_\nu \tag{3.37}$$

which implies $T_\mu T_\nu^{-1} = T_{\mu'} T_{\nu'}^{-1}$.

Analogously we see that

$$\widetilde{T}_\lambda \widetilde{T}_\delta = \widetilde{T}_{\lambda + \delta} \tag{3.38}$$

for all $\lambda, \delta \in X_*(T)$ (not just in $P^+$).

In conclusion the map $\lambda \mapsto \widetilde{T}_\lambda$ embeds the group ring $\mathbb{Q}'[X_*(T)]$ into the Hecke-algebra $H'_I$ and we will think of $\mathbb{Q}'[X_*(T)]$ as the subspace of $H'_I$ with basis $T_\lambda,\ \lambda \in X_*(T)$, in this way.

Remember that the first decomposition of $H_I$ was obtained from the semidirect product decomposition $\widetilde{W} = W_a \ltimes \Omega$ and that we originally had another decomposition $\widetilde{W} = X_*(T) \ltimes W_0$. The latter one yields the following decomposition:

**Lemma 3.2.12 ([HO97, (1.18)])** *$H'_I$ has the following tensor product decomposition as a vector space:*

$$H'_I \cong \mathbb{Q}'[X_*(T)] \otimes H'_0 \cong H'_0 \otimes \mathbb{Q}'[X_*(T)]. \tag{3.39}$$

*In particular, since $H'_0$ is a finite-dimensional algebra, $H'_I$ is a finitely generated module over $\mathbb{Q}'[X_*(T)]$, and the sets $\{\widetilde{T}_\lambda T_w \mid \lambda \in X_*(T), w \in W_0\}$ and $\{T_w \widetilde{T}_\lambda \mid \lambda \in X_*(T), w \in W_0\}$ are $\mathbb{Q}'$-bases of $H'_I$.*

The multiplication constants in this decomposition are given by the so-called push relation or Bernstein-Zelevinsky relation (cf. [HO97, (1.19)]):

$$\widetilde{T}_\lambda T_{s_i} - T_{s_i} \widetilde{T}_{\lambda s_i} = T_{s_i} \widetilde{T}_\lambda - \widetilde{T}_{\lambda s_i} T_{s_i} = (q-1) \frac{\widetilde{T}_\lambda - \widetilde{T}_{\lambda s_i}}{1 - \widetilde{T}_{-\alpha_i^\vee}} \tag{3.40}$$

for $\lambda \in X_*(T)$ and $s_i = s_{\alpha_i} \in S_0$. Note that the division can actually be performed in $\mathbb{Q}'[X_*(T)]$.

The group $W_0$ acts on the ($\mathbb{Z}$-)lattice $X_*(T)$ as a group of module isomorphisms and hence this action extends to an action of $W_0$ on $\mathbb{Q}'[X_*(T)]$ as a group of algebra isomorphisms. The center of $H'_I$ can now be described via the invariants of this action, a result which is originally due to Bernstein (cf. [Lus83, Thm. (8.1)]). To that end define for $\lambda \in P^+$

$$z_\lambda := \sum_{\lambda' \in \lambda W_0} \widetilde{T}_{\lambda'}. \tag{3.41}$$

**Theorem 3.2.13 ([Lus83, Thm. (8.1)],[RR03, Thm. (4.12)])** *As a vector space, the center $Z$ of $H_I'$ has the basis $z_\lambda,\ \lambda \in P^+$.*

*Proof.* The elements $z_\lambda,\ \lambda \in P^+$, have mutually disjoint supports (in the standard basis of $\mathbb{Q}'[X_*(T)]$) and are hence linearly independent.

By use of formula 3.40 we compute for $\lambda \in P^+$ and $s_i = s_{\alpha_i} \in S_0$:

$$\begin{aligned}
z_\lambda T_{s_i} &= \sum_{\lambda' \in \lambda W_0} \widetilde{T}_{\lambda'} T_{s_i} \\
&= \sum_{\lambda' \in \lambda W_0} \left( (q-1) \frac{\widetilde{T}_{\lambda'} - \widetilde{T}_{\lambda' s_i}}{1 - \widetilde{T}_{-\alpha_i^\vee}} + T_{s_i} \widetilde{T}_{\lambda' s_i} \right) \\
&= \sum_{\lambda' \in \lambda W_0} \left( (q-1) \frac{\widetilde{T}_{\lambda'} - \widetilde{T}_{\lambda' s_i}}{1 - \widetilde{T}_{-\alpha_i^\vee}} \right) + T_{s_i} z_\lambda \\
&= T_{s_i} z_\lambda.
\end{aligned} \tag{3.42}$$

Hence $z_\lambda$ commutes with the $T_s,\ s \in S_0$, and so with the $T_w,\ w \in W_0$. Since they obviously also commute with the $\widetilde{T}_\mu,\ \mu \in X_*(T)$, and these two types of elements generate $H_I'$, the $z_\lambda$ actually lie in the center.

On the other hand let

$$z = \sum_{w' \in W_0, \lambda' \in X_*(T)} c_{\lambda', w'} \widetilde{T}_{\lambda'} T_{w'} \tag{3.43}$$

be an element of the center and choose some $w \in W_0$ maximal with respect to the Bruhat order with the property that $c_{\lambda,w} \neq 0$ for some $\lambda \in X_*(T)$. Furthermore choose some arbitrary $\mu \in P^+$ such that $c_{\lambda+\mu-\mu w} = 0$ which exists since $z$ has finite support.

We consider the equation

$$z = \widetilde{T}_{-\mu} z \widetilde{T}_\mu = \sum_{w', \lambda'} c_{\lambda', w'} \widetilde{T}_{\lambda' - \mu} T_{w'} \widetilde{T}_\mu. \tag{3.44}$$

Now write

$$T_{w'} \widetilde{T}_\mu = \sum_{\nu, v} d_{\nu, v}(w') \widetilde{T}_\nu T_v \tag{3.45}$$

for suitable $d_{\nu,v}(w')$. If we set $w' = u s_i$ with $s_i \in S_0$ and $\ell(w') = \ell(u+1)$ we see

$$T_{w'} \widetilde{T}_\mu = T_u T_{s_i} \widetilde{T}_\mu = T_u \left( \widetilde{T}_{\mu s_i} T_s + \underbrace{\frac{\widetilde{T}_\mu - \widetilde{T}_{\mu s_i}}{1 - \widetilde{T}_{-\alpha_i^\vee}}}_{\in \mathbb{Q}' X_*(T)} \right) \tag{3.46}$$

and so $d_{\mu w', w'}(w') = 1$, $d_{\nu, w'}(w') = 0$ for $\nu \neq \mu w'$ and $d_{\nu,v}(w') \neq 0$ only if $v \leq w'$ in the Bruhat order. Plugging this in we get

$$z = \sum_{w', \lambda'} c_{\lambda', w'} \widetilde{T}_{\lambda'} T_{w'} = \sum_{w', \lambda'} \sum_{v, \nu} c_{\lambda', w'} d_{\nu, v}(w') \widetilde{T}_{\lambda' + \nu - \mu} T_v. \tag{3.47}$$

Now we compare the coefficients of $\widetilde{T}_\lambda T_w$ on both sides: Remember that $w$ was maximal in the Bruhat order such that $c_{\lambda,w} \neq 0$ for some $\lambda$, so $c_{\lambda',w'} d_{\nu,w}(w') \neq 0$ only if $w \leq w'$ in the Bruhat order and $w \not< w'$ so $w = w'$. In this case $d_{\nu,w}(w) \neq 0$ only for $\nu = \mu w$ and hence for $\widetilde{T}_{\lambda'+\nu-\mu}$ to equal $\widetilde{T}_\lambda$ we must have $\lambda' = \lambda + \mu - \mu w$. This yields

$$c_{\lambda,w} = c_{\lambda+\mu-\mu,w} d_{\mu w,w}(w) = 0 \tag{3.48}$$

which is a contradiction. Hence $z$ is in fact an element of $\mathbb{Q}'[X_*(T)]$ and the first computation shows that it is a linear combination of the orbit sums of $W_0$ on $X_*(T)$. □

The last theorem shows that $Z := Z(H'_I)$ is isomorphic to the ring $\mathbb{Q}'[X_*(T)]^{W_0}$ of $W_0$-invariants in $\mathbb{Q}'[X_*(T)]$. The ring $\mathbb{Q}'[X_*(T)]$ is the group ring of the finitely generated free $\mathbb{Z}$-module $X_*(T)$ and hence isomorphic to the ring of Laurent polynomials in $\mathrm{rank}(X_*(T))$ many indeterminates on which $W_0$ acts as a group of algebra automorphisms. Studying the invariants of actions of this kind is the subject of the field of *multiplicative invariant theory* which was essentially founded by Daniel R. Farkas in the 1980s (cf [Far84, Far85, Far86]). We will not go into the details of this theory and instead opt to simply cite the results which are of relevance to our current situation.

Let us start with the following easy observation:

**Lemma 3.2.14** *The center $Z$ of $H'$ is a finitely generated $\mathbb{Q}'$-algebra and $\mathbb{Q}'[X_*(T)]$ is finitely generated as a $Z$-module.*

*Proof.* This is always true in the situation $A/A^G$ where $A$ is some finitely generated commutative algebra and $G$ is a finite group of algebra automorphisms by the Noether normalization theorem. □

**Corollary 3.2.15** *$H'_I$ is finitely generated as a $Z$-module.*

*Proof.* $H'_I$ is finitely generated as a $\mathbb{Q}'[X_*(T)]$-module since we have the decomposition $H'_I \cong \mathbb{Q}'[X_*(T)] \otimes H_0$ and $H_0$ is finite-dimensional. $\mathbb{Q}'[X_*(T)]$ is itself finitely generated over $Z$ by the above lemma so the assertion follows. □

Using the theory of multiplicative invariants we can achieve a more detailed description of $Z$. The following theorem can be understood as an analogue of the Shephard-Todd-Chevalley theorem in classical invariant theory.

**Theorem 3.2.16 ([Lor06, Thm. 8.1.1],[Far84])** *Let $L$ be a $\mathbb{Z}$-lattice and $G \leq \mathrm{GL}(L)$ a finite group. The following holds for the ring of invariants $\mathbb{Q}'[L]^G$:*

1. *If $G$ acts as a reflection group on $L$ then $\mathbb{Q}'[L]^G$ is isomorphic to the semigroup algebra $\mathbb{Q}'[M]$ for some submonoid $M$ of $\mathbb{Q}'[L]^G$.*

2. *The algebra $\mathbb{Q}'[L]^G$ is a polynomial ring if and only if $G$ is the Weyl group of a root system and $L$ is isomorphic (as a $\mathbb{Z}G$-module) to the weight lattice of said root system.*

Remember that the weight lattice of a root system $\Psi$ is given as

$$P(\Psi) = \{\beta \in \mathbb{Q} \otimes X^*(T) \mid \langle \alpha^\vee, \beta \rangle \in \mathbb{Z} \ \forall \ \alpha \in \Psi\} = \langle \Psi^\vee \rangle_{\mathbb{Z}}^{\#}. \tag{3.49}$$

In particular we see that the weight-lattice becomes the (analogously defined) coweight-lattice if we switch from the root system $\Psi$ to its dual (while the Weyl group does not change). Hence exchanging 'weight lattice' for 'coweight lattice' in the above theorem does not alter the statement.

The theoretical statement of Theorem 3.2.16 can be made explicit in the sense that there are constructive descriptions of (minimal) generators for the monoid $M$ and algorithms to compute these as well as module generators for $\mathbb{Q}'[L]$ over $\mathbb{Q}'[L]^G$. An implementation for Magma ([BCP97]) can be found in Marc S. Renault's PhD-thesis [Ren02].

The (finite) Weyl group $W_0$ always acts on $X_*(T)$ and $X^*(T)$ as well as their distinguished sub- and superlattices as a reflection group, so we are always in the situation of Theorem 3.2.16. The question remains for which groups we are in the situation of 3.2.16(2), so for which groups the center of $H_I'$ is a polynomial ring.

**Proposition 3.2.17** *If $\mathbb{G}$ is almost simple and adjoint, then $Z$ is a polynomial ring.*

*Proof.* If $\mathbb{G}$ is adjoint we have $P(\Phi) = \langle \Phi \rangle_{\mathbb{Z}} = X^*(T)$ which implies

$$X_*(T) = (X^*(T))^{\#} = P(\Phi)^{\#} \tag{3.50}$$

which is the coweight lattice of $\Phi$ or (by the above observation) the weight lattice of the dual of $\Phi$. Hence we are in the situation of Theorem 3.2.16(2) and $Z \cong (\mathbb{Q}'X_*(T))^{W_0}$ is in fact a polynomial ring. □

**Proposition 3.2.18** *If $\mathbb{G}$ is almost simple and simply connected, then $Z$ is a polynomial ring if and only if $\Phi$ is one of $G_2, F_4, E_8$ or $C_n,\ n \geq 1$.*

*Proof.* For $G_2$, $F_4$ and $E_8$ the group $\mathbb{G}$ is always simply connected as well as adjoint so the fact that $Z$ is a polynomial ring was already clear from the last proposition. For $C_n$ in the simply connected case we have $X_*(T) = \langle \Phi^\vee \rangle_{\mathbb{Z}}$ which is isomorphic to the root lattice of type $B_n$ (the dual of $C_n$) which is itself isomorphic to the weight lattice of type $C_n$ and we are once again in the situation of 3.2.16(2). Alternatively one notes that for $C_n$ in the simply connected case we have $X_*(T) = \langle \omega_1, ..., \omega_{n-1}, 2\omega_n \rangle_{\mathbb{Z}}$ where $\omega_i$ denotes the $i$-th fundamental coweight, so $X_*(T)$ is a stretched (co)weight lattice and we deduce the assertion from [Far84, Thm. 16].

For the 'only if'-part of the assertion we refer to [Ham16] or equivalently [Ham14]. Alternatively we refer once more to [Far84, Thm. 16] and note that $C_n$ is the only (non adjoint) case where $X_*(T)$ is a stretched coweight lattice. □

In the case where $\mathbb{G}$ is simply connected and $Z$ is not a polynomial ring, explicit generators for $Z$ as a semigroup algebra can be found in [Ham16]. We will not treat the remaining cases (i.e. $A_n,\ n+1$ not prime and $D_n,\ n \geq 4$) but merely note that in the computationally feasible situations the abovementioned algorithms quickly produce a suitable generating set for $Z$.

So far we have only considered the Hecke algebra of $\mathbb{G}$ with respect to the Iwahori subgroup $I$, but our previous result (Lemma 3.2.3) now allows us to get some insight into the Hecke algebras with respect to parahoric subgroups of $\mathbb{G}(F)$.

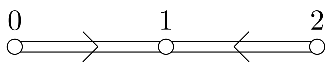
**Proposition 3.2.19** *Let $K$ be a parahoric subgroup of $\mathbb{G}(F)$. Then $H'_K$ is finitely generated as an algebra and a set of generators can be explicitly computed.*

*Proof.* Remember that by Lemma 3.2.3 the algebra $H'_K$ is isomorphic to the condensation $H'_K \cong e_K H'_I e_K$, where $e_K$ is the idempotent corresponding to the characteristic function on $K$. Now let $E_Z$ be a finite set of generators for $Z$ and $E_M$ be a finite set of module generators of $H'_I$ over $Z$ (both of which can be explicitly constructed as before). Then $e_K Z e_K$ is generated by $e_K z e_K = e_K z = z e_K,\ z \in E_Z$, since $Z$ is central and obviously $e_K H' I e_K$ is generated as a module over $e_K Z e_K$ by the $e_K m e_K,\ m \in E_M$. This proves the assertion. □

Let us look at an example to illustrate this:

**Example 3.2.20** *Let $\mathbb{G}$ be a split form of $\mathrm{Sp}_4$ over a local field $F$ with residue class field order $q$. Then $\mathbb{G}$ is simply connected of type $C_2$ with extended Dynkin diagram $\widetilde{C}_2$*

$\widetilde{C}_2$ :




*Since $\mathbb{G}$ is simply connected we have $\Omega = \{1\}$ and furthermore we get $X_*(T) = \langle \omega_1, 2\omega_2 \rangle_{\mathbb{Z}}$ where $\omega_1, \omega_2$ are the fundamental coweights. The Hecke algebra of $\mathbb{G}$ with respect to the Iwahori subgroup $I$ is then isomorphic to the $\mathbb{Q}$-algebra with generators $T_0, T_1, T_2$ (corresponding to the 3 simple reflections in the (extended) affine Weyl group) subject (only) to the relations*

$$\begin{aligned} &T_i^2 = (q-1)T_i + q,\ 0 \leq i \leq 2 \\ &T_0T_1T_0T_1 = T_1T_0T_1T_0 \\ &T_0T_2 = T_1T_0 \\ &T_1T_2T_1T_2 = T_2T_1T_2T_1. \end{aligned} \tag{3.51}$$

*Reduced expressions for $t_{\omega_1}$ and $t_{2\omega_2}$ in the Weyl group are given by*

$$\begin{aligned} t_{\omega_1} &= s_0 s_1 s_2 s_1, \\ t_{2\omega_2} &= (s_0 s_1 s_2)^2. \end{aligned} \tag{3.52}$$

*Hence the image of the embedding of $\mathbb{Q}[X_*(T)]$ into $H_I$ (we have $\rho \in X^*(T)$ so we do not need a square root of $q$) is the subalgebra of $H_I$ generated by $\widetilde{T}_{\omega_1} = q^{-2}T_0T_1T_2T_1$, $\widetilde{T}_{2\omega_2} = q^{-3}(T_0T_1T_2)^2$ and their respective inverses.*

*The center $Z$ of $H_I$ is the polynomial ring in the orbit sums corresponding to $\omega_1$ and $2\omega_2$ under the finite Weyl group $W_0 \cong D_8$ (both orbits contain 4 cocharacters). Both $\mathbb{Q}[X_*(T)]/Z$ and $H_I/\mathbb{Q}[X_*(T)]$ are free modules of rank $8 = |W_0|$, so $H_I$ is generated over $Z$ by 64 elements.*

*If $K$ is the maximal compact subgroup of $\mathbb{G}(F)$ containing $I$ and corresponding to the subset $s_0, s_2$ (so $W_K \cong C_2 \times C_2$), we compute $[K : I] = q^2 + 2q + 1$ and*

$$e_K = \frac{1}{q^2 + 2q + 1}(1 + T_0 + T_2 + T_0T_2). \tag{3.53}$$

*Condensating the generators from above we get 66 generators for the algebra $H_K \cong e_K H_I e_K$. However, this system is far from being minimal. In fact one checks (e.g. with Magma ([BCP97])) that the 66 generators and hence all of $H_K$ is contained in the algebra generated by*

$$e_K T_1 e_K,\ e_K T_1 T_2 T_1 e_K,\ e_K T_1 T_0 T_1 e_K \tag{3.54}$$

*and no proper subset of these generates $H_K$.*

We want to conclude the section by describing the structure of the Hecke algebra with respect to a hyperspecial maximal compact subgroup. These algebras - which are also known as the *spherical Hecke algebras* - were extensively studied by Ichirô Satake (cf. [Sat63]) and will play an important role in the following chapters. Moreover they are obvioulsy of interest to us since we already saw that an integral form of an algebraic group defines a hyperspecial maximal compact subgroup at all but finitely many places.

**Theorem 3.2.21 ([Sat63, Thm. 1])** *Let $K = IW_0I$ be a hyperspecial maximal compact subgroup of $\mathbb{G}(F)$. The algebra $H_K$ is a commutative integral domain of transcendence degree* $\operatorname{rank}(\Phi)$ *with basis*

$$\mathbb{1}_{Kt_\omega K} = \mathbb{1}_{K\omega(\pi)K},\ \omega \in P^+. \tag{3.55}$$

*With respect to this basis the multiplication in $H_K$ is as follows:*

$$\mathbb{1}_{Kt_\omega K}\mathbb{1}_{Kt_{\omega'} K} = \mathbb{1}_{Kt_{\omega'} K}\mathbb{1}_{Kt_\omega K} = \mathbb{1}_{Kt_{\omega+\omega'} K} + \sum_{\lambda \prec \omega+\omega'} a_\lambda \mathbb{1}_{Kt_\lambda K}, \tag{3.56}$$

*for certain $a_\lambda$ and $\lambda \prec \omega + \omega'$ if and only if $\omega + \omega' - \lambda$ is a sum of positive coroots.*

*In particular if $\omega_1, ..., \omega_r \in P^+$ generate $P^+$ as a semigroup then the elements $\mathbb{1}_{Kt_{\omega_1}K}, ..., \mathbb{1}_{Kt_{\omega_r}K}$ generate $H_K$ as an algebra and conversely if $\omega_1, ..., \omega_r \in P^+$ are linearly independent then $\mathbb{1}_{Kt_{\omega_1}K}, ..., \mathbb{1}_{Kt_{\omega_r}K}$ are algebraically independent.*

If $\mathbb{G}$ is adjoint, the set $P^+$ is simply the "integral cone" on the fundamental coweights,

$$P^+ = \left\{ \sum_{\alpha \in \Delta} x_\alpha \omega_\alpha^\vee \mid x_\alpha \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Delta \right\}, \tag{3.57}$$

where $\langle \omega_\alpha^\vee, \beta \rangle = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$.

Thus the Hecke algebra $H_K$, for hyperspecial $K$, is a polynomial ring in the rank($\Phi$) indeterminates $\mathbb{1}_{Kt_{\omega_\alpha^\vee}K}$, $\alpha \in \Delta$.

## 3.3. The Action of the Hecke Algebra on Modular Forms

Having defined the Hecke algebra $H_K$ of an open compact subgroup $K \subset \mathbb{G}(\hat{k})$ we return to our original object of interest, the space $M := M(V, K)$ of modular forms of weight $V$ and level $K$, and consider the following action of $H_K$ on $M(V, K)$:

Let $K\gamma K \in \mathbb{G}(\hat{k})/\!\!/K$, so that the characteristic function on $K\gamma K$ is an element of the standard basis of $H_K$, and write $K\gamma K = \bigsqcup_{i=1}^s K\gamma_i$. Then we get a linear operator (Hecke operator) $T(\gamma) = T(K\gamma K)$ on $M(V, K)$ by setting for $f \in M(V, K)$ and $g \in \mathbb{G}(\hat{k})$:

$$f(T(\gamma))(g) := \sum_{i=1}^{s} f(\gamma_i g). \tag{3.58}$$

The function $fT(\gamma)$ remains left $K$-invariant since an element of $K$ only permutes the cosets $K\gamma_i$, $1 \leq i \leq s$, and $fT(\gamma)$ also transforms suitably under multiplication with elements of $\mathbb{G}(k)$. Hence it is again an element of $M(V, K)$ and we obtain, by linear extension, a map

$$H_K \to \mathrm{End}_k(M). \tag{3.59}$$

This is an algebra homomorphism since for $\gamma, \lambda \in \mathbb{G}(\hat{k})$ with

$$K\gamma K = \bigsqcup_{i=1}^{s} K\gamma_i, \ K\lambda K = \bigsqcup_{i=1}^{t} K\lambda_i \tag{3.60}$$

we have

$$\begin{aligned}((fT(K\gamma K))T(K\lambda K))(x) &= \sum_i (fT(K\gamma K))(\lambda_i x) \\ &= \sum_{i,j} f(\gamma_j \lambda_i x) \\ &= (fT((K\gamma K)(K\lambda K)))(x)\end{aligned} \tag{3.61}$$

by Proposition 3.2.6 and furthermore $1_{H_K} = K = K1K = K1$ maps to the identity homomorphism in $\mathrm{End}_k(M)$.

Since we defined a positive definite inner product on $M$, it is a natural question to ask whether this structure as a bilinear space is in any form compatible with the action of $H_K$. It is answered by the following proposition which is stated without proof in [Gro99].

**Proposition 3.3.1 ([Gro99, Prop. (6.9)])** *The adjoint operator of $T(\gamma)$ is given by $\mu_{\hat{k}}(\gamma)T(\gamma^{-1})$ (as an element of $\mathrm{End}(M)$).*

*Proof.* Let us first choose a system of representatives $\gamma_i,\ 1 \leq i \leq t$, such that $\mathbb{G}(\hat{k}) = \bigsqcup_{i=1}^t K\gamma_i\mathbb{G}(k)$. We need to show that

$$\langle fT(\gamma), f'\rangle = \mu_{\hat{k}}(\gamma)\langle f, f'T(\gamma^{-1})\rangle \tag{3.62}$$

for all $f, f' \in M(V, K)$ and it is obviously enough to do so for $f, f'$ in a basis of $M(V, K)$. Hence we may assume that $f(\gamma_j) = f'(\gamma_i) = 0$ for $j \neq x, i \neq y$ and some $x, y \in \{1, ..., t\}$.

Now let $K\gamma K = \bigsqcup_{i=1}^s Kg_i = \bigsqcup_{j=1}^{s'} g'_j K$ be the decompositions of $K\gamma K$ into right cosets and left cosets, respectively. To shorten the notation we will denote the property $Kg_j'^{-1}\gamma_x\mathbb{G}(k) = K\gamma_y\mathbb{G}(k)$ as $A(j)$ and $Kg_i\gamma_y\mathbb{G}(k) = K\gamma_x\mathbb{G}(k)$ as $B(i)$ for $j = 1, ..., s'$ and $i = 1, ..., s$. We compute

$$\begin{aligned}\langle fT(\gamma), f'\rangle_M &= \frac{1}{|\Gamma_y|\mu_{\hat{k}}(\gamma_y)} \sum_{i=1}^s \langle f(g_i\gamma_y), f'(\gamma_y)\rangle \\ &= \frac{1}{|\Gamma_y|\mu_{\hat{k}}(\gamma_y)} \sum_{B(i)} \langle f(g_i\gamma_y), f'(\gamma_y)\rangle \\ &= \frac{1}{|\Gamma_y||\Gamma_x|\mu_{\hat{k}}(\gamma_y)} \sum_{B(i)} \sum_{h\in\Gamma_x} \langle f(g_i\gamma_y), f'(\gamma_y)\rangle\end{aligned} \tag{3.63}$$

and analogously

$$\begin{aligned}\mu_{\hat{k}}(\gamma)\langle f, f'T(K\gamma^{-1}K)\rangle_M &= \frac{\mu_{\hat{k}}(\gamma)}{|\Gamma_x|\mu_{\hat{k}}(\gamma_x)} \sum_{j=1}^{s'} \langle f(\gamma_x), f'(g_j'^{-1}\gamma_x)\rangle \\ &= \frac{\mu_{\hat{k}}(\gamma)}{|\Gamma_x|\mu_{\hat{k}}(\gamma_x)} \sum_{A(j)} \langle f(\gamma_x), f'(g_j'^{-1}\gamma_x)\rangle \\ &= \frac{\mu_{\hat{k}}(\gamma)}{|\Gamma_y||\Gamma_x|\mu_{\hat{k}}(\gamma_x)} \sum_{A(j)} \sum_{h\in\Gamma_y} \langle f(\gamma_x), f'(g_j'^{-1}\gamma_x)\rangle.\end{aligned} \tag{3.64}$$

We now fix, for $j \in \{1, ..., s'\}$ with $A(j)$, a $k_j \in K$ and $b_j \in \mathbb{G}(k)$ with $g_j'^{-1}\gamma_x = k_j\gamma_y b_j$. For $i \in \{1, ..., s\}$ with $B(i)$ we fix a $k_i' \in K$ and $a_i \in \mathbb{G}(k)$ with $g_i\gamma_y = k_i'\gamma_x a_i$. Note that then $g_j'^{-1}\gamma_x = k\gamma_y b$ implies $b = hb_j =: b_j(h), k = \gamma_y h\gamma_y^{-1}k_j =: k_j(h)$ for some $h \in \Gamma_y$, and $g_i\gamma_y = k'\gamma_x a$ implies $a = h'a_i =: a_i(h'), k' = \gamma_x h'\gamma_x^{-1}k_i' =: k_i'(h')$ for some $h' \in \Gamma_x$.

Now for $h \in \Gamma_y$ and $j \in \{1, ..., s\}$ with $A(j)$ we have $g_j'^{-1}\gamma_x = k_j(h)\gamma_y b_j(h)$ and hence $\gamma_x b_j(h)^{-1} = g_j' k_j(h)\gamma_y$. Due to the decomposition of $K\gamma K$ chosen above there is a unique $i \in \{1, ..., s\}$ and $k'^{-1} \in K$ such that $g_j' k_j(h) = k'^{-1}g_i$ and we have $k'\gamma_x b_j(h)^{-1} = g_i\gamma_y$. In particular $B(i)$ holds and hence $k' = k_i'(h'), b_j(h)^{-1} = a_i(h')$ for a unique $h' \in \Gamma_x$. This yields a bijection

$$\Gamma_y \times \{j \in \{1, .., s'\} \mid A(j)\} \to \Gamma_x \times \{i \in \{1, .., s\} \mid B(i)\}. \tag{3.65}$$

Furthermore if $(h, j)$ is mapped to $(h', i)$ under this bijection we have

$$\begin{aligned}
\frac{\mu_{\hat{k}}(\gamma)}{\mu_{\hat{k}}(\gamma_x)}\langle f(\gamma_x), f'(g_j'^{-1}\gamma_x)\rangle &= \frac{\mu_{\hat{k}}(\gamma)}{\mu_{\hat{k}}(g_j' k_j(h)\gamma_y b_j(h))}\langle f(\gamma_x), f'(k_j(h)\gamma_y b_j(h))\rangle \\
&= \frac{\mu_{\hat{k}}(\gamma)}{\mu_{\hat{k}}(g_j')\mu_{\hat{k}}(\gamma_y)\mu(b_j)}\langle f(\gamma_x), f'(k_j(h)\gamma_y b_j(h))\rangle \\
&= \frac{\mu_{\hat{k}}(\gamma)\mu(b_j)}{\mu_{\hat{k}}(\gamma)\mu_{\hat{k}}(\gamma_y)\mu(b_j)}\langle f(\gamma_x b_j(h)^{-1}), f'(\gamma_y)\rangle \\
&= \frac{1}{\mu_{\hat{k}}(\gamma_y)}\langle f(k_i'(h')\gamma_x a_i(h')), f'(\gamma_y)\rangle \\
&= \frac{1}{\mu_{\hat{k}}(\gamma_y)}\langle f(g_i\gamma_y), f'(\gamma_y)\rangle,
\end{aligned} \tag{3.66}$$

which, together with the first two computations, proves our assertion. □

**Corollary 3.3.2 ([Gro99, Prop(6.11)])** *The anti-involution $T \mapsto T'$, mapping each operator to its adjoint, of the endomorphism ring $\mathrm{End}_k(M)$ fixes the image of $H_K$ and $M$ is a semisimple $H_K$-module.*

*Proof.* The fact that the image of $H_K$ is stabilized by the involution is a consequence of the previous proposition. Now let $N \leq M$ be an $H_K$-submodule of $M$. We need to show that $N$ has an $H_K$-stable complement in $M$.

To that end note that $M = N \oplus N^\perp$ as a vector space and for $f \in N, f' \in N^\perp$ and $T$ in the image of $H_K$ we have

$$\langle f'T, f\rangle_M = \langle f', fT'\rangle_M = 0 \tag{3.67}$$

since $N$ is $H_K$-stable and hence $fT' \in N$. This holds for all $f$ and $f'$ hence $N^\perp$ is again $H_K$-stable and thus the desired complement. □

We conclude this section by presenting an algorithm for computing the action of a Hecke operator $T(\gamma)$ on the space $M(V, K)$ where $K$ is related to an integral form $\mathbb{G}_L$ via

$$K = \prod_{\mathfrak{p}} \mathbb{G}_L(\mathcal{O}_{\mathfrak{p}}) \tag{3.68}$$

and $\gamma$ is only supported at a single prime $\mathfrak{p}$. Since every coset $K\gamma' K$ with $\gamma' \in \mathbb{G}(\hat{k})$ has a representative that is only supported at finitely many primes this is no essential restriction.

**Algorithm 3.3.3** $\textsc{HeckeOperator}(L, \mathfrak{p}, \gamma, \rho)$

***Input:*** *An $\mathcal{O}_k$-lattice $L$ (in a faithful $\mathbb{G}$-module), a prime ideal $\mathfrak{p} \subset \mathcal{O}_k$, an element $\gamma \in \mathbb{G}(k_{\mathfrak{p}})$, and the representation $\rho : \mathbb{G} \to \mathrm{GL}(V)$.*

***Output:*** *Representatives $(L = L_1, ..., L_h)$ for the isomorphism classes in the genus of $L$, elements $\sigma_i \in \mathbb{G}(\hat{k})$ with $L_i = L\sigma_i$, a basis of the space $M(V, K)$ where $K = \prod_{\mathfrak{p}} \mathbb{G}_L(\mathcal{O}_{\mathfrak{p}})$ and a matrix describing the action of $T(\gamma)$[1] on $M(V, K)$ with respect to this basis.*

*Initialize* $\mathcal{G} \leftarrow [(L, 1_{\mathbb{G}(\hat{k})})]$, $\mathcal{B} \leftarrow [\mathrm{Basis}(V^{\rho(\mathrm{Stab}_{\mathbb{G}(k)}(L))})]$, *and* $\mathrm{Images} \leftarrow []$.
*Fix a system* $R$ *of representatives of* $K_{\mathfrak{p}} \backslash K_{\mathfrak{p}} \gamma K_{\mathfrak{p}}$.[2]
***for*** $(L', \mu) \in \mathcal{G}$ ***do***
    *Initialize* $\mathrm{I} \leftarrow [[0 \in V$ *for* $1 \le i \le |\mathrm{Bases}[j]|]]$ *for* $1 \le j \le |\mathcal{G}|]$.
    ***for*** $\sigma_{\mathfrak{p}} \in R$ ***do***
        *Construct* $L'' := L\sigma_{\mathfrak{p}}\mu$ *and set* $\mathrm{new} \leftarrow \mathrm{true}$.
        ***for*** $(N, \nu) \in \mathcal{G}$ ***do***
            ***if*** $N \cong_{\mathbb{G}(k)} L''$ ***then***
                $\mathrm{new} \leftarrow \mathrm{false}$.
                *Find* $g \in \mathbb{G}(k)$ *with* $Ng = L''$ *and set* $j$ *the position of* $(N, \nu)$ *in* $\mathcal{G}$.
                *For* $1 \le i \le |I[j]|$ *set* $I[j][i] \leftarrow I[j][i] + \mathcal{B}[j][i] \cdot \rho(g)$.
                *Break* $(N, \nu)$.
            ***end if***
        ***end for***
        ***if*** $\mathrm{new}$ ***then***
            *Append* $(L'', \sigma_{\mathfrak{p}}\mu)$ *to* $\mathcal{G}$, *append* $\mathrm{Basis}(V^{\rho(\mathrm{Stab}_{\mathbb{G}(k)}(L''))})$ *to* $\mathcal{B}$ *and* $I$.
            *Append* $|\mathrm{Basis}(V^{\rho(\mathrm{Stab}_{\mathbb{G}(k)}(L''))})|$ *many zeroes to all elements of* $\mathrm{Images}$.
        ***end if***
    ***end for***
    *Append* $I$ *to* $\mathrm{Images}$.
***end for***
***if*** $\sum_{(L', \mu) \in \mathcal{G}} |\mathrm{Stab}_{\mathbb{G}(k)}(L')|^{-1} \neq \mathrm{mass}(\mathrm{genus}(L))$ ***then***
    *Find a lattice* $L'' = L\sigma$ *(for some* $\sigma \in \mathbb{G}(\hat{k})$*) whose isomorphism class is not yet represented in* $\mathcal{G}$.

[1]We identify $\gamma \in \mathbb{G}(k_{\mathfrak{p}})$ with its image under the embedding into $\mathbb{G}(\hat{k})$.
[2]See Chapter 4 for details

24: *Append* $(L'', \sigma)$ *to* $\mathcal{G}$, *append* $\mathrm{Basis}(V^{\rho(\mathrm{Stab}_{\mathbb{G}(k)}(L''))})$ *to* $\mathcal{B}$.
25: *Append* $|\mathrm{Basis}(V^{\rho(\mathrm{Stab}_{\mathbb{G}(k)}(L''))})|$ *many zeroes to all elements of* Images.
26: *Continue with* $(L'', \sigma)$ *at 3.*
27: ***end if***
28: $h \leftarrow []$
29: ***for*** $1 \le i \le |\mathcal{G}|, 1 \le j \le |\mathcal{B}[j]|$ ***do***
30: *For* $1 \le k \le \mathcal{G}$ *write* $\mathrm{Images}[k][i][j]$ *as a linear combination of* $\mathcal{B}[k]$, *concatenate these basis representations and append the result to* $h$.
31: ***end for***
32: $T \leftarrow$ *the matrix whose rows are the elements of* $h$.
33: ***return*** $\mathcal{G}, \mathcal{B}, T$

### 3.3.1. Integrality of Hecke Operators

Consider the situation where $K = \prod_{\mathfrak{q}} K_{\mathfrak{q}}$ is a product of local factors (each of which is then open and compact in $\mathbb{G}(k_{\mathfrak{q}})$). Fix a prime $\mathfrak{p}$ and set $K' := \prod_{\mathfrak{q} \neq \mathfrak{p}} K_{\mathfrak{q}}$ which is an open and compact subgroup of $\prod_{\mathfrak{q} \neq \mathfrak{p}} \mathbb{G}(k_{\mathfrak{q}}) = \mathbb{G}(\hat{k}')$.

We define the space

$$M(V, K, \mathfrak{p}) := \left\{ f : K' \backslash \mathbb{G}(\hat{k}) / \mathbb{G}(k) \to V \otimes k_{\mathfrak{p}} \mid f(\kappa_{\mathfrak{p}} \gamma) = f(\gamma) \kappa_{\mathfrak{p}}^{-1} \right\}, \tag{3.69}$$

and consider the homomorphism

$$\begin{aligned} \phi : M(V, K) \otimes k_{\mathfrak{p}} \to M(V, K, \mathfrak{p}),\ f \mapsto \phi(f), \\ \text{where } \phi(f)(\gamma) = f(\gamma) \gamma_{\mathfrak{p}}^{-1}. \end{aligned} \tag{3.70}$$

Then $\phi$ is not only an isomorphism of $k_{\mathfrak{p}}$-vector spaces but is also compatible with the action of $H_{K'} \otimes k_{\mathfrak{p}}$, the Hecke algebra over $k_{\mathfrak{p}}$ with respect to $K'$ (which naturally embeds into $H_K \otimes k_{\mathfrak{p}}$). By virtue of this isomorphism we will identify $M(V, K) \otimes k_{\mathfrak{p}}$ with $M(V, K, \mathfrak{p})$.

Consider now any $\mathcal{O}_{\mathfrak{p}}$-lattice $L_{\mathfrak{p}} \subset V \otimes k_{\mathfrak{p}}$ that is stable under $K_{\mathfrak{p}}$. Such a lattice always exists since $K_{\mathfrak{p}}$ is compact. Then $L_{\mathfrak{p}}$ gives rise to an $\mathcal{O}_{\mathfrak{p}}$-lattice $M(L_{\mathfrak{p}}) \subset M(V, K, \mathfrak{p})$ containing precisely those maps in $M(V, K, \mathfrak{p})$ that take values only in $L_{\mathfrak{p}}$. By construction the space $M(L_{\mathfrak{p}})$ remains fixed under any $\mathcal{O}_{\mathfrak{p}}$-linear combinations of Hecke operators $T(\gamma)$ with $\gamma \in \mathbb{G}(\hat{k}') /\!/ K'$. For the action of the Hecke algebra this has the following consequence (cf. [Gro99, Prop. 8.9]):

**Corollary 3.3.4** *Let* $\gamma \in \mathbb{G}(\hat{k})$ *be supported only at the primes in the finite subset* $S$ *of the set of all (finite) primes of* $k$. *Then the eigenvalues of* $T(\gamma)$ *acting on the space* $M(V, K)$ *are* $S$*-integers, i.e. they are integral away from* $S$.

*Proof.* The operator $T(\gamma)$ fixes the lattice $M(L_{\mathfrak{q}})$ for all $\mathfrak{q} \notin S$. $\square$

### 3.3.2. Modular Forms of Trivial Weight

Consider the case where $V = k$ is the trivial representation and use the inner product $\langle v, v' \rangle = vv'$ (so $\mu \equiv 1$). Furthermore fix a system of representatives $\mathbb{G}(\hat{k}) = \bigsqcup_{i=1}^{t} K\lambda_i\mathbb{G}(k)$ and set $\Gamma_i := \mathbb{G}(k) \cap \lambda_i^{-1}K\lambda_i$. Now $M(V, K)$ is just the space of functions $f : \Sigma_K \to k$ and we may choose as a basis the functions $f_i$ with

$$f_i(\lambda_j) = \delta_{i,j}. \tag{3.71}$$

These are pairwise orthogonal with respect to $\langle -, - \rangle_M$ and fulfill

$$\langle f_i, f_i \rangle_M = \frac{1}{|\Gamma_i|}. \tag{3.72}$$

Set $e := \sum_{i=1}^{t} f_i$, then the following holds:

$$\langle e, e \rangle_M = \sum_{i=1}^{t} \frac{1}{|\Gamma_i|}, \tag{3.73}$$

which is just the mass of the genus of the lattice $L$ if $K = \prod_{\mathfrak{p}} K_{\mathfrak{p}}$ with $K_{\mathfrak{p}} = \mathbb{G}_L(\mathcal{O}_{\mathfrak{p}})$. The action of the Hecke algebra is easily seen to be

$$eT(\gamma) = e \cdot \operatorname{degree}(K\gamma K) \tag{3.74}$$

where $\operatorname{degree}(K\gamma K) = |K\backslash K\gamma K|$ denotes the number of right $K$-cosets in $K\gamma K$.

Hence $ke$ is a simple one-dimensional subspace of $M$ with $H_K$-stable complement

$$(ke)^{\perp} = \left\{ \sum_{i=1}^{t} a_i f_i \mid \sum_{i=1}^{t} \frac{a_i}{|\Gamma_i|} = 0 \right\}. \tag{3.75}$$

For future reference we want to write down the action of $H_K$ on $M$ as explicitly as possible.

**Lemma 3.3.5** *For $\gamma \in \mathbb{G}(\hat{k})$ let $K\gamma K = \bigsqcup_{j=1}^{s} K\gamma_j$. Then $T(\gamma)$ acts on $M(V, K)$ by the matrix $(\alpha_{i,j})$ with respect to the basis $\{f_i\}$ where*

$$\alpha_{i,j} = |\{1 \leq k \leq s \mid \gamma_k\lambda_j \in K\lambda_i\mathbb{G}(k)\}|. \tag{3.76}$$

*Proof.* We have $(fT(\gamma)) = \sum_{i=1}^{t} a_i f_i$ with $a_i = (fT(\gamma))(\lambda_j)$. Thus we compute

$$\begin{aligned} \alpha_{i,j} &= (f_iT(\gamma))(\lambda_j) = \sum_{k=1}^{s} f_i(\gamma_k\lambda_j) \\ &= \sum_{k=1}^{s} \delta(K\lambda_i\mathbb{G}(k), K\gamma_k\lambda_j\mathbb{G}(k)) = |\{1 \leq k \leq s \mid \gamma_k\lambda_j \in K\lambda_i\mathbb{G}(k)\}|. \end{aligned} \tag{3.77}$$

□

The matrices describing the action of the Hecke operators (in this basis) are sometimes called *Brandt matrices* since they generalize the classical Brandt matrices - which describe the action of certain Hecke operators in the case where $\mathbb{G}$ is the unit group of a definite quaternion algebra - to arbitrary algebraic modular forms.

If $K$ was obtained as the stabilizer of a lattice $L$ (so $K = \prod_{\mathfrak{p}} K_{\mathfrak{p}}$ with $K_{\mathfrak{p}} = \mathbb{G}_L(\mathcal{O}_{\mathfrak{p}})$) this reads as follows.

**Corollary 3.3.6** *Assume that $L$ is as above and set $L_i := L\lambda_i$, then $\{L_i \mid 1 \leq i \leq s\}$ is a system of representatives of the isomorphism classes in the genus of $L$ and in the notation from above we have*

$$\alpha_{i,j} = |\{1 \leq k \leq s \mid L\gamma_k\lambda_j \cong L_i\}| = |\{L' \in L\gamma K\lambda_j \mid L' \cong L_i\}|. \tag{3.78}$$

*Proof.* We have $L\gamma_k\lambda_j \cong L_i = L\lambda_i$ if and only if there is a $g \in \mathbb{G}(k)$ with $L\gamma_k\lambda_j = L\lambda_i g$ which means that $\gamma_k\lambda_j$ and $\lambda_i g$ differ only by an element of $K$ (since $K$ is the stabilizer of $L$) or equivalently $K\gamma_k\lambda_j = K\lambda_i g$, which implies $K\gamma_k\lambda_j\mathbb{G}(k) = K\lambda_i\mathbb{G}(k)$. On the other hand if $K\lambda_i\mathbb{G}(k) = K\gamma_k\lambda_j\mathbb{G}(k)$, then $L\gamma_k\lambda_j \cong L_i$, which proves the assertion. □

Since the representing matrices (with respect to the basis $\{f_i\}$) of the Hecke operators $T(\gamma)$, $\gamma \in \mathbb{G}(\hat{k})$, have integral entries the following is easily seen.

**Corollary 3.3.7** *The eigenvalues of the operators $T(\gamma)$, $\gamma \in \mathbb{G}(\hat{k})$, on $M(k, K)$ are algebraic integers.*

# 4. Coset Decompositions in Semisimple Groups over Local Fields

In this chapter we will consider the decomposition of double cosets in an algebraic group over a local field into right (or left) cosets. A lot of the results are taken from the articles [LP02] and [Lan01] but we will also formulate some further results which we will need in the following chapters. We will only prove results which are new or stated without proof in [Lan01]. Note that for technical reason we will restate the results in terms of right rather than left cosets. We revisit the notation of Subsection 3.2.3 (which was itself essentially identical to the notation in [LP02] and [Lan01]) and restate it verbatim for the reader's convenience.

Let $F$ be a local field of characteristic 0 with ring of integers $\mathcal{O}_F$, uniformizer $\pi$ and finite residue class field of order $q$ and characteristic $p$.
Furthermore let $\mathbb{G}$ be a connected, semisimple, linear algebraic group defined and split over $F$. There is a Chevalley group scheme $\underline{G}$ over $\mathcal{O}_F$ such that $K := \underline{G}(\mathcal{O}_F) \leq \underline{G}(F) = \mathbb{G}(F)$ is a hyperspecial maximal compact subgroup and such that the special fiber $\underline{G}_{\mathcal{O}_F/\pi\mathcal{O}_F}$ is again semisimple of the same type as $\mathbb{G}$ (cf. Theorem B.2.11).

Let $\underline{T} \leq \underline{G}$ be a split maximal torus scheme (whose generic fiber $\underline{T}_F$ we call $\mathbb{T}$). We set $\mathbb{N}_{\mathbb{T}}$ the normalizer of $\mathbb{T}$ in $\mathbb{G}$, i.e. $\mathbb{N}_{\mathbb{T}}(A) := N_{\mathbb{G}(A)}(\mathbb{T}(A))$ for all $A \in F\text{-}\mathsf{Alg}$. Furthermore let $X^*(\mathbb{T}) = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ and $X_*(\mathbb{G}_m, \mathbb{T})$ be the character and cocharacter module of $\mathbb{T}$, respectively. Let $\Phi \subset X^*(\mathbb{T})$ be the (finite) set of roots (i.e. the non-trivial weights occuring in the adjoint representation). We choose some positive subset $\Phi^+ \subset \Phi$ (or equivalently a Borel subgroup $\mathbb{T}(F) \subset B \subset \mathbb{G}(F)$) and denote by $\Delta$ the corresponding simple (or indecomposable) roots. Dually to this, let $\Phi^\vee \subset X_*(\mathbb{T})$ be the set of coroots and $\alpha \mapsto \alpha^\vee$ the usual bijective correspondence.

Given $\alpha \in \Phi$ we denote by $x_\alpha : \mathbb{G}_a \to \underline{U_\alpha}$ the isomorphism between $\mathbb{G}_a$ and the one-dimensional unipotent subgroup scheme $\underline{U_\alpha} \leq \underline{G}$ (whose generic fiber $(\underline{U_\alpha})_F$ we will call $\mathbb{U}_\alpha$). The morphism $x_\alpha$ when considered as a map $F \to \mathbb{U}_\alpha(F)$ restricts to $\mathcal{O}_F$ with $\underline{U_\alpha}(\mathcal{O}_F) = \mathbb{U}_\alpha(F) \cap K$.

The (finite) Weyl group of $\mathbb{G}$, defined as $\mathbb{N}_{\mathbb{T}}(F)/\mathbb{T}(F) = (\mathbb{N}_{\mathbb{T}}(F) \cap K)/\underline{T}(\mathcal{O}_F)$, will here be denoted by $W_0$ (to avoid ambiguity), while we use the symbol $\widetilde{W}$ for the extended affine Weyl group $\mathbb{N}_{\mathbb{T}}(F)/\underline{T}(\mathcal{O}_F)$. Then both $W_0$ and $\widetilde{W}$ act as groups of affine transformations on the vector space $X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $W_0$ is precisely the stabilizer of $0 \in X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ in $\widetilde{W}$. Furthermore there is an isomorphism

$$\widetilde{W} \cong X_*(\mathbb{T}) \rtimes W_0 \tag{4.1}$$

where we embed $X_*(\mathbb{T})$ into $\widetilde{W}$ as a normal subgroup of translations (acting in the obvious way on $X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$). In this sense set $t_\lambda = t(\lambda) \in \widetilde{W}$ the translation corresponding to $\lambda \in X_*(\mathbb{T})$ which yields the following identity:

$$w^{-1} t_\lambda w = t_{\lambda w}, \tag{4.2}$$

for $w \in W_0$ and $\lambda \in X_*(\mathbb{T})$.

We follow the description in Appendix B. The Weyl group $W_0$ is a finite Coxeter group with set of involutive generators $S_0 = \{w_\alpha \mid \alpha \in \Delta\}$, where $w_\alpha$ simply denotes the reflection through the vanishing hyperplane of the root $\alpha \in \Phi$. Now decompose $\Phi = \Phi_1 \cup ... \cup \Phi_m$ into irreducible root systems with corresponding simple systems $\Delta_i$, $1 \leq i \leq m$ (such that each $\Phi_i$ is the root system of an almost simple component of $\mathbb{G}$. If we put $\alpha_{0,i}$ the (unique) highest root of $\Phi_i$ (with respect to the simple system $\Delta_i$) we can form a larger Coxeter group with generators $\widetilde{S} := S_0 \cup \{t_{\alpha_{0,i}^\vee} w_{\alpha_{0,i}}\}$ which is isomorphic to the affine Weyl group $W_{af}$ associated to $\Phi$, which we will now think of as a subgroup of $\widetilde{W}$ via this isomorphism.

We make the canonical choice for an Iwahori subgroup of $G$ by letting $I$ equal the subgroup generated by $\underline{T}(\mathcal{O}_F)$, the groups $x_\alpha(\mathcal{O}_F) = \underline{U_\alpha}(\mathcal{O}_F)$ for $\alpha \in \Phi^+$ and the groups $x_\alpha(\pi\mathcal{O}_F)$ for $\alpha \in \Phi^- = -\Phi^+$. This is the canonical choice for $I$ since it is just the inverse image of the Borel subgroup associated to $\Phi^+$ under the reduction modulo $\pi$ from $K$ to the special fiber of $\underline{G}$ (cf. [Tit79]). Under these definitions the triple $(\mathbb{G}(F), I, \mathbb{N}_{\mathbb{T}}(F))$ is a generalized Tits system.

The group $\widetilde{W}$ is an extension of $W_{af}$ by a group $\Omega$ which one can find as follows: Put $\widetilde{I}$ the normalizer of $I$ in $\mathbb{G}(F)$ then

$$\Omega = (\mathbb{N}_{\mathbb{T}}(F) \cap \widetilde{I})/\underline{T}(\mathcal{O}_F) \subset \widetilde{W}. \tag{4.3}$$

The group $\Omega$ is finite Abelian and canonically isomorphic to $X_*(\mathbb{T})/\Lambda$, where $\Lambda \leq_{\mathbb{Z}} X_*(\mathbb{T})$ is the lattice generated by the coroots $\Phi^\vee$ (cf. Section B.3) (in particular $\Omega$ is trivial if $\mathbb{G}$ is simply connected). It normalizes $W_{af}$ and we get a split extension

$$\widetilde{W} \cong W_{af} \rtimes \Omega. \tag{4.4}$$

As usual we can consider the length function $w \mapsto \ell(w)$ on the Coxeter group $W_{af}$ (with respect to the generating system $\widetilde{S}$). This length function extends to $\widetilde{W}$ by setting $\ell(\rho w) := \ell(w\rho) := \ell(w)$ for $w \in W_{af}$ and $\rho \in \Omega$. In this sense we will call an expression $w = w_1 ... w_r \rho$ with $w_i \in \widetilde{S}$, $1 \leq i \leq r$, and $\rho \in \Omega$ reduced if $\ell(w) = r$.

## 4.1. Double Cosets for Coxeter Subgroups of the Affine Weyl Group

Let $W_1$ be a subgroup of $W_{af}$ with $W_1 = \langle S \rangle$ where $S = W_1 \cap \widetilde{S}$. Remember that such a subgroup is called a special (or standard parabolic) subgroup and is a Coxeter group in its own right where the length function of $W_1$ is just the restriction of the length function of $W_{af}$ to $W_1$. If $W_2 \leq W_1$ is another special subgroup (generated by $S' = W_2 \cap \widetilde{S}$) we set

$$[W_2 \backslash W_1] := \left\{ w \in W_1 \ : \ \ell(w'w) = \ell(w') + \ell(w) \text{ for all } w' \in W_2 \right\}. \tag{4.5}$$

Note that the elements of $[W_2 \backslash W_1]$ are just the representatives of $W_2 \backslash W_1$ of minimal length (cf. [Car89, §2.5]).

For the remainder of this section we will consider two special subgroups $W_1, W_2$ of $W_{af}$ with respective sets of generators $S_1$ and $S_2$ and intersection $W_{1,2} := W_1 \cap W_2$ (another special subgroup generated by $S_1 \cap S_2$). Furthermore define for $\sigma \in \widetilde{W}$ the group $W_2^{W_1\sigma}$ as $W_2 \cap \sigma^{-1} W_1 \sigma$ which is just the stabilizer of the coset $W_1\sigma$ in $W_2$. Finally we will choose a system $[W_1 \backslash \widetilde{W} / W_2] \subset \widetilde{W}$ of representatives for $W_1 \backslash \widetilde{W} / W_2$ of minimal length, i.e. each $\sigma \in [W_1 \backslash \widetilde{W} / W_2]$ is of minimal length in $W_1 \sigma W_2$ (a priori there is no reason to assume that these are unique).

One of the main results of [Lan01] is the fact that for $\sigma \in [W_1 \backslash \widetilde{W} / W_2]$, the group $W_2^{W_1\sigma}$ is a special subgroup of $W_2$. We will first state the series of assertions leading to this result and then formulate some additional applications of this theory.

The first statement is a slight generalization of the deletion condition for Coxeter groups.

**Proposition 4.1.1 ([Lan01, Prop. 4.2])** *Let $w \in \widetilde{W}$ with reduced expression $w = \rho s_1 ... s_r$ (so $\rho \in \Omega$, $s_i \in \widetilde{S}$ and $r = \ell(w)$). Then for all $s \in \widetilde{S}$ exactly one of the following holds:*

1. *$\ell(ws) = \ell(w) + 1$.*
2. *There is an $1 \leq i \leq r$ such that $w = \rho s_1 ... \hat{s}_i ... s_r s$ (in which case $\ell(ws) = \ell(w) - 1$).*

As a consequence we get the following lemma which is of fundamental importance for this section.

**Lemma 4.1.2 ([Lan01, Lemma 4.3])** *Let $w, w' \in \widetilde{W}$ with $\ell(w'w) = \ell(w') + \ell(w)$ and $s \in \widetilde{S}$ with $\ell(ws) = \ell(w) + 1$. Then exactly one of the following holds:*

*1.* $\ell(w'ws) = \ell(w'w) + 1$.

*2.* $w'w = \hat{w}'ws$ *for some* $\hat{w}' \in \widetilde{W}$ *with* $\ell(\hat{w}') < \ell(w')$.

*Furthermore if* $w' = \rho s_1 ... s_r$ *is a minimal expression then* $\hat{w}' = \rho s_1 ... \hat{s_i} ... s_r$ *for some* $1 \leq i \leq r$. *This implies that if* $w'$ *is an element of some special subgroup of* $\widetilde{W}$, *then* $\hat{w}'$ *is an element of that subgroup, too.*

**Proposition 4.1.3 ([Lan01, Prop. 4.5])** *For* $\sigma \in [W_1 \backslash \widetilde{W} / W_2]$ *the group*

$$W_2^{W_1 \sigma} = W_2 \cap W_1^{\sigma} \tag{4.6}$$

*is a special subgroup of* $W_2$.

In particular this means that the expression $[W_2^{W_1\sigma} \backslash W_2]$ is well-defined and we have the following length additivity property:

**Theorem 4.1.4 ([Lan01, Thm. 4.6])** *Fix two elements* $\sigma \in [W_1 \backslash \widetilde{W} / W_2]$ *and* $\tau \in [W_2^{W_1\sigma} \backslash W_2]$ *then*

$$\ell(w\sigma\tau) = \ell(w) + \ell(\sigma) + \ell(\tau) = \ell(w) + \ell(\sigma\tau) \tag{4.7}$$

*for all* $w \in W_1$.

Among other things we can deduce the following alternative description of $[W_2^{W_1\sigma} \backslash W_2]$.

**Corollary 4.1.5** *For* $\sigma \in [W_1 \backslash \widetilde{W} / W_2]$ *we have*

$$[W_2^{W_1\sigma} \backslash W_2] = \{\tau \in W_2 \mid \sigma\tau \textit{ is of shortest length in } W_1\sigma\tau\}. \tag{4.8}$$

*Proof.* Set

$$X := \{\tau \in W_2 \mid \sigma\tau \text{ is of shortest length in } W_1\sigma\tau\}. \tag{4.9}$$

Now for $\tau \in [W_2^{W_1\sigma} \backslash W_2]$ and $w \in W_1$ we have $\ell(w\sigma\tau) = \ell(w) + \ell(\sigma\tau) \geq \ell(\sigma\tau)$ by Theorem 4.1.4 so $[W_2^{W_1\sigma} \backslash W_2] \subset X$.

On the other hand write $\tau \in X$ as $w\tau'$ with $\tau' \in [W_2^{W_1\sigma} \backslash W_2]$ and $w \in W_2^{W_1\sigma}$. Then $W_1\sigma\tau = W_1\sigma w\tau' = W_1\sigma\tau'$ hence $w'\sigma\tau' = \sigma\tau$ for some $w' \in W_1$ which implies

$$\ell(\sigma\tau) = \ell(w'\sigma\tau') = \ell(w') + \ell(\sigma\tau') \geq \ell(\sigma\tau'). \tag{4.10}$$

But $\sigma\tau$ was of shortest length in $W_1\sigma\tau$ so $w' = 1$ and $\tau = \tau' \in [W_2^{W_1\sigma} \backslash W_2]$. □

**Corollary 4.1.6 ([Lan01, Cor. 4.7])** *Let* $\sigma \in [W_1 \backslash \widetilde{W} / W_2]$ *then* $\sigma$ *is the unique element of minimal length in* $W_1\sigma W_2$.

We now want to study what happens if we consider double cosets with respect to the intersection $W_{1,2}$ and how these correspond to the double cosets with respect to $W_1$ or $W_2$. Since we need the results in slightly larger generality let $W_2' = W_2\Omega_2$ with $\Omega_2 \leq \Omega$ fixing $S_2$. Then $W_{1,2} = W_1 \cap W_2 = W_1 \cap W_2'$.

**Lemma 4.1.7** *We have $\ell(\sigma w_1) = \ell(w_1\sigma) = \ell(w_1) + \ell(\sigma)$ for all $w_1 \in W_1$ and all $\sigma \in [W_{1,2}\backslash W_2'/W_{1,2}]$, where $[W_{1,2}\backslash W_2'/W_{1,2}] = [W_{1,2}\backslash \widetilde{W}/W_{1,2}] \cap W_2'$ corresponds to the double cosets with respect to $W_{1,2}$ that are contained in $W_2'$.*

*Proof.* We will show this by induction on $\ell(w_1)$ where the assertion is trivial for $\ell(w_1) = 0$. Let $w_1 = w_1's$ with $s \in S_1$. Then $\ell(\sigma w_1') = \ell(\sigma) + \ell(w_1')$ by induction and via Lemma 4.1.2 we have either $\ell(\sigma w_1's) = \ell(\sigma w_1') + 1$ (which we want to show) or $\sigma w_1' = \sigma' w_1's$ with $\ell(\sigma') < \ell(\sigma)$ and $\sigma' \in W_2'$. But in the latter case we have $\sigma = \sigma' w_1' s w_1'^{-1}$ and $w_1' s w_1'^{-1} = \sigma'^{-1}\sigma \in W_1 \cap W_2'$ which contradicts the assumption that $\sigma$ is of minimal length in its double coset with respect to $W_{1,2}$.

The other equality follows analogously. □

Using this length additivity we can show that the representatives of minimal length with respect to $W_{1,2}$ are already of minimal length with respect to the larger group $W_1$ as long as they are contained in $W_2'$.

**Lemma 4.1.8** *The following holds: $\sigma \in [W_{1,2}\backslash W_2'/W_{1,2}]$ already implies $\sigma \in [W_1\backslash \widetilde{W}/W_1]$.*

*Proof.* We need to show that $\ell(w_1\sigma w_1') \geq \ell(\sigma)$ for all $w_1, w_1' \in W_1$ and will do so by induction on $\min(\ell(w_1), \ell(w_1'))$. If this minimum is 0 the assertion follows from the length additivity property in Lemma 4.1.7 so let $\min(\ell(w_1), \ell(w_1')) > 0$ be realized (without loss of generality) at $w_1$. Then we can write $w_1 = s\widetilde{w_1}$ with $s \in S_1$ and $\ell(\widetilde{w_1}) = \ell(w_1) - 1$. Assume $\ell(w_1\sigma w_1') < \ell(\sigma)$ while (by induction) $\ell(\widetilde{w_1}\sigma w_1') \geq \ell(\sigma)$. We see

$$\ell(\sigma) > \ell(w_1\sigma w_1') = \ell(s\widetilde{w_1}\sigma w_1') \geq \ell(\widetilde{w_1}\sigma w_1') - 1 \geq \ell(\sigma) - 1 \tag{4.11}$$

and hence $\ell(\widetilde{w_1}\sigma w_1') = \ell(\sigma)$. Thus

$$\ell(\sigma) = \ell(\widetilde{w_1}\sigma w_1') \geq \ell(\sigma w_1') - \ell(\widetilde{w_1}) = \ell(\sigma) + \ell(w_1') - \ell(\widetilde{w_1}) \tag{4.12}$$

which implies $\ell(w_1') \leq \ell(\widetilde{w_1}) = \ell(w_1) - 1$ but we had $\ell(w_1') \geq \ell(w_1)$ which is a contradiction. Hence $\ell(w_1\sigma w_1') \geq \ell(\sigma)$ which completes the proof. □

Using this lemma and Theorem 4.1.4 we easily get the following result on the intersection of $W_1$ and $W_1^\sigma$ with $\sigma \in [W_{1,2}\backslash W_2'/W_{1,2}]$.

**Corollary 4.1.9** *In the same notation as above: For $\sigma \in [W_{1,2}\backslash W_2'/W_{1,2}]$ the group $W_1 \cap W_1^\sigma$ is a special subgroup of $W_1$.*

Not only are the elements of $[W_{1,2}\backslash W_2'/W_{1,2}]$ of shortest length in their $W_1$-$W_1$-double coset but no two of them define the same one.

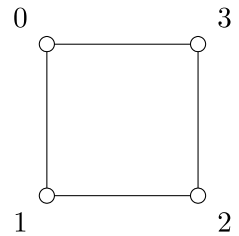
**Corollary 4.1.10** *For $\sigma, \sigma' \in [W_{1,2}\backslash W_2'/W_{1,2}]$ we have $W_1\sigma W_1 = W_1\sigma' W_1$ if and only if $\sigma = \sigma'$.*

*Proof.* This follows from Lemma 4.1.8 together with the fact, that the elements of $[W_1\backslash \widetilde{W}/W_1]$ are the unique elements of shortest length in their respective double coset by Lemma 4.1.6. $\square$

Note that it is not possible to generalize the last result to also include the case where $W_1$ contains a non-trivial subgroup of $\Omega$ as the following example shows.

**Example 4.1.11** *Consider the extended affine Weyl group with Dynkin diagram*

$\widetilde{A}_3$ :




*and finite group $\Omega = \langle \omega \rangle \cong C_2$ interchanging $s_0$ with $s_2$ and $s_1$ with $s_3$. We set $W_1' := \langle s_0, s_2, \omega \rangle$ and $W_2 = W_2' = \langle s_1, s_3 \rangle$. Then $W_1' \cap W_2' = \{1\}$ and hence $[W_{1,2}\backslash W_2/W_{1,2}] = W_2$, but $s_1$ and $s_3$ define the same $W_1' - W_1'-$coset since $s_3 = \omega s_1 \omega \in W_1' s_1 W_1'$.*

## 4.2. Double Coset Decompositions

Having a lot of information on the structure of $\widetilde{W}$ at hand we now come to decomposing double cosets of the form $P_1 w P_2$ into $P_1$-right cosets where $P_1$ and $P_2$ are two subgroups of $\mathbb{G}(F)$ containing the Iwahori subgroup $I$.

We will first restate some classical results on the structure of $\mathbb{G}(F)$ most of which are due to Iwahori and Matsumoto (cf. [IM65]). We will need some more notation and will denote the group $(\mathbb{N}_{\mathbb{T}}(F) \cap P)/\underline{T}(\mathcal{O}_F) \leq \widetilde{W}$ by $W_P$ for a subgroup $P \leq \mathbb{G}(F)$ containing $I$ (if $W_P$ is finite then $P$ is a parahoric subgroup). Furthermore set $\Omega_P := \Omega \cap W_P$, $S_P := \widetilde{S} \cap W_P$ and $W_P' = \langle S_P \rangle = W_P \cap W_{af}$. Then $W_P'$ is a special subgroup of $W_{af}$, $S_P$ is fixed by $\Omega_P$ under conjugation, and $W_P = \Omega_P W_P'$.

**Proposition 4.2.1 ([LP02, 2.6])** *Let $P_1$ and $P_2$ be two subgroups of $\mathbb{G}(F)$ containing $I$.*

1. $P_1 = \bigsqcup_{w \in W_{P_1}} IwI$.

2. *For $g, g' \in \widetilde{W}$ we have $P_1 g P_2 = P_1 g' P_2$ if and only if $W_{P_1} g W_{P_2} = W_{P_1} g' W_{P_2}$. Hence*

$$\mathbb{G}(F) = \bigsqcup_{\sigma \in [W_{P_1} \backslash \widetilde{W} / W_{P_2}]} P_1 \sigma P_2. \tag{4.13}$$

For $P_1 = \mathbb{G}(F)$ Proposition 4.2.1(1) is just the *affine Bruhat decomposition* (cf. Lemma 3.2.7).

The decomposition of the cosets $IsI$ with $s \in \widetilde{S}$ is a well known result which is once again due to Iwahori and Matsumoto (cf. [IM65, Cor. 2.7]).

**Proposition 4.2.2** *Let $\alpha \in \Delta$.*

- $Iw_\alpha I = \bigsqcup_{v \in R} I w_\alpha x_\alpha(v)$.
- $Iw_{\alpha_0} t(\alpha_0^\vee) I = \bigsqcup_{v \in R} I w_{\alpha_0} t(\alpha_0^\vee) x_{-\alpha_0}(\pi v)$

Knowing this it is fairly straight forward to give a decomposition for arbitrary double cosets with respect to $I$. To that end we fix for $s \in \widetilde{S}$ a lifting $\hat{s} \in \mathbb{G}(F)$ and define for $s$ and $v \in R$,

$$g_s(v) := \begin{cases} \hat{s} x_\alpha(v) & \text{if } s = w_\alpha \text{ for some } \alpha \in \Delta \\ \hat{s} x_{-\alpha_0}(\pi v) & \text{if } s = w_{\alpha_0} t(\alpha_0^\vee). \end{cases} \tag{4.14}$$

Restating the above proposition in this language yields $IsI = \bigsqcup_{v \in R} I g_s(v)$ for all $s \in \widetilde{S}$.

We extend our notation to all of $\widetilde{W}$ by choosing for $w \in \widetilde{W}$ a reduced expression $w = \rho s_{w,1} ... s_{w,\ell(w)}$ as a word in $\widetilde{S}$ and $\Omega$ and set for $v \in R^{\ell(w)}$

$$g_w(v) = \hat{\rho} g_{s_{w,1}}(v_1) ... g_{s_{w,\ell(w)}}(v_{\ell(w)}) \tag{4.15}$$

where $\hat{\rho}$ is an arbitrary lift of $\rho$ to $\mathbb{G}(F)$.

Using Proposition 3.2.8 one easily obtains the following result for the double cosets with respect to $I$:

**Corollary 4.2.3** *Let $w \in \widetilde{W}$.*

- $[IwI : I] = q^{\ell(w)}$.

- $IwI = \bigsqcup_{v \in R^{\ell(w)}} Ig_w(v)$.

Combining this with the Bruhat decomposition yields the following result for the indices of $I$ in parahoric subgroups:

**Corollary 4.2.4** *Let $P \geq I$ be a parahoric subgroup and set $\Omega_P = \Omega \cap W_P$ and $W'_P := W_{af} \cap W_P$. Then $W_P = \Omega_P W'_P = W'_P \Omega_P$ and $W'_P$ is a special subgroup of $W_{af}$. We compute:*

$$\begin{aligned}[P : I] &= \sum_{w \in W_P} [IwI : I] = \sum_{w \in W_P} q^{\ell(w)} \\ &= |\Omega_P| \cdot \sum_{w \in W'_P} q^{\ell(w)}\end{aligned} \tag{4.16}$$

Using this it is not hard to deduce the decomposition of parahoric subgroups with respect to other parahoric subgroups.

**Lemma 4.2.5** *Let $P_2 \leq P_1$ be two parahoric subgroups and let $\Xi$ be a system of representatives of $\Omega_{P_2} \backslash \Omega_{P_1}$. Then we have*

- $[P_1 : P_2] = [\Omega_{P_1} : \Omega_{P_2}] \cdot \sum_{w \in [W_{P'_2} \backslash W_{P'_1}]} q^{\ell(w)}$.

- $P_1 = \bigsqcup_{\rho \in \Xi} \bigsqcup_{w \in [W_{P'_2} \backslash W_{P'_1}]^\rho} \bigsqcup_{v \in R^{\ell(w)}} P_2 g_{\rho w}(v)$.

*Proof.* The index is easily computed as follows:

$$\begin{aligned}[P_1 : P_2] &= \frac{[P_1 : I]}{[P_2 : I]} \\ &= \frac{|\Omega_{P_1}| \cdot \sum_{w \in W'_{P_1}} q^{\ell(w)}}{|\Omega_{P_2}| \cdot \sum_{w' \in W'_{P_2}} q^{\ell(w')}} \\ &= [\Omega_{P_1} : \Omega_{P_2}] \frac{\sum_{w \in [W'_{P_2} \backslash W'_{P_1}]} \sum_{w' \in W'_{P_2}} q^{\ell(w'w)}}{\sum_{w' \in W'_{P_2}} q^{\ell(w')}} \\ &= [\Omega_{P_1} : \Omega_{P_2}] \frac{\sum_{w \in [W'_{P_2} \backslash W'_{P_1}]} \sum_{w' \in W'_{P_2}} q^{\ell(w')+\ell(w)}}{\sum_{w' \in W'_{P_2}} q^{\ell(w')}} \\ &= [\Omega_{P_1} : \Omega_{P_2}] \sum_{w \in [W'_{P_2} \backslash W'_{P_1}]} q^{\ell(w)}.\end{aligned} \tag{4.17}$$

It remains to prove that $P_1$ is in fact the union over the cosets given in the equation. This is enough since we have already computed the index and it coincides with the number of single cosets in the union.

To that end let $g \in P_1$, then $IgI = IwI$ for some $w \in W_{P_1}$. Write $w = \rho'\eta w'$ with $\rho' \in \Omega_{P_2}$, $\eta \in \Xi$ and $w' = w_2^\eta w_1^\eta$ with $w_2 \in W_2$ and $w_1 \in [W_{P_2'} \backslash W_{P_1'}]$. Then we compute

$$P_2 gI = P_2(IgI) = P_2 wI = P_2\rho'\eta w_2^\eta w_1^\eta I = P_2 w_2 \eta w_1^\eta I = P_2 \eta w_1^\eta I. \tag{4.18}$$

Furthermore

$$\begin{aligned} P_2 g \subset P_2 gI = P_2\eta w_1^\eta I &= P_2(I\eta w_1^\eta I) \\ &= P_2 \left( \bigcup_{v \in R^{\ell(w_1^\eta)}} I g_{\eta w_1^\eta}(v) \right) \\ &= \bigcup_{v \in R^{\ell(w_1^\eta)}} P_2 g_{\eta w_1^\eta}(v) \\ &\subset \bigcup_{\rho \in \Xi} \bigcup_{w \in [W_{P_2'} \backslash W_{P_1'}]^\rho} \bigcup_{v \in R^{\ell(w)}} P_2 g_{\rho w}(v), \end{aligned} \tag{4.19}$$

which shows that $P_2 g$ actually occurs in the union and thus proves the assertion. $\square$

We now come back to the original task of decomposing a coset of the form $P_1\sigma P_2$ into $P_1$-right cosets. To that end let $P_1$ and $P_2$ be two subgroups of $\mathbb{G}(F)$ containing the Iwahori subgroup $I$ and (following the ideas in [Lan01]) suppose for the time being that $W_{P_1}, W_{P_2} \subset W_{af}$. By Proposition 4.2.1 we have

$$\mathbb{G}(F) = \bigsqcup_{\sigma \in [W_1 \backslash \widetilde{W} / W_2]} P_1 \sigma P_2 \tag{4.20}$$

so it suffices to give a decomposition of $P_1\sigma P_2$ into $P_1$-right cosets for $\sigma \in [W_1 \backslash \widetilde{W} / W_2]$.

**Lemma 4.2.6 ([Lan01, Lemma 5.1])** *For $\sigma \in [W_1 \backslash \widetilde{W} / W_2]$ we have*

$$P_1 \sigma P_2 = \bigsqcup_{\tau \in [W_2^{W_1^\sigma} \backslash W_2]} P_1 \sigma \tau I. \tag{4.21}$$

**Theorem 4.2.7 ([Lan01, Thm. 5.2])** *For $\sigma \in [W_1 \backslash \widetilde{W} / W_2]$ we have*

$$P_1 \sigma P_2 = \bigsqcup_{\tau \in [W_2^{W_1^\sigma} \backslash W_2]} \bigsqcup_{v \in R^{\ell(\sigma\tau)}} P_1 g_{\sigma\tau}(v). \tag{4.22}$$

If we drop the condition that $W_{P_1}$ and $W_{P_2}$ are contained in $W_{af}$ the situation becomes considerably more complicated. So let from now on $P_1$ and $P_2$ be two arbitrary subgroups of $\mathbb{G}(F)$ containing $I$. Adding to our existing notation we denote by $P_i'$ the group $IW_{P_i}'I$ for $i = 1, 2$ and we choose a system

$[W_{P_1}\backslash\widetilde{W}/W_{P_2}]$ of representatives of $W_{P_1}\backslash\widetilde{W}/W_{P_2}$ of shortest lengths. Note that this is no longer unique.

Fix a $\sigma \in [W_{P_1}\backslash\widetilde{W}/W_{P_2}]$, denote by $\Omega^{\sigma}_{P_2}$ the stabilizer of $\sigma$ in $\Omega_{P_2}$, so $\Omega^{\sigma}_{P_2} = \{\rho \in \Omega_{P_2} \ : \ \sigma^{\rho} = \sigma\}$, set $\Omega^{\sigma}_{P_1,P_2} := \Omega_{P_1} \cap \Omega^{\sigma}_{P_2}$ and choose a system of representatives $J^{\sigma}_{P_1,P_2}$ for $\Omega^{\sigma}_{P_1,P_2}\backslash\Omega_{P_2}$.

**Lemma 4.2.8 ([Lan01, Lemma 5.6])** *The double coset $P_1\sigma P_2$ can be decomposed into $P_1 - I$-double cosets as follows:*

$$P_1\sigma P_2 = \bigsqcup_{\eta\in J^{\sigma}_{P_1,P_2}} \bigsqcup_{\tau\in[W_2'^{W_1'^{\sigma}}\backslash W_2']} P_1\sigma\tau\eta I. \tag{4.23}$$

**Lemma 4.2.9 ([Lan01, Thm. 5.7])** *The double coset $P_1\sigma P_2$ can be decomposed into $P_1$-right cosets as follows:*

$$P_1\sigma P_2 = \bigsqcup_{\eta\in J^{\sigma}_{P_1,P_2}} \bigsqcup_{\tau\in[W_2'^{W_1'^{\sigma}}\backslash W_2']} \bigsqcup_{v\in R^{\ell(\sigma\tau)}} P_1 g_{\sigma\tau\eta}(v). \tag{4.24}$$

# 5. A Variation on Eichler's Method

Eichler's method, which we introduced in Chapter 2, allowed us to compare the masses of two genera of integral forms of a given group $\mathbb{G}$ by considering their respective stabilizers in the adelic points of $\mathbb{G}$. Here we want to take this idea one step further and leverage the adjacency relation in the affine building to compute Hecke operators (in an often more efficient way than before).

We adopt the notation from Chapter 3: $\mathbb{G}$ is a semisimple linear algebraic group which is defined over the totally real number field $k$ and we again assume $\mathbb{G}(k_\infty)$ to be compact so that every arithmetic subgroup of $\mathbb{G}(k)$ is finite and the number of isomorphism classes in a given genus of integral forms of $\mathbb{G}$ is finite, too. Where necessary, we want to think of integral forms of $\mathbb{G}$ as lattices via a fixed (faithful) representation $\mathbb{G} \hookrightarrow \mathrm{GL}_n$ and we will write "$\cong_{\mathbb{G}(A)}$" for "isomorphic with respect to the $A$-rational points" (if this is not clear from the context), where $A$ is an arbitrary commutative $k$-algebra.

## 5.1. Intertwining Operators and Eichler Elements

Let us denote by $V$ an irreducible finite-dimensional representation of $\mathbb{G}$ defined over some finite extension of $k$ and fix two open and compact subgroups $K_1$ and $K_2$ of $\mathbb{G}(\hat{k})$. We want to consider the spaces of algebraic modular forms of weight $V$ and levels $K_1$ and $K_2$, respectively, $M_i := M(V, K_i),\ i = 1, 2$.

The group $K_1 \cap K_2$ is again open and compact, hence of finite index in both $K_1$ and $K_2$. We fix coset representatives $m_i \in K_2,\ i \in I$, and $l_j \in K_1,\ j \in J$, such that

$$\begin{aligned} K_1 &= \bigsqcup_{j \in J} (K_1 \cap K_2) l_j = \bigsqcup_{j \in J} l_j^{-1} (K_1 \cap K_2), \\ K_2 &= \bigsqcup_{i \in I} (K_1 \cap K_2) m_i = \bigsqcup_{i \in I} m_i^{-1} (K_1 \cap K_2). \end{aligned} \tag{5.1}$$

The following is a simple observation but puts an easy upper bound on the amount of computations we have to do.

**Remark 5.1.1** *Let* $\mathbb{G}(\hat{k}) = \bigsqcup_{s \in S} K_1 \gamma_s \mathbb{G}(k)$*, then*

$$\mathbb{G}(\hat{k}) = \bigcup_{s \in S, j \in J} K_2 l_j \gamma_s \mathbb{G}(k). \tag{5.2}$$

*Thus the set* $\{l_j \gamma_s \ : \ j \in J, s \in S\}$ *contains a system of representatives for* $K_2 \backslash \mathbb{G}(\hat{k}) / \mathbb{G}(k)$*.*

Remember that in the case where $K_1$ and $K_2$ are the stabilizers of two lattices $L$ and $M$, Eichler's method can be used to obtain the mass of the genus of $M$ from the mass of the genus of $L$ via

$$\text{mass}(\text{genus}(M)) = \text{mass}(\text{genus}(L)) \cdot \frac{[K_1 : K_1 \cap K_2]}{[K_2 : K_1 \cap K_2]}. \tag{5.3}$$

We want to take this idea one step further and try to employ the connection between $K_1$ and $K_2$ to compute the action of certain Hecke operators.

**Definition 5.1.2** *We define*

$$T_2^1 = T_{K_2}^{K_1} = T(K_1, K_2) : M_1 \to M_2, \ f \mapsto f', \tag{5.4}$$

*where*

$$f'(\gamma) = \sum_{i \in I} f(m_i \gamma) \text{ for all } \gamma \in \mathbb{G}(\hat{k}). \tag{5.5}$$

*We call* $T_2^1$ *the intertwining operator (with respect to* $K_1$ *and* $K_2$ *or from* $M_1$ *to* $M_2$*).*

$T_2^1$ is independent of the choice of the $m_i$ since $f$ is invariant under left-multiplication of the argument by elements of $K_1$ and hence, in particular, by elements of $K_1 \cap K_2$. Furthermore $T_2^1$ is well-defined since for $f \in M_1, \gamma \in \mathbb{G}(\hat{k})$, $g \in \mathbb{G}(k)$ and $u \in K_2$ we have

$$\begin{aligned}(fT_2^1)(u\gamma g) &= \sum_{i \in I} f(m_i u \gamma g) = \left( \sum_{i \in I} f(m_i u \gamma) \right) g \\ &= \left( \sum_{i \in I} f(m_i \gamma) \right) g = ((fT_2^1)(\gamma))g,\end{aligned} \tag{5.6}$$

where the second to last equality holds since $K_2$ acts on the cosets $(K_1 \cap K_2)m_i$, $i \in I$, and hence permutes the $m_i$ up to left-multiplication by elements of $K_1 \cap K_2 \subset K_1$ which do not alter the value of $f \in M(V, K_1)$.

**Remark 5.1.3** *Let us assume that* $K_1$ *and* $K_2$ *arise from integral forms* $\mathbb{G}_{L_1}$ *and* $\mathbb{G}_{L_2}$ *with* $L_2 \leq L_1$*, and that* $K_1$ *acts transitively on the elements of* genus$(L_2)$ *contained in* $L_1$*. This is the case for example, if* $L_1$ *and* $L_2$ *define vertices of a common chamber at all of the finitely many primes where they* *do not coincide. In this situation the operator* $T_1^2$ *(for* $V$ *the trivial representation and with respect to the standard bases) is simply the adjacency matrix of the weighted bipartite graph whose vertices are the isomorphism classes in* $\mathrm{genus}(L_1)$ *and* $\mathrm{genus}(L_2)$ *and whose arcs* $L_1' \to L_2'$ $(L_i' \in \mathrm{genus}(L_i), i = 1, 2)$ *have weight the number of sublattices of* $L_1'$ *isomorphic to* $L_2'$*.*

In the case where $K_1$ and $K_2$ come from integral forms $\mathbb{G}_{L_1}$ and $\mathbb{G}_{L_2}$, we describe an algorithm to compute the intertwining operator. Note that, while we need to compute representatives of the isomorphism classes in $\mathrm{genus}(L_1)$ beforehand, a system of representatives for the classes in $\mathrm{genus}(L_2)$ arises as a byproduct of this computation. Furthermore the applicability of this method depends on our ability to decompose $K_1$ into $(K_1 \cap K_2)$-right cosets, which we know how to do at least in the case where $K_{1,\mathfrak{p}} \cap K_{2,\mathfrak{p}}$ contains an Iwahori subgroup at all of the finitely many primes $\mathfrak{p}$ at which $K_{1,\mathfrak{p}} \neq K_{2,\mathfrak{p}}$ (see Lemma 4.2.5).

**Algorithm 5.1.4** $\textsc{IntertwiningOperator}(L_1, L_2, \rho)$

***Input:*** *Two* $\mathcal{O}_k$*-lattices* $L_1, L_2$ *(in possibly distinct* $\mathbb{G}$*-modules affording faithful representations) and the representation* $\rho : \mathbb{G} \to \mathrm{GL}(V)$*.*

***Output:*** *Representatives* $(L_i = L_i^{(1)}, ..., L_i^{(h_i)})$ *for the isomorphism classes in the genus of* $L_i$*,* $i = 1, 2$*, elements* $\sigma_i^{(j)} \in \mathbb{G}(\hat{k})$ *with* $L_i^{(j)} = L_i \sigma_i^{(j)}$*, bases of the spaces* $M(V, K_i)$ *where* $K_i = \prod_{\mathfrak{p}} \mathbb{G}_{L_i}(\mathcal{O}_{\mathfrak{p}})$ *and a matrix describing the homomorphism* $T(K_2, K_1) : M(V, K_2) \to M(V, K_1)$ *with respect to these bases.*

*Initialize* $\mathcal{G}_1 \leftarrow \textsc{ComputeGenus}(L_1)$
$\mathcal{B}_1 \leftarrow [\mathrm{Basis}(V^{\rho(\mathrm{Stab}_{\mathbb{G}(k)}(L_1^{(j)}))})$ *for* $1 \leq j \leq h_1]$.
$\mathcal{G}_2 \leftarrow [(L_2, 1_{\mathbb{G}(\hat{k})})]$, $\mathcal{B}_2 \leftarrow [\mathrm{Basis}(V^{\rho(\mathrm{Stab}_{\mathbb{G}(k)}(L_2))})]$.
$\mathrm{Images} \leftarrow []$.
*Fix a system* $R$ *of representatives of* $(K_1 \cap K_2) \backslash K_1$.
***for*** $(L', \mu) \in \mathcal{G}_1$ ***do***
    *Initialize* $\mathrm{I} \leftarrow [[0 \in V$ *for* $1 \leq i \leq |\mathrm{Bases2}[j]|]$ *for* $1 \leq j \leq |\mathcal{G}_2|]$.
    ***for*** $\sigma \in R$ ***do***
        *Construct* $N := L_2 \sigma \mu$ *and set* $\mathrm{new} \leftarrow \mathrm{true}$.
        ***for*** $(N', \nu) \in \mathcal{G}_2$ ***do***
            ***if*** $N \cong_{\mathbb{G}(k)} N'$ ***then***
                $\mathrm{new} \leftarrow \mathrm{false}$.
                *Find* $g \in \mathbb{G}(k)$ *with* $N'g = N$.
                $j \leftarrow$ *The position of* $(N', \nu)$ *in* $\mathcal{G}_2$.
                *For* $1 \leq i \leq |I[j]|$ *replace* $I[j][i] \leftarrow I[j][i] + \mathcal{B}_2[j][i] \cdot \rho(g)$.
                *Break* $(N', \nu)$.
            ***end if***
        ***end for***
        ***if*** new ***then***
            *Append* $(N, \sigma\mu)$ *to* $\mathcal{G}_2$*, append* $\mathrm{Basis}(V^{\rho(\mathrm{Stab}_{\mathbb{G}(k)}(N))})$ *to* $\mathcal{B}_2$ *and* $I$*.*
            *Append* $|\mathrm{Basis}(V^{\rho(\mathrm{Stab}_{\mathbb{G}(k)}(N))})|$ *many zeroes to all elements of* Images.

**end if**
**end for**
*Append* $I$ *to* Images.
**end for**
$T \leftarrow []$
**for** $1 \leq i \leq |\mathcal{G}_2|, 1 \leq j \leq |\mathcal{B}_2[j]|$ **do**
*For* $1 \leq k \leq \mathcal{G}_1$ *write* Images$[k][i][j]$ *as a linear combination of* $\mathcal{B}_1[k]$*, concatenate these basis representations and append the result to* $T$*.*
**end for**
$T_1^2 \leftarrow$ *the matrix whose rows are the elements of* $T$*.*
**return** $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_2, T_1^2$

The operators $T_2^1$ and $T_1^2 = T_{K_1}^{K_2}$ are connected in a way that becomes apparent if we consider the spaces $M_1$ and $M_2$ endowed with their inner products defined in Equation 3.9.

**Theorem 5.1.5** *The operators $T_2^1$ and $T_1^2$ are adjoint to each other. That is, for all $f \in M_1$ and $f' \in M_2$ we have*

$$\langle f T_2^1, f' \rangle_{M_2} = \langle f, f' T_1^2 \rangle_{M_1}. \tag{5.7}$$

*Proof.* The proof resembles the one for the adjoint Hecke operator. We will first introduce some notation. Let

$$\begin{aligned} \mathbb{G}(\hat{k}) &= \bigsqcup_{s \in S} K_1 \gamma_s \mathbb{G}(k) \\ &= \bigsqcup_{r \in R} K_2 \delta_r \mathbb{G}(k) \end{aligned} \tag{5.8}$$

and set

$$\Delta_r := \mathbb{G}(k) \cap \delta_r^{-1} K_2 \delta_r, \ \Gamma_s := \mathbb{G}(k) \cap \gamma_s^{-1} K_1 \gamma_s \text{ for } r \in R \text{ and } s \in S. \tag{5.9}$$

Since we need to check the equality in our claim only on bases of $M_1$ and $M_2$, respectively, we may assume that $f$ is only supported on $K_1 \gamma_{s_0} \mathbb{G}(k)$ for some $s_0 \in S$ and $f'$ is only supported on $K_2 \delta_{r_0} \mathbb{G}(k)$ for some $r_0 \in R$.

We compute

$$\begin{aligned}
\langle fT_2^1, f'\rangle &= \sum_{r\in R} \frac{1}{|\Delta_r|\cdot\mu_{\mathbb{A}}(\delta_r)}\langle (fT_2^1)(\delta_r), f'(\delta_r)\rangle \\
&= \frac{1}{|\Delta_{r_0}|\cdot\mu_{\mathbb{A}}(\delta_{r_0})}\langle (fT_2^1)(\delta_{r_0}), f'(\delta_{r_0})\rangle \\
&= \frac{1}{|\Delta_{r_0}|\cdot\mu_{\mathbb{A}}(\delta_{r_0})}\sum_{i\in I}\langle f(m_i\delta_{r_0}), f'(\delta_{r_0})\rangle \\
&= \frac{1}{|\Delta_{r_0}|\cdot\mu_{\mathbb{A}}(\delta_{r_0})}\sum_{A(i)}\langle f(m_i\delta_{r_0}), f'(\delta_{r_0})\rangle \\
&= \frac{1}{|\Gamma_{s_0}|\cdot|\Delta_{r_0}|\cdot\mu_{\mathbb{A}}(\delta_{r_0})}\sum_{x\in\Gamma_{s_0}}\sum_{A(i)}\langle f(m_i\delta_{r_0}), f'(\delta_{r_0})\rangle,
\end{aligned} \tag{5.10}$$

where $A(i)$ denotes the property $K_1 m_i\delta_{r_0}\mathbb{G}(k) = K_1\gamma_{s_0}\mathbb{G}(k)$ and analogously

$$\begin{aligned}
\langle f, f'T_1^2\rangle &= \frac{1}{|\Gamma_{s_0}|\cdot\mu_{\mathbb{A}}(\gamma_{s_0})}\sum_{B(j)}\langle f(\gamma_{s_0}), f'(l_j\gamma_{s_0})\rangle \\
&= \frac{1}{|\Delta_{r_0}|\cdot|\Gamma_{s_0}|\cdot\mu_{\mathbb{A}}(\gamma_{s_0})}\sum_{y\in\Delta_{r_0}}\sum_{B(j)}\langle f(\gamma_{s_0}), f'(l_j\gamma_{s_0}),
\end{aligned} \tag{5.11}$$

where $B(j)$ denotes the property $K_2 l_j\gamma_{s_0}\mathbb{G}(k) = K_2\delta_{r_0}\mathbb{G}(k)$.

Now for $i \in I$ with $A(i)$ we fix a $\kappa_i^{(1)} = \kappa_i^{(1)}(1) \in K_1$ and a $g_i = g_i(1) \in \mathbb{G}(k)$ with $m_i\delta_{r_0} = \kappa_i^{(1)}\gamma_{s_0}g_i$. Analogously for $j \in J$ with $B(j)$ we fix a $\kappa_j^{(2)} = \kappa_j^{(2)}(1) \in K_2$ and a $h_j = h_j(1) \in \mathbb{G}(k)$ with $l_j\gamma_{s_0} = \kappa_j^{(2)}\delta_{r_0}h_j$. We note that $m_i\delta_{r_0} = \kappa^{(1)}\gamma_{s_0}g = \kappa_i^{(1)}\gamma_{s_0}g_i$ for some $\kappa^{(1)} \in K_1$, $g \in \mathbb{G}(k)$ implies $x := \gamma_{s_0}^{-1}(\kappa^{(1)})^{-1}\kappa^{(1)}\gamma_{s_0} = gg_i^{-1} \in \mathbb{G}(k) \cap \gamma_{s_0}^{-1}K_1\gamma_{s_0} = \Gamma_{s_0}$ and $g = xg_i =: g_i(x)$ as well as $\kappa^{(1)} = \kappa_i^{(1)}\gamma_{s_0}x^{-1}\gamma_{s_0}^{-1} =: \kappa_i^{(1)}(x)$. After an analogous observation regarding the property $B(j)$ we define for $y \in \Delta_{r_0}$ the elements $h_j(y) := yh_j$ and $\kappa_j^{(2)}(y) := \kappa_j^{(2)}\delta_{r_0}y^{-1}\delta_{r_0}^{-1}$.

Now let $i \in I$ with $A(i)$ and $x \in \Gamma_{s_0}$. Then we have $m_i\delta_{r_0} = \kappa_i^{(1)}(x)\gamma_{s_0}g_i(x)$, hence $m_i\delta_{r_0}g_i(x)^{-1} = \kappa_i^{(1)}(x)\gamma_{s_0}$ and there is a unique $j \in J$ and $\sigma \in K_1 \cap K_2$ with $\kappa_i^{(1)}(x) = \sigma l_j$. Since

$$\underbrace{\sigma^{-1}m_i}_{\in K_2}\,\delta_{r_0}\,\underbrace{g_i(x)^{-1}}_{\in\mathbb{G}(k)} = l_j\gamma_{s_0} \tag{5.12}$$

the property $B(j)$ is fulfilled and there is a unique $y' \in \Delta_{r_0}$ such that $\sigma^{-1}m_i = \kappa_j^{(2)}(y')$ and $g_i(x)^{-1} = h_j(y')$. This observation yields a bijection between

$$\{i \in I \mid A(i)\} \times \Gamma_{s_0} \to \{j \in J \mid B(j)\} \times \Delta_{r_0}. \tag{5.13}$$

Furthermore if $(i,x)$ is mapped to $(j,y)$ under this bijection we compute

$$\begin{aligned}
\frac{1}{\mu_{\mathbb{A}}(\delta_{r_0})}\langle f(m_i\delta_{r_0}), f'(\delta_{r_0})\rangle &= \frac{1}{\mu_{\mathbb{A}}(m_i^{-1}\kappa_i^{(1)}(x)\gamma_{s_0}g_i(x))}\langle f(\kappa_i^{(1)}(x)\gamma_{s_0}g_i(x)), f'(\delta_{r_0})\rangle \\
&= \frac{1}{\mu_{\mathbb{A}}(\gamma_{s_0})\mu(g_i(x))}\langle f(\gamma_{s_0})g_i(x), f'(\delta_{r_0})\rangle \\
&= \frac{\mu(g_i(x))}{\mu_{\mathbb{A}}(\gamma_{s_0})\mu(g_i(x))}\langle f(\gamma_{s_0}), f'(\delta_{r_0})g_i(x)^{-1}\rangle \\
&= \frac{1}{\mu_{\mathbb{A}}(\gamma_{s_0})}\langle f(\gamma_{s_0}), f'(\delta_{r_0}g_i(x)^{-1})\rangle \\
&= \frac{1}{\mu_{\mathbb{A}}(\gamma_{s_0})}\langle f(\gamma_{s_0}), f'(\kappa_j^{(2)}(y)\delta_{r_0}h_j(y))\rangle \\
&= \frac{1}{\mu_{\mathbb{A}}(\gamma_{s_0})}\langle f(\gamma_{s_0}), f'(l_j\gamma_{s_0})\rangle.
\end{aligned} \tag{5.14}$$

Summing these terms up over all $i \in I$ with $A(i)$ and $x \in \Gamma_{s_0}$ together with our first computation yields the assertion. □

Since the adjoint operator of $T$ is uniquely determined by $T$, the above theorem shows that in applications it suffices to compute just one of the two. This observation is particularly useful in light of the next two results.

**Lemma 5.1.6** *The function* $\sum_{i\in I, j\in J}\mathbb{1}_{K_1 m_i l_j}$ *is an element of* $H_{K_1}$.

*Proof.* The function is obviously compactly supported and $K_1$-left invariant. Now let $\kappa \in K_1$, then $\kappa^{-1}$ induces a bijection $\sigma \in \mathrm{Sym}(J)$ such that there are elements $\beta_j \in K_1 \cap K_2$ with $l_j\kappa^{-1} = \beta_j l_{\sigma(j)}$ since the $l_j$ form a complete system of representatives of $(K_1 \cap K_2)\backslash K_1$. In turn every $\beta_j$ induces a permutation $\tau_j \in \mathrm{Sym}(I)$ such that there are elements $\epsilon_{i,j} \in K_1 \cap K_2$ with $m_i\beta_j = \epsilon_{i,j}m_{\tau_j(i)}$ by the same argument. Hence

$$\begin{aligned}
\sum_{i\in I, j\in J}\mathbb{1}_{K_1 m_i l_j}(\gamma\kappa) &= \sum_{i\in I, j\in J}\mathbb{1}_{K_1 m_i l_j\kappa^{-1}}(\gamma) \\
&= \sum_{i\in I, j\in J}\mathbb{1}_{K_1 m_i \beta_j l_{\sigma(j)}}(\gamma) \\
&= \sum_{i\in I, j\in J}\mathbb{1}_{K_1 \epsilon_{i,j} m_{\tau_j(i)} l_{\sigma(j)}}(\gamma) \\
&= \sum_{i\in I, j\in J}\mathbb{1}_{K_1 m_{\tau_j(i)} l_{\sigma(j)}}(\gamma) \\
&= \sum_{i\in I, j\in J}\mathbb{1}_{K_1 m_i l_j}(\gamma),
\end{aligned} \tag{5.15}$$

which implies $K_1$-right invariance. □

**Definition 5.1.7** *We will call the function*

$$\nu_{1,2} := \nu(K_1, K_2) := \sum_{i \in I, j \in J} \mathbb{1}_{K_1 m_i l_j} \in H_{K_1} \tag{5.16}$$

*the Eichler element (of $H_{K_1}$) with respect to $K_2$.*

Since $(fT_2^1 T_1^2)(\gamma) = \sum_{i \in I, j \in J} f(m_i l_j \gamma)$ for all $f \in M_1$ and $\gamma \in \mathbb{G}(\hat{k})$, the above lemma immediately implies the following corollary.

**Corollary 5.1.8** *The linear operator $T_2^1 T_1^2 : M_1 \to M_1$ is self-adjoint and acts as the element $\nu_{1,2}$ of the Hecke algebra $H_{K_1}$ on the space $M_1$.*

Since $K_1 m_i l_j \subset K_1 m_i K_1$ and $m_i \in K_2$, the Eichler element $\nu_{1,2}$ is only supported on the double cosets $K_1 m K_1$ with $m \in K_2$. On the other hand write $m \in K_2$ as $\kappa m_i$ for some $i \in I$ and $\kappa \in K_1 \cap K_2$, then $K_1 m K_1 = K_1 m_i K_1$, which shows that each of these cosets actually appears in the support of $\nu_{1,2}$. Let us denote (by slight abuse of notation) the set of double cosets in $\mathbb{G}(\hat{k}) /\!\!/ K_1$ which have a representative in $K_2$ by $K_2 /\!\!/ K_1$. Then we have shown that

$$\nu_{1,2} = \sum_{i \in I, j \in J} \mathbb{1}_{K_1 m_i l_j} = \sum_{K_1 \kappa K_1 \in K_2 /\!\!/ K_1} \nu_{1,2}(\kappa) \mathbb{1}_{K_1 \kappa K_1} \tag{5.17}$$

and furthermore

$$\begin{aligned} \nu_{1,2}(\kappa) &= |\{(i,j) \in I \times J \ : \ \kappa \in K_1 m_i l_j\}| \\ &= \frac{|\{(i,j) \in I \times J \ : \ K_1 m_i l_j \subset K_1 \kappa K_1\}|}{|K_1 \backslash K_1 \kappa K_1|} \\ &= \frac{|J| \cdot |\{i \in I \ : \ K_1 m_i \subset K_1 \kappa K_1\}|}{|K_1 \backslash K_1 \kappa K_1|} \\ &= \frac{[K_1 : K_1 \cap K_2] \cdot |\{i \in I \ : \ K_1 m_i \subset K_1 \kappa K_1\}|}{|K_1 \backslash K_1 \kappa K_1|} \end{aligned} \tag{5.18}$$

If the intersection of $K_1$ and $K_2$ is small, $\nu_{1,2}$ will have very large support and hence will in general not be of particular interest. However, if $K_1 \cap K_2$ has (in some sense) small index in both $K_1$ and $K_2$ the operator $\nu_{1,2}$ will be a linear combination of only a few elements of the standard basis of $H_{K_1}$, and thus $T(\nu_{1,2})$ might yield some information on the action of $H_{K_1}$ on $M_1$.

Let us now assume that $K_i = \prod_{\mathfrak{q}} K_{i,\mathfrak{q}}$, $i = 1, 2$, are both products of local factors $K_{i,\mathfrak{q}}$, where $\mathfrak{q}$ runs over the finite places of $k$ (this is the case for example when $K_1$ and $K_2$ arise as stabilizers of lattices). Furthermore assume that there is a finite place $\mathfrak{p}$ of $k$ such that $\mathbb{G}$ is split at $\mathfrak{p}$ and $K_{1,\mathfrak{p}}$ and $K_{2,\mathfrak{p}}$ are two parahoric subgroups of $\mathbb{G}(k_{\mathfrak{p}})$ containing a common Iwahori subgroup $I$ while $K_{1,\mathfrak{q}} = K_{2,\mathfrak{q}}$

for all $\mathfrak{q} \neq \mathfrak{p}$. Let $\widetilde{W}$ be the extended affine Weyl group of $\mathbb{G}(k_{\mathfrak{p}})$ (with respect to a suitable torus whose integral points are contained in $I$) and $W_1, W_2 \leq \widetilde{W}$ the subgroups corresponding to $K_{1,\mathfrak{p}}$ and $K_{2,\mathfrak{p}}$ via

$$K_{i,\mathfrak{p}} = \bigsqcup_{w \in W_i} IwI, \; i = 1, 2. \tag{5.19}$$

Furthermore set $W_{1,2} := W_1 \cap W_2$.

If $W_1 \subset W_{af}$ is a special subgroup of $W_{af}$ we have

$$K_2 /\!\!/ K_1 = \{K_1 \sigma K_1 \mid \sigma \in [W_{1,2} \backslash W_2 / W_{1,2}]\} \tag{5.20}$$

(where we embed the $\sigma \in [W_{1,2} \backslash W_2 / W_{1,2}]$ into $\mathbb{G}(\hat{k})$ in the usual way) and these cosets are pairwise distinct by Corollary 4.1.10. In this case we can give an explicit formula for the values $\nu_{1,2}(\kappa)$ from above.

**Theorem 5.1.9** *If $W_1 \subset W_{af}$ the following holds:*

$$\nu_{1,2} = \sum_{i \in I, j \in J} \mathbb{1}_{K_1 m_i l_j} = \sum_{\kappa \in [W_{1,2} \backslash W_2 / W_{1,2}]} [IW_1^{W_1 \kappa} I : I(W_1^{W_1 \kappa} \cap W_2) I] \mathbb{1}_{K_1 \kappa K_1}. \tag{5.21}$$

*Proof.* First note that $W_1^{W_1 \kappa}$ is a special subgroup of $W_1$ by Corollary 4.1.9, hence $IW_1^{W_1 \kappa} I$ and $I(W_1^{W_1 \kappa} \cap W_2) I$ are in fact subgroups of $\mathbb{G}(k_{\mathfrak{p}})$ for $\kappa \in [W_{1,2} \backslash W_2 / W_{1,2}]$. Furthermore, since $K_1$ and $K_2$ coincide away from $\mathfrak{p}$, we can perform our computation locally and we set $P_i := K_{i,\mathfrak{p}}, \; i = 1, 2$.

Now let $\kappa \in [W_{1,2} \backslash W_2 / W_{1,2}]$, then we need to show that $\nu_{1,2}(\kappa) = [IW_1^{W_1 \kappa} I : I(W_1^{W_1 \kappa} \cap W_2) I]$. We already noticed that

$$\nu_{1,2}(\kappa) = \frac{[P_1 : P_1 \cap P_2] \cdot |\{i \in I \; : \; P_1 m_i \subset P_1 \kappa P_1\}|}{|P_1 \backslash P_1 \kappa P_1|}. \tag{5.22}$$

Note that the set in the numerator makes sense since we can assume that the $m_i$ are only supported at $\mathfrak{p}$.

We will now make the terms appearing in this description more explicit.

First of all we have

$$[P_1 : P_1 \cap P_2] = [IW_1 I : I(W_1 \cap W_2) I] = \frac{[IW_1 I : I]}{[IW_{1,2} I : I]}. \tag{5.23}$$

By Lemma 4.1.8 we see that $\kappa$ is also of shortest length in the double coset $W_1 \kappa W_1$, hence by Theorem 4.2.7 and Corollary 4.2.4

$$|P_1 \backslash P_1 \kappa P_1| = q^{\ell(\kappa)} \cdot \sum_{w \in [W_1^{W_1 \kappa} \backslash W_1]} q^{\ell(w)} = q^{\ell(\kappa)} [IW_1 I : IW_1^{W_1 \kappa} I], \tag{5.24}$$

where $q$ is the order of the residue class field at $\mathfrak{p}$.

Now consider the set $A := \{i \in I \ : \ P_1 m_i \subset P_1 \kappa P_1\}$ whose cardinality gives the last term in the above expression. Let $x \in P_2$, then $P_1 x \subset P_1 \kappa P_1$ if and only if $P_1 x P_1 = P_1 \kappa P_1$ and by Corollary 4.1.10 this is the case if and only if already $P_{1,2} x P_{1,2} = P_{1,2} \kappa P_{1,2}$. In particular this means that $i \in A$ if and only if $P_{1,2} m_i \subset P_{1,2} \kappa P_{1,2}$ which implies

$$|A| = |P_{1,2} \backslash P_{1,2} \kappa P_{1,2}| = q^{\ell(\kappa)} \sum_{w' \in [W_{1,2}^{W_{1,2}\kappa} \backslash W_{1,2}]} q^{\ell(w')} = q^{\ell(\kappa)} [IW_{1,2}I : IW_{1,2}^{W_{1,2}\kappa} I]. \tag{5.25}$$

Note that

$$W_{1,2}^{W_{1,2}\kappa} = (W_1 \cap W_2) \cap (W_1 \cap W_2)^\kappa = W_{1,2} \cap W_1^\kappa \tag{5.26}$$

since $\kappa \in W_2$ and hence $W_2^\kappa = W_2$.

Now we put all of this together and see

$$\begin{aligned}
\nu_{1,2}(\kappa) &= \frac{[P_1 : P_1 \cap P_2] \cdot |\{i \in I \ : \ P_1 m_i \subset P_1 \kappa P_1\}|}{|P_1 \backslash P_1 \kappa P_1|} \\
&= \frac{[IW_1 I : I] \cdot |P_{1,2} \backslash P_{1,2} \kappa P_{1,2}| \cdot [IW_1^{W_1\kappa} I : I]}{q^{\ell(\kappa)} [IW_1 I : I] \cdot [IW_{1,2} I : I]} \\
&= \frac{q^{\ell(\kappa)} [IW_{1,2} I : IW_{1,2}^{W_{1,2}\kappa} I] \cdot [IW_1^{W_1\kappa} I : I]}{q^{\ell(\kappa)} \cdot [IW_{1,2} I : I]} \\
&= \frac{[IW_{1,2} I : I] \cdot [IW_1^{W_1\kappa} I : I]}{[IW_{1,2}^{W_{1,2}\kappa} I : I] \cdot [IW_{1,2} I : I]} \\
&= \frac{[IW_1^{W_1\kappa} I : I]}{[IW_{1,2}^{W_{1,2}\kappa} I : I]} \\
&= [IW_1^{W_1\kappa} I : IW_{1,2}^{W_{1,2}\kappa} I].
\end{aligned} \tag{5.27}$$

But this was exactly our assertion. □

Note that the condition $W_1 \leq W_{af}$ is necessary for the result to hold as is shown by Example 4.1.11. The condition is always fulfilled if $\mathbb{G}$ is simply connected (in which case $\widetilde{W} = W_{af}$) or if $P_1$ is a hyperspecial maximal compact subgroup.

Now we want to consider the situation where $W_1 \not\leq W_{af}$. To that end we change the notation slightly and consider two parahoric subgroups $P_1', P_2'$ with $P_i' = IW_i' I = IW_i \Omega_i I$, where $W_i \leq W_{af}$ and $\Omega_i \leq \Omega$ fixes $S_i = S \cap W_i$. Furthermore we set $P_{1,2} = P_1 \cap P_2 = I(W_1 \cap W_2)I = I(W_{1,2})I$ and $P_{1,2}' = P_1' \cap P_2' = IW_{1,2}\Omega_{1,2} I$. Finally we choose representatives for certain coset decompositions as follows

$$
\begin{aligned}
P'_1 &= \bigsqcup_j P'_{1,2} l'_j, \\
P_1 &= \bigsqcup_j P_{1,2} l_j, \\
P'_2 &= \bigsqcup_i P'_{1,2} m'_i, \\
P_2 &= \bigsqcup_i P_{1,2} m_i.
\end{aligned}
\tag{5.28}
$$

We choose representatives $\rho$ for $\Omega_1$, $\sigma$ for $\Omega_{1,2}$ and summarize all of this in the following diagram:

$$
\begin{tikzcd}
P'_1 \arrow[d, dash, "\rho"'] \arrow[dr, dash, "l'_j"'] & & P'_2 \arrow[dl, dash, "m'_i"] \arrow[d, dash] \\
P_1 \arrow[dr, dash, "l_j"'] & P'_{1,2} \arrow[d, dash, "\sigma"] & P_2 \arrow[dl, dash, "m_i"] \\
& P_{1,2} &
\end{tikzcd}
$$

Now consider the element

$$
\widetilde{\nu'_{1,2}} := \sum_{\sigma,\sigma' \in \Omega_{1,2}} \sum_{i,j} \mathbb{1}_{P'_1 \sigma m'_i \sigma' l'_j} \in H_{P_1}. \tag{5.29}
$$

We see that

$$
\begin{aligned}
\widetilde{\nu'_{1,2}} &= |\Omega_{1,2}| \sum_{\sigma' \in \Omega_{1,2}} \sum_{i,j} \mathbb{1}_{P'_1 m'_i \sigma' l'_j} \in H_{P'_1} \\
&= |\Omega_{1,2}|^2 \sum_{i,j} \mathbb{1}_{P'_1 m'_i l'_j} \in H_{P_1} \\
&= |\Omega_{1,2}|^2 \nu(P'_1, P'_2).
\end{aligned}
\tag{5.30}
$$

In particular, if we want to study $\nu(P'_1, P'_2)$ we can study $\widetilde{\nu_{1,2}}$ instead.

**Theorem 5.1.10** *For $\kappa \in [W_{1,2} \backslash W'_2 / W_{1,2}]$ set*

$$
t_\kappa := [I W_1^{W_1 \kappa} I : I(W_1^{W_1 \kappa} \cap W_2) I], \tag{5.31}
$$

*the coefficient of $\mathbb{1}_{P_1 \kappa P_1}$ in $\nu(P_1, P'_2)$. Then the following holds:*

$$
\nu(P'_1, P'_2) = |\Omega_{1,2}|^{-2} \sum_{\kappa \in [W_{1,2} \backslash W'_2 / W_{1,2}]} t_\kappa |\Omega_1^\kappa| \mathbb{1}_{P'_1 \kappa P'_1}, \tag{5.32}
$$

*where $\Omega_1^\kappa$ denotes the stabilizer of $\kappa$ in $\Omega_1$.*

*Proof.* As noted before we will compute $\widetilde{\nu'_{1,2}}$. Notice that $\{P_{1,2}\sigma l'_j \mid \sigma \in \Omega_{1,2}, j\} = \{P_{1,2}l_j\rho \mid \rho \in \Omega_1, j\}$ and compute for arbitrary $x$:

$$\begin{aligned}
\widetilde{\nu'_{1,2}}(x) &= \sum_{\sigma\in\Omega_{1,2},\rho\in\Omega_1} \sum_{i,j} \mathbb{1}_{P'_1\sigma m'_i l_j\rho}(x) \\
&= \sum_{\sigma\in\Omega_{1,2},\rho,\rho'\in\Omega_1} \sum_{i,j} \mathbb{1}_{P_1\rho'\sigma m'_i l_j\rho}(x) \\
&= \sum_{\sigma\in\Omega_{1,2},\rho,\rho'\in\Omega_1} \sum_{i,j} \mathbb{1}_{\rho' P_1\sigma m'_i l_j\rho}(x) \\
&= \sum_{\sigma\in\Omega_{1,2},\rho,\rho'\in\Omega_1} \sum_{i,j} \mathbb{1}_{P_1\sigma m'_i l_j}(\rho' x\rho) \\
&= \sum_{\rho,\rho'\in\Omega_1} \nu(P_1, P'_2)(\rho' x\rho) \\
&= \sum_{\kappa\in[W_{1,2}\backslash W'_2/W_{1,2}]} \sum_{\rho,\rho'\in\Omega_1} t_\kappa \mathbb{1}_{P_1\kappa P_1}(\rho' x\rho).
\end{aligned} \tag{5.33}$$

Now we decompose $P_1\kappa P_1 = \bigsqcup_r P_1\kappa_r$ and see

$$\begin{aligned}
\sum_{\rho,\rho'\in\Omega_1} \mathbb{1}_{P_1\kappa P_1}(\rho' x\rho) &= \sum_{\rho,\rho'\in\Omega_1} \sum_r \mathbb{1}_{\rho' P_1\kappa_r}(x\rho) \\
&= \sum_r \sum_{\rho\in\Omega_1} \mathbb{1}_{P'_1\kappa_r\rho}(x) \\
&= |\Omega_1^\kappa| \mathbb{1}_{P_1\kappa P_1}.
\end{aligned} \tag{5.34}$$

The last equality holds due to Lemma 4.2.9. Putting all of this together we achieve the result. ☐

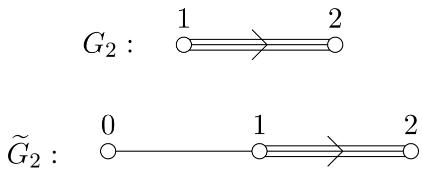
It is noteworthy that the double cosets appearing in the sum in the last theorem are no longer necessarily distinct; in fact two cosets $P'_1\kappa P'_1$ and $P'_1\kappa' P'_1$ with $\kappa, \kappa' \in [W_{1,2}\backslash W'_2/W_{1,2}]$ coincide if and only if $\kappa = \sigma\kappa'\sigma'$ for suitable $\sigma, \sigma' \in \Omega_1$.

## 5.2. The Eichler Subalgebra

In this section we want to take a look at some explicit groups and describe what insight into the (local) Hecke algebras one can achieve by applying the method described above. For the time being we will only consider the situation at hyperspecial primes.

### 5.2.1. The Group $G_2$

Let $\mathbb{G}$ (still defined over the totally real number field $k$ and compact at $k_\infty$) be of type $G_2$ and let $K$ be an open and compact subgroup of $\mathbb{G}(\hat{k})$ such that $K = K_\mathfrak{p} \times K_{\mathfrak{p}'}$ for some finite prime $\mathfrak{p}$ where $\mathbb{G}$ is split and $K_\mathfrak{p}$ is hyperspecial. The ordinary Dynkin diagram of $\mathbb{G}$ and the extended Dynkin diagram of $\mathbb{G}$ at $\mathfrak{p}$ have the following forms:




Let $I$ be an Iwahori subgroup of $\mathbb{G}(k_\mathfrak{p})$ contained in $K_\mathfrak{p}$ and $T$ a (split maximal) torus whose integral part is contained in $I$. Since $\mathbb{G}$ is simply connected, the extended affine Weyl group is just the affine Weyl group and we have

$$\widetilde{W} = W_0 \ltimes X_*(T) \cong W_{af}, \tag{5.35}$$

where $W_0$ is the finite Weyl group with involutive generators $s_1$ and $s_2$ and $W_{af}$ is generated by $s_1, s_2$ and $s_0 = s_{\alpha_0} t_{\alpha_0^\vee}$.

We set $P_0 := K_\mathfrak{p}$, then $P_0 = IW_0I$ and there are two more maximal compact subgroups of $\mathbb{G}(k_\mathfrak{p})$ containing $I$, namely

$$P_1 := IW_1I = I\langle s_0, s_2\rangle I \text{ and } P_2 := IW_2I = I\langle s_0, s_1\rangle I. \tag{5.36}$$

The Hecke algebra $H_{K_\mathfrak{p}}$ is generated by the double cosets corresponding to the translations $t_{\omega_i^\vee}$, $i = 1, 2$, where $\omega_i^\vee$ is the $i$-th fundamental coweight. Following [LP02, 2.5] we find reduced expressions

$$\begin{aligned} t_{\omega_1^\vee} &= s_1 s_2 s_1 s_2 s_1 s_0, \\ t_{\omega_2^\vee} &= s_2 s_1 s_2 s_1 s_2 s_0 s_1 s_2 s_1 s_0, \end{aligned} \tag{5.37}$$

whence

$$\begin{aligned} P_0 t_{\omega_1^\vee} P_0 &= P_0 s_0 P_0, \\ P_0 t_{\omega_2^\vee} P_0 &= P_0 s_0 s_1 s_2 s_1 s_0 P_0. \end{aligned} \tag{5.38}$$

Now let $K_i := P_i \times K_{\mathfrak{p}'}$, $i = 1, 2$, be two other open compact subgroups of $\mathbb{G}(\hat{k})$ which coincide with $K$ away from $\mathfrak{p}$ and set $\nu_i := \nu(K, K_i) \in H_K$, the Eichler element of $H_K$ with respect to $K_i$. By the choice of our compact groups we are in the situation of Theorem 5.1.9 and we can explicitly give the elements $\nu_i$ as linear combinations of double cosets. One would hope that $H_{K_\mathfrak{p}}$ is generated by $\nu_1$ and $\nu_2$, however, we compute the following.

**Theorem 5.2.1** *The subalgebra of $H_{K_\mathfrak{p}}$ that is generated by $\nu_1$ and $\nu_2$ conincides with the subalgebra generated by* $\mathrm{id}_{P_0 t_{\omega_1^\vee} P_0} = \mathrm{id}_{P_0 s_0 P_0}$.

*Proof.* The double cosets in $W_{0,1}\backslash W_1/W_{0,1}$ are represented by

$$[W_{0,1}\backslash W_1/W_{0,1}] = \{1, s_0\} \tag{5.39}$$

and $W_0 \cap W_0^{s_0} = \langle s_2 \rangle = W_0 \cap W_0^{s_0} \cap W_1$. Hence by Theorem 5.1.9

$$\nu_1 = [P_0 : (P_0 \cap P_1)] \cdot \mathbb{1}_K + \mathbb{1}_{Ks_0K} = \frac{q^6 - 1}{q - 1} \cdot \mathbb{1}_K + \mathbb{1}_{Ks_0K}, \tag{5.40}$$

where $q$ is the order of the residue class field at $\mathfrak{p}$. In particular we see that the subalgebra generated by $\nu_1$ is the same as the one generated by $Ks_0K$.

The double cosets in $W_{0,2}\backslash W_2/W_{0,2}$ are represented by

$$[W_{0,1}\backslash W_1/W_{0,1}] = \{1, s_0\} \tag{5.41}$$

and $W_0 \cap W_0^{s_0} = \langle s_2 \rangle, W_0 \cap W_0^{s_0} \cap W_2 = \{1\}$. Hence by Theorem 5.1.9

$$\nu_2 = [P_0 : (P_0 \cap P_2)] \cdot \mathbb{1}_K + [(P_0 \cap P_1) : I] \cdot \mathbb{1}_{Ks_0K} = \frac{q^6 - 1}{q - 1} \cdot \mathbb{1}_K + (q+1) \cdot \mathbb{1}_{Ks_0K}. \tag{5.42}$$

Thus $\nu_1$ and $\nu_2$ generate the same subalgebra of $H_{K_\mathfrak{p}}$ as $\mathrm{id}_{P_0 s_0 P_0}$. □

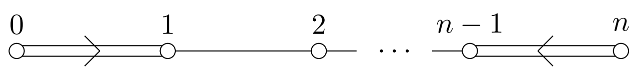
While $\nu_1$ and $\nu_2$ do not generate the full Hecke algebra, we attain two methods of computing the action of $Ks_0K$ on a space of algebraic modular forms. Since $[P_0 : (P_0 \cap P_1)] = [P_0 : (P_0 \cap P_2)]$, the computational complexity of these two computations essentially only depends on the values $|K_i\backslash\mathbb{G}(\hat{k})/\mathbb{G}(k)|$, $i = 1, 2$, so if we know these beforehand we can choose the smaller one to do our computations.

### 5.2.2. The Groups $C_n$

Let $\mathbb{G}$ now be simply connected of type $C_n$, $n \geq 2$, and let $K$ be an open and compact subgroup of $\mathbb{G}(\hat{k})$ such that $K = K_\mathfrak{p} \times K_{\mathfrak{p}'}$ for some finite prime $\mathfrak{p}$ where $\mathbb{G}$ is split and $K_\mathfrak{p}$ is hyperspecial. The ordinary Dynkin diagram of $\mathbb{G}$ and the extended Dynkin diagram of $\mathbb{G}$ at $\mathfrak{p}$ have the following forms:

$C_n$ :



$\widetilde{C}_n$ :


Let $I$ be an Iwahori subgroup of $\mathbb{G}(k_\mathfrak{p})$ contained in $K_\mathfrak{p}$ and $T$ a (split maximal) torus whose integral part is contained in $I$. Since $\mathbb{G}$ is simply connected, the extended affine Weyl group is just the affine Weyl group and we have

$$\widetilde{W} = W_0 \ltimes X_*(T) \cong W_{af}, \tag{5.43}$$

where $W_0$ is the finite Weyl group with involutive generators $s_1, ..., s_n$ and $W_{af}$ is generated by $s_1, ..., s_n$ and $s_0 = s_{\alpha_0} t_{\alpha_0^\vee}$.

The local Hecke algebra $H_{K_\mathfrak{p}}$ is generated by the double cosets corresponding to the translations $t_i := t_{\omega_i^\vee},\ i = 1, ..., n-1$, and $t_n := t_{2\omega_n^\vee}$, where $\omega_i^\vee$ is the $i$-th fundamental coweight. We set $P_0 := K_\mathfrak{p}$ and will show that $H_{K_\mathfrak{p}} = H_{P_0}$ is also generated by the Eichler elements corresponding to the other maximal subgroups of $\mathbb{G}(k_\mathfrak{p})$ containing $I$.

To that end set $W_i := \langle s_0, ..., \hat{s}_i, ..., s_n \rangle \leq \widetilde{W}$ and $P_i := IW_iI$ for $0 \leq i \leq n$. Corresponding to these groups we have the (local) Eichler elements

$$\nu_i := \nu(P_0, P_i) = \sum_{\kappa \in [W_{0,i} \backslash W_i / W_{0,i}]} [IW_0^{W_0\kappa} I : I(W_0^{W_0\kappa} \cap W_i)I] \mathbb{1}_{P_0 \kappa P_0}. \tag{5.44}$$

To prove our assertion we will show that the sum defining $\nu_i$ contains exactly the double cosets $P_0 t_j P_0,\ j \leq i$, and $P_0$. Let us first introduce some new notation: Denote the product $s_i s_{i+1} ... s_j$ for $0 \leq i \leq j \leq n$ by $s_{[i,j]}$ (as an element of the extended affine Weyl group) and set $s_{[i]} := s_{[0,i]}$.

**Lemma 5.2.2** *The double cosets in $W_{0,i} \backslash W_i / W_{0,i}$ are represented by*

$$1, s_{[0]}, s_{[1]}s_{[0]}, ..., s_{[i]}s_{[i-1]}...s_{[1]}s_{[0]} \tag{5.45}$$

*Proof.* $W_i$ is a Coxeter group with Dynkin diagram of type $B_i \times B_{n-i}$ and $W_{0,i}$ is the standard parabolic subgroup with Dynkin subdiagram $A_{i-1} \times B_{n-i}$. In particular we can ignore the $B_{n-i}$ part and focus on the double cosets of $A_{i-1} = \langle s_1, ..., s_{i-1} \rangle$ in $B_i = \langle s_0, ..., s_{i-1} \rangle$.

The Coxeter group of type $B_i$ is isomorphic to $\mathbb{F}_2 \wr S_i$ under the following identification of generators: $s_0 \leftrightarrow e_1 \in \mathbb{F}_2^i,\ s_j \leftrightarrow (j, j+1) \in S_i$. Hence the double cosets in $A_{i-1} \backslash B_i / A_{i-1}$ are represented by the elements $v \in \mathbb{F}_2^i$ and $v, v'$ lie in the same double coset if and only if $v, v'$ have the same number of nonzero entries. Furthermore we easily see by induction that

$$s_{[j-1]}s_{[j-2]}...s_{[1]}s_{[0]} = \underbrace{\left( \sum_{k=1}^{j} e_k \right)}_{\in \mathbb{F}_2^i} \underbrace{\sigma}_{\in S_i} \tag{5.46}$$

for some $\sigma \in S_i$. Thus the double cosets indeed have the representatives given above. $\square$

This means that in the decomposition of $\nu_i$ into double cosets there is exactly one coset that did not show up in the decomposition of $\nu_{i-1}$. Inductively we can hence obtain the operators $P_0, P_0 s_0 P_0, P_0 s_{[1]} s_{[0]} P_0, ...$ from the Eichler elements $\nu_0, \nu_1, \nu_2, ....$ It remains to be shown that these double cosets are precisely those of the translations $t_i$.

**Lemma 5.2.3** *We have* $\omega_i^\vee = \alpha_0^\vee + \sum_{k=1}^{i-1} \alpha_0^\vee s_{[1,k]}$ *and* $s_{\alpha_0} = s_{[1,n]}(s_{[1,n-1]})^{-1}$.

*Proof.* We consider the following realization of a root system of type $C_n$ in the Euclidean space $\mathbb{R}^n$ with the standard inner product. The set of roots is

$$\Phi := \{\pm e_i \pm e_j \mid 1 \le i < j \le n\} \cup \{\pm 2e_i \mid 1 \le i \le n\} \tag{5.47}$$

and the simple roots are

$$\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, ..., \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n. \tag{5.48}$$

The coweights can be realized in the same vector space by the usual construction

$$\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha. \tag{5.49}$$

It is easily seen that $\alpha_0 = 2e_1$ and since for $1 \le i \le n-1$ we have $e_i s_{\alpha_i} = e_{i+1}$, the second assertion is easily verified. Furthermore we check that

$$\langle \alpha_0^\vee + \sum_{k=1}^{i-1} (\alpha_0^\vee s_{[1,k]}), \alpha_j \rangle = \langle \sum_{k=1}^{i} e_k, \alpha_j \rangle = \begin{cases} 1 & i = j \neq n \\ 2 & i = j = n \\ 0 & i \neq j \end{cases} \tag{5.50}$$

whence $\alpha_0^\vee + \sum_{k=1}^{i-1} (\alpha_0^\vee s_{[1,k]})$ is just $\omega_i^\vee$ for $i < n$ and $2\omega_n^\vee$ for $i = n$. □

For the next essential result we first need some rules for computing with the symbols $s_{[i,j]}$.

**Lemma 5.2.4** *The following holds:*

1. $s_{[j]} s_k = s_{k+1} s_{[j]}$ *for* $2 \le j \le n-1$ *and* $k < j$.
2. $s_{[j]}^2 = s_1 s_{[j]} s_{[j-1]}$ *for* $1 \le j \le n-1$.

*Proof.* 1. Since $s_i s_{i'} = s_{i'} s_i$ if $|i - i'| > 1$, we have

$$\begin{aligned} s_{[j]} s_k &= s_{[k+1]} s_k s_{[k+2,j]} = s_{[k-1]} s_k s_{k+1} s_k s_{[k+2,j]} \\ &= s_{[k-1]} s_{k+1} s_k s_{k+1} s_{[k+2,j]} = s_{k+1} s_{[j]}. \end{aligned} \tag{5.51}$$

2. We will prove this by induction on $j$. For $j = 1$ we have

$$s_{[1]}^2 = s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 = s_1 s_{[1]} s_{[0]}. \tag{5.52}$$

For the step from $j$ to $j+1$ we compute

$$\begin{aligned} s_{[j+1]}^2 &= s_{[j]}s_{j+1}s_{[j+1]} \overset{(1)}{=} s_{[j]}s_{[j+1]}s_j \\ &= s_{[j]}^2 s_{j+1}s_j = s_1 s_{[j]}s_{[j-1]}s_{j+1}s_j \\ &= s_1 s_{[j]}s_{j+1}s_{[j-1]}s_j = s_1 s_{[j+1]}s_{[j]}. \end{aligned} \tag{5.53}$$

□

**Lemma 5.2.5** *We have*

$$W_0 t_i = W_0 s_{[i-1]}s_{[i-2]}...s_{[1]}s_{[0]}. \tag{5.54}$$

*Proof.* We will show this by induction on $i$. For $i=1$ we have

$$W_0 t_1 = W_0 t(\alpha_0^\vee) = W_0 s_{\alpha_0}s_0 = W_0 s_0. \tag{5.55}$$

Now consider the step from $i$ to $i+1$. First of all we compute

$$\begin{aligned} t_{i+1} &\overset{5.2.3}{=} t(\alpha_0^\vee + \sum_{k=1}^{i}(\alpha_0^\vee s_{[1,k]})) \\ &= t(\alpha_0^\vee)t(\alpha_0^\vee s_{[1,1]})...t(\alpha_0^\vee s_{[1,i]}) \\ &= t(\alpha_0^\vee)t(\alpha_0^\vee)^{s_{[1,1]}}...t(\alpha_0^\vee)^{s_{[1,i]}} \\ &= (s_{\alpha_0}s_0)(s_{\alpha_0}s_0)^{s_{[1,1]}}...(s_{\alpha_0}s_0)^{s_{[1,i]}} \\ &= t_i(s_{\alpha_0}s_0)^{s_{[1,i]}}. \end{aligned} \tag{5.56}$$

By Lemma 5.2.3 we have $s_{\alpha_0}^{s_{[1,k]}} = s_{k+1}...s_n...s_{k+1}$, which commutes with all $s_j$, $0 \le j < k$, and thus by induction

$$\begin{aligned} W_0 t_{i+1} &= W_0 t_i (s_{\alpha_0}s_0)^{s_{[1,i]}} \\ &= W_0 s_{[i-1]}s_{[i-2]}...s_{[1]}s_{[0]}s_{\alpha_0}^{s_{[1,i]}}s_0^{s_{[1,i]}} \\ &= W_0 s_{[i-1]}s_{[i-2]}...s_{[1]}s_{[0]}s_0^{s_{[1,i]}} \\ &= W_0(s_{[i]}s_{[i-1]}...s_{[1]}s_{[0]})s_{[1,i]}. \end{aligned} \tag{5.57}$$

Since this is almost our assertion it remains to be verified that

$$W_0(s_{[i]}s_{[i-1]}...s_{[1]}s_{[0]})s_j = W_0(s_{[i]}s_{[i-1]}...s_{[1]}s_{[0]}) \tag{5.58}$$

for all $1 \le j \le i$. For $j=1$ we see

$$\begin{aligned} W_0(s_{[i]}s_{[i-1]}...s_{[1]}s_{[0]})s_1 &= W_0 s_{[i]}s_{[i-1]}...s_{[2]}s_0 s_1 s_0 s_1 \\ &= W_0 s_{[i]}s_{[i-1]}...s_{[2]}s_1 s_0 s_1 s_0 \\ &\overset{5.2.4(1)}{=} W_0 s_{[i]}s_{[i-1]}...s_{[3]}s_2 s_{[2]}s_{[1]}s_{[0]} \\ &\vdots \\ &\overset{5.2.4(1)}{=} W_0 s_i s_{[i]}s_{[i-1]}...s_{[3]}s_{[2]}s_{[1]}s_{[0]} \\ &= W_0 s_{[i]}s_{[i-1]}...s_{[3]}s_{[2]}s_{[1]}s_{[0]}. \end{aligned} \tag{5.59}$$

In the same vein we continue for $1 < j \leq i$:

$$\begin{aligned}
W_0(s_{[i]}s_{[i-1]}...s_{[1]}s_{[0]})s_j &= W_0s_{[i]}s_{[i-1]}...s_{[j]}s_{[j-1]}s_js_{[j-2]}...s_{[0]} \\
&= W_0s_{[i]}s_{[i-1]}...s_{[j+1]}s_{[j]}^2s_{[j-2]}...s_{[0]} \\
&\overset{5.2.4(2)}{=} W_0s_{[i]}s_{[i-1]}...s_{[j+1]}s_1s_{[j]}s_{[j-1]}...s_{[0]} \\
&\overset{5.2.4(1)}{=} W_0s_{[i]}s_{[i-1]}...s_{[j+2]}s_2s_{[j+1]}s_{[j]}s_{[j-1]}...s_{[0]} \qquad (5.60)\\
&\vdots \\
&= W_0s_{i-j+1}s_{[i]}s_{[i-1]}...s_{[j+2]}s_{[j+1]}s_{[j]}s_{[j-1]}...s_{[0]} \\
&= W_0s_{[i]}s_{[i-1]}...s_{[1]}s_{[0]}.
\end{aligned}$$

This concludes the proof. □

In conclusion we get the following result on the local Hecke algebras.

**Corollary 5.2.6** *The vector space spanned by $\mathbb{1}_{P_0}$ and the local Eichler elements $\nu_1, ..., \nu_i$ is the same as the one spanned by $\mathbb{1}_{P_0}$ and $\mathbb{1}_{P_0t_1P_0}, ..., \mathbb{1}_{P_0t_iP_0}$ for all $1 \leq i \leq n$. In particular we see that the local Hecke algebra (with respect to $P_0$) is generated by the Eichler elements.*

### 5.2.3. The Other Simply Connected Groups

In the last two subsections we studied the Eichler elements for the groups for which we wish to perform explicit computations later. Here we want to describe the local situtation for the other almost simple, simply connected groups. Let $\mathbb{G}$ now be almost simple and otherwise as before, and $\mathfrak{p}$ a prime of $k$ where $\mathbb{G}$ is split. We choose an Iwahori subgroup $I \leq \mathbb{G}(k_\mathfrak{p})$ and denote the usual generators of the affine Weyl group (with respect to a suitably chosen torus) by $s_0, ..., s_n$. For $0 \leq i \leq n$ we set $W_i$ the subgroup of $W_{af}$ generated by all the $s_j$ except for $s_i$,

$$W_i = \langle s_0, ..., s_{i-1}, s_{i+1}, ..., s_n \rangle, \qquad (5.61)$$

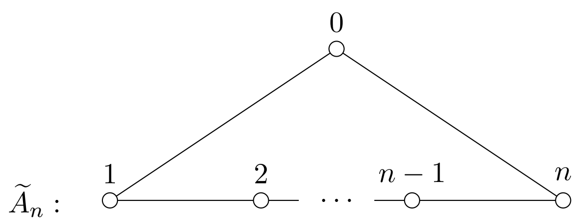
and set $P_i := IW_iI$, the maximal parahoric subgroup corresponding to $W_i$. Then $P_0$ is a hyperspecial maximal compact subgroup.

We are interested in the subalgebra of $H(\mathbb{G}(k_\mathfrak{p}), P_0)$ that is generated by the Eichler elements $\nu(P_0, P_i)$, $1 \leq i \leq n$, and we call it the Eichler subalgebra. This is precisely the subalgebra whose action we can obtain by only computing intertwining operators with respect to maximal compact subgroups. We already saw that for $\mathbb{G}$ of type $G_2$, the Eichler subalgebra is a polynomial ring in one variable, generated by $\mathbb{1}_{P_0s_0P_0}$, while for $\mathbb{G}$ of type $C_n$ the Eichler subalgebra coincides with the full Hecke algebra. Note that we cannot expect this to happen for any other groups (with the possible exceptions of $F_4$ and $E_8$) since $H(\mathbb{G}(k_\mathfrak{p}), P_0)$ is minimally generated by more than $n$ elements (since $\mathbb{G}$ is simply connected and not adjoint) whereas the Eichler subalgebra is (not necessarily minimally) generated by $n$ elements. However, we may still obtain a subalgebra that is large enough to be of interest for computational purposes.

Seeing that for $V = k$ the trivial module we can identify $M(V, K)$ with the space of $k$-valued functions on $K\backslash\mathbb{G}(\hat{k})/\mathbb{G}(k)$, the elements in the Eichler subalgebra are characterized as those Hecke operators (or Brandt matrices) whose intrinsic combinatorics are already completely determined by the combinatorics of the chambers containing a given hyperspecial point.

The following results describe the Eichler subalgebra in terms of generators and we choose generators of the form $\mathbb{1}_{P_0 w P_0}$ with $w \in W_{af}$ instead of sums over double cosets (we will see that this is always possible). Let us start with the three infinite series we have not considered so far.

**Theorem 5.2.7** *Let $\mathbb{G}$ be of type $A_n$. We label the nodes of the extended Dynkin diagram as follows:*




*Let us denote the product $s_i s_{i+1} ... s_j$ with $0 \leq i < j \leq n$ by $s_{[i,j]}$. Then the Eichler subalgebra of $H(\mathbb{G}(k_\mathfrak{p}), P_0)$ is generated by the (pairwise distinct) $P_0$-double cosets of the following elements of $W_{af}$:*

$$s_{[0,i-1]}(s_n s_{[0,i-2]})...(s_{[n-i+2,n]} s_0), \ 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor. \tag{5.62}$$

*These are precisely the double cosets of the following translations:*

$$t\left(\omega_i^\vee + \omega_{n+1-i}^\vee\right), \ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \ \textit{and } t(2\omega_{\frac{n+1}{2}}^\vee) \textit{ for odd } n. \tag{5.63}$$

*Proof.* The Eichler element $\nu(P_0, P_i)$ is supported precisely on the double cosets of the elements in $[(W_0 \cap W_i)\backslash W_i/(W_0 \cap W_i)]$ and for symmetry reasons we only need to look at $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$.

We identify $W_i$ with $S_{n+1}$, the symmetric group on $n+1$ elements via

$$s_{i-1} \mapsto (1,2), s_{i-2} \mapsto (2,3), ..., s_0 \mapsto (i, i+1), s_n \mapsto (i+1, i+2), ..., s_{i+1} \mapsto (n, n+1). \tag{5.64}$$

Under this identification $W_0 \cap W_i$ is the stabilizer of $\underline{i} = \{1, ..., i\}$ in the standard action of $S_{n+1}$ on the set of subsets of $\underline{n+1} = \{1, ..., n+1\}$ of cardinality $i$. In particular, two elements $\sigma, \sigma' \in W_i$ are in the same double coset with respect to $W_0 \cap W_i$ if and only if $|\sigma \underline{i} \cap \underline{i}| = |\sigma' \underline{i} \cap \underline{i}|$. We compute

$$\begin{aligned}
&1_{W_i}\underline{i} = \underline{i} \\
&s_0\underline{i} = \underline{i-1} \cup \{i+1\} \\
&s_0 s_1 s_n s_0 \underline{i} = \underline{i-2} \cup \{i+1, i+2\} \\
&(s_0 s_1 s_2)(s_n s_0 s_1)(s_{n-1} s_n s_0)\underline{i} = \underline{i-3} \cup \{i+1, i+2, i+3\} \\
&\vdots \\
&s_{[0,i-1]}(s_n s_{[0,i-2]}) ... (s_{[n-i+2,n]} s_0)\underline{i} = \{i+1, ..., 2i\}.
\end{aligned} \tag{5.65}$$

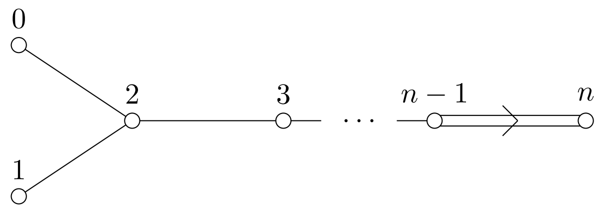
Thus we have $[(W_0 \cap W_1)\backslash W_1/(W_0 \cap W_1)] = \{\mathrm{id}, s_0\}$ and for $2 \leq i \leq \frac{n+1}{2}$ we get

$$\begin{aligned}
&[(W_0 \cap W_i)\backslash W_i/(W_0 \cap W_i)] \\
=&[(W_0 \cap W_{i-1})\backslash W_{i-1}/(W_0 \cap W_{i-1})] \cup \{s_{[0,i-1]}(s_n s_{[0,i-2]}) ... (s_{[n-i+2,n]} s_0)\}.
\end{aligned} \tag{5.66}$$

The assertion follows. In order to identify the translations whose double cosets we have found, one takes an explicit realization of the $A_n$-root system. $\square$

**Theorem 5.2.8** *Let $\mathbb{G}$ be of type $B_n$, $n \geq 3$. We label the nodes of the extended Dynkin diagram as follows:*

$\widetilde{B}_n$ :




*Let us here denote by $s_{[i,j]}$ with $n \geq i > j \geq 0$ the product $s_i s_{i-1} ... s_j$ ($s_{[i,i]} := s_i$) and by $v_i$, $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, the element $v_i := s_{[2i,1]} s_{[2i+1,2]} s_0$. Then the Eichler subalgebra of $H(\mathbb{G}(k_\mathfrak{p}), P_0)$ is generated by the (pairwise distinct) $P_0$-double cosets of the following elements of $W_{af}$:*

$$s_0, s_0 s_2 s_3 ... s_{n-1} s_{[n,2]} s_0, \text{ and the elements } s_0 v_1 v_2 ... v_i,\ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1. \tag{5.67}$$

*These double cosets are precisely those of the following translations:*

$$P_0 t(\omega_{2i}^\vee) P_0,\ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \text{ and } P_0 t(2\omega_1^\vee) P_0. \tag{5.68}$$

*Proof.* First of all we compute

$$\begin{aligned}
&[(W_0 \cap W_1)\backslash W_1/(W_0 \cap W_1)] = \{\mathrm{id}, s_0, s_0 s_2 s_3 \ldots s_{n-1} s_{[n,2]} s_0\} \text{ and} \\
&[(W_0 \cap W_2)\backslash W_2/(W_0 \cap W_2)] = [(W_0 \cap W_3)\backslash W_3/(W_0 \cap W_3)] = \{\mathrm{id}, s_0\},
\end{aligned} \tag{5.69}$$

which gives us the first two generators above. For $4 \le i \le n$ we have

$$[(W_0 \cap W_i)\backslash W_i/(W_0 \cap W_i)] = [\langle s_1, \ldots, s_{i-1}\rangle \backslash \langle s_0, \ldots, s_{i-1}\rangle / \langle s_1, \ldots, s_{i-1}\rangle], \tag{5.70}$$

so we need to compute the double cosets of a parabolic subgroup of type $A_{i-1}$ in the Coxeter group of type $D_i$. The group $\langle s_0, \ldots, s_{i-1}\rangle$ is isomorphic to $V \rtimes S_i$ where $V \le \mathbb{F}_2^i$ is the $i-1$-dimensional subspace that is orthogonal to the all-ones vector under the standard inner product. The explicit isomorphism is given by

$$s_0 \mapsto e_1 + e_2, s_1 \mapsto (1,2), s_2 \mapsto (2,3), \ldots, s_{i-1} \mapsto (i-1, i) \tag{5.71}$$

and thus $\langle s_1, \ldots, s_{i-1}\rangle$ maps to the subgroup isomorphic to $S_i$. In particular the double cosets are represented by elements of $V$ and two vectors $v, v'$ represent the same double coset if and only if the have they same number of nonzero entries. Hence there are $\lfloor \frac{i}{2} \rfloor + 1$ double cosets, represented by

$$\begin{aligned}
&\mathrm{id} \leftrightarrow 0 \in V \\
&s_0 \leftrightarrow e_1 + e_2 \in V \\
&s_0 v_1 \leftrightarrow \sum_{j=1}^{4} e_j \in V \\
&\vdots \\
&s_0 v_1 \ldots v_{\lfloor \frac{i}{2} \rfloor - 1} \leftrightarrow \sum_{j=1}^{2\lfloor \frac{i}{2} \rfloor} e_j \in V.
\end{aligned} \tag{5.72}$$

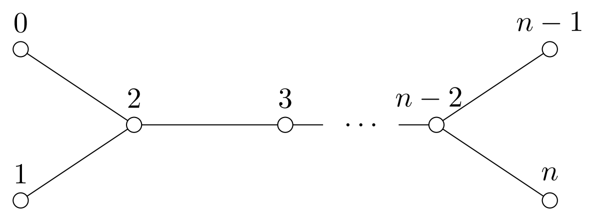
In conclusion we get

$$\begin{aligned}
&[(W_0 \cap W_4)\backslash W_4/(W_0 \cap W_4)] = \{\mathrm{id}, s_0, s_0 v_1\} \text{ and} \\
&[(W_0 \cap W_i)\backslash W_i/(W_0 \cap W_i)] \\
=&[(W_0 \cap W_{i-2})\backslash W_{i-2}/(W_0 \cap W_{i-2})] \cup \{s_0 v_1 \ldots v_{\lfloor \frac{i}{2} \rfloor - 1}\} \text{ for } i > 4.
\end{aligned} \tag{5.73}$$

The assertion follows. In order to identify the translations whose double cosets we have found, one takes an explicit realization of the $B_n$-root system. See Theorem 5.2.16 for details. □

**Theorem 5.2.9** *Let $\mathbb{G}$ be of type $D_n$, $n \ge 4$. We label the nodes of the extended Dynkin diagram as follows:*

$\widetilde{D}_n$ :



*Let us, as above, denote by $s_{[i,j]}$ with $n \geq i \geq j \geq 0$ the product $s_i s_{i-1} ... s_j$ ($s_{[i,i]} := s_i$) and by $v_i$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, the element $v_i := s_{[2i,1]} s_{[2i+1,2]} s_0$. Then the Eichler subalgebra of $H(\mathbb{G}(k_{\mathfrak{p}}), P_0)$ is generated by the (pairwise distinct) $P_0$-double cosets of the following elements of $W_{af}$:*

$$\begin{aligned} &s_0, s_0 s_2 s_3 ... s_{n-2} s_{[n,2]} s_0, \text{ the elements } s_0 v_1 v_2 ... v_i, \ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ &\text{and finally, if } n \text{ is even, } (s_0 v_1 ... v_{\frac{n}{2}-2}) s_{[n-2,1]} s_n s_{[n-2,2]} s_0. \end{aligned} \tag{5.74}$$

*These double cosets are precisely those of the following translations:*

$$\begin{aligned} &P_0 t(\omega_{2i}^\vee) P_0, \ 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \ P_0 t(2\omega_1^\vee) P_0 \text{ and either} \\ &P_0 t(2\omega_{n-1}^\vee) P_0 \text{ and } P_0 t(2\omega_n^\vee) P_0 \text{ if } n \text{ is even, or} \\ &P_0 t(\omega_{n-1}^\vee + \omega_n^\vee) P_0 \text{ if } n \text{ is odd.} \end{aligned} \tag{5.75}$$

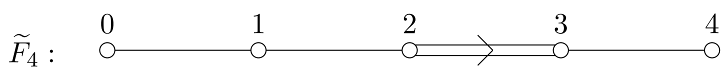
*Proof.* As for $B_n$ we first compute

$$\begin{aligned} &[(W_0 \cap W_1) \backslash W_1 / (W_0 \cap W_1)] = \{\mathrm{id}, s_0, s_0 s_2 s_3 ... s_{n-2} s_{[n,2]} s_0\} \text{ and} \\ &[(W_0 \cap W_2) \backslash W_2 / (W_0 \cap W_2)] = [(W_0 \cap W_3) \backslash W_3 / (W_0 \cap W_3)] = \{\mathrm{id}, s_0\}, \end{aligned} \tag{5.76}$$

which gives us the first two generators. For $4 \leq i \leq n-2$ we are in the same situation as for $B_n$ so we need to compute the double cosets of $A_{i-1}$ in $D_i$, which we already did. For $W_n$ we are in the situation of $[A_{n-1} \backslash D_n / A_{n-1}]$ which we already considered as well. Finally for $W_{n-1}$ we are in the same situation as for $W_n$ with the generator $s_{n-1}$ exchanged for the generator $s_n$. Thus we get one additional representative if and only if $s_{n-1}$ actually appears in one of the representatives of $[(W_0 \cap W_n) \backslash W_n / (W_0 \cap W_n)]$ ,which is the case precisely for even $n$, where we obtain the additional representative just as stated by exchanging $s_{n-1}$ for $s_n$ in the last representative of $[(W_0 \cap W_n) \backslash W_n / (W_0 \cap W_n)]$. The assertion follows. In order to identify the translations whose double cosets we have found, one takes an explicit realization of the $D_n$-root system. See Theorem 5.2.17 for details □

Let us now handle the remaining exceptional cases.

**Theorem 5.2.10** *Let $\mathbb{G}$ be of type $F_4$. We label the nodes of the extended Dynkin diagram as follows:*




*Since $\mathbb{G}$ is adjoint as well as simply connected, the Hecke algebra $H(\mathbb{G}(k_{\mathfrak{p}}), P_0)$ is a polynomial ring in 4 variables and the Eichler subalgebra is generated by the double cosets corresponding to*

$$s_0 \text{ and } s_0 s_1 s_2 s_3 s_2 s_1 s_0. \tag{5.77}$$

*These are the cosets of the translations*

$$t(\omega_1^\vee) \text{ and } t(\omega_4^\vee). \tag{5.78}$$

*Proof.* We easily compute

$$[(W_0 \cap W_i)\backslash W_i/(W_0 \cap W_i)] = \{\mathrm{id}, s_0\} \text{ for } i = 1, 2, 3, \tag{5.79}$$

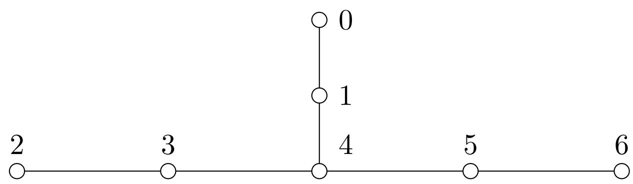
and

$$[(W_0 \cap W_4)\backslash W_4/(W_0 \cap W_4)] = \{\mathrm{id}, s_0, s_0s_1s_2s_3s_2s_1s_0\}. \tag{5.80}$$

One checks that these double cosets correspond to the given translations by considering an explicit realization of an $F_4$-root system. □

**Theorem 5.2.11** *Let $\mathbb{G}$ be of type $E_6$. We label the nodes of the extended Dynkin diagram as follows:*

$\widetilde{E}_6$ :




*The Eichler subalgebra is generated by the double cosets corresponding to*

$$s_0 \text{ and } s_0s_1s_4s_3s_5s_4s_1s_0. \tag{5.81}$$

*These are the double cosets of the translations*

$$t(\omega_1^\vee) \text{ and } t(\omega_2^\vee + \omega_6^\vee). \tag{5.82}$$

*Proof.* We easily compute

$$[(W_0 \cap W_i)\backslash W_i/(W_0 \cap W_i)] = \{\mathrm{id}, s_0\} \text{ for } i = 1, 3, 4, 5, \tag{5.83}$$

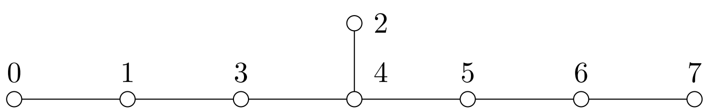
and

$$[(W_0 \cap W_i)\backslash W_i/(W_0 \cap W_i)] = \{\mathrm{id}, s_0, s_0s_1s_4s_3s_5s_4s_1s_0\} \text{ for } i = 2, 6. \tag{5.84}$$

One checks that these double cosets correspond to the given translations by considering an explicit realization of an $E_6$-root system. □

**Theorem 5.2.12** *Let $\mathbb{G}$ be of type $E_7$. We label the nodes of the extended Dynkin diagram as follows:*

$\widetilde{E}_7$ :



*The Eichler subalgebra is generated by the double cosets corresponding to*

$$\begin{aligned} &s_0, s_0s_1s_3s_4s_5s_2s_4s_3s_1s_0 \textit{ and} \\ &w_7 := s_0s_1s_3s_4s_2s_5s_6s_4s_5s_3s_4s_2s_1s_3s_4s_5s_6s_0s_1s_3s_4s_2s_5s_4s_3s_1s_0 \end{aligned} \tag{5.85}$$

*These are the double cosets of the translations*

$$t(\omega_1^\vee), t(\omega_5^\vee) \textit{ and } t(2\omega_6^\vee). \tag{5.86}$$

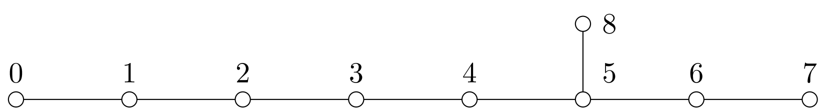
*Proof.* We compute

$$\begin{aligned} &[(W_0 \cap W_i)\backslash W_i/(W_0 \cap W_i)] = \{\mathrm{id}, s_0\} \text{ for } i = 1,2,3,4,5 \\ &[(W_0 \cap W_5)\backslash W_6/(W_0 \cap W_6)] = \{\mathrm{id}, s_0, s_0s_1s_3s_4s_2s_5s_4s_3s_1s_0\} \text{ and} \\ &[(W_0 \cap W_7)\backslash W_7/(W_0 \cap W_7)] = \{\mathrm{id}, s_0, s_0s_1s_3s_4s_2s_5s_4s_3s_1s_0, w_7\}. \end{aligned} \tag{5.87}$$

The corresponding translations can be found by choosing an explicit realization of a root system of type $E_7$. □

**Theorem 5.2.13** *Let $\mathbb{G}$ be of type $E_8$. We label the nodes of the extended Dynkin diagram as follows:*

$\widetilde{E}_8$ :




*The Eichler subalgebra is generated by the double cosets corresponding to*

$$s_0 \textit{ and } s_0s_1s_2s_3s_4s_5s_6s_8s_5s_4s_3s_2s_1s_0. \tag{5.88}$$

*These are the double cosets of the translations*

$$t(\omega_2^\vee) \textit{ and } t(\omega_1^\vee). \tag{5.89}$$

*Proof.* We compute

$$\begin{aligned} &[(W_0 \cap W_i)\backslash W_i/(W_0 \cap W_i)] = \{\mathrm{id}, s_0\} \text{ for } i = 1,2,3,4,5,6,8 \text{ and} \\ &[(W_0 \cap W_7)\backslash W_7/(W_0 \cap W_7)] = \{\mathrm{id}, s_0s_1s_2s_3s_4s_5s_6s_8s_5s_4s_3s_2s_1s_0\} \end{aligned} \tag{5.90}$$

The corresponding translations can be found by choosing an explicit realization of a root system of type $E_8$. □

Let us summarize the results of this section in the following theorem.

**Theorem 5.2.14** *The Hecke algebra $H_{P_0}$ is always a commutative integral domain of transcendence degree* rank($\mathbb{G}$) *(even a polynomial ring in* rank($\mathbb{G}$) *indeterminates for $\mathbb{G}$ of type $G_2, F_4, E_8$ or $C_n, n \geq 2$). The Eichler subalgebra always constitutes a polynomial ring inside $H_{P_0}$ and the number of indeterminates is given in the following table:*

| *Dynkin type* | *Number of indeterminates* |
|---|---|
| $A_n,\ n \geq 1$ | $\lfloor \frac{n+1}{2} \rfloor$ |
| $B_n,\ n \geq 3$ | $\lfloor \frac{n}{2} \rfloor + 1$ |
| $C_n,\ n \geq 2$ | $n$ |
| $D_n,\ n \geq 4,\ n\ even$ | $\frac{n}{2} + 2$ |
| $D_n,\ n \geq 5,\ n\ odd$ | $\lfloor \frac{n}{2} \rfloor + 1$ |
| $E_6$ | 2 |
| $E_7$ | 3 |
| $E_8$ | 2 |
| $F_4$ | 2 |
| $G_2$ | 1 |

### 5.2.4. Some Adjoint Groups

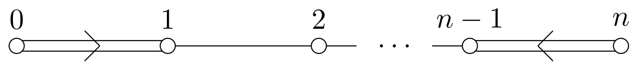
Let $\mathbb{G}$ now be almost simple of adjoint type (and otherwise let $\mathbb{G}$ have the same properties as before). We once more choose a prime $\mathfrak{p}$ at which $\mathbb{G}$ is split and we also choose an Iwahori subgroup $I$ of $\mathbb{G}(k_\mathfrak{p})$. As before we denote the generators of the affine Weyl group (with respect to a suitably chosen torus) by $s_0, s_1, ..., s_n$ and the extended affine Weyl group is now generated by $s_0, ..., s_n$ and a finite Abelian group $\Omega$ fixing $\{s_0, ..., s_n\}$ under conjugation. Let $W_0 = \langle s_1, ..., s_n \rangle$ and $P_0 := IW_0I$; then $P_0$ is again a hyperspecial maximal compact subgroup and we are interested in the (local) Hecke algebra $H(\mathbb{G}(k_\mathfrak{p}), P_0)$. As in the simply connected case we define the Eichler subalgebra of $H(\mathbb{G}(k_\mathfrak{p}), P_0)$ as the algebra generated by the Eichler elements $\nu(P_0, P')$, where $P'$ runs through the other maximal compact subgroups of $\mathbb{G}(k_\mathfrak{p})$ containing $I$.

While the local situation looks more complicated than in the simply connected case, since the extended affine Weyl group is no longer a Coxeter group, the chances of the Eichler subalgebra coinciding with the full Hecke algebra are actually better. This has two reasons: On the one hand there are now potentially more maximal compact subgroups containing $I$ than in the simply connected case and on the other hand the Hecke algebra $H(\mathbb{G}(k_\mathfrak{p}), P_0)$ is a polynomial ring in $n$ variables given by the double cosets with respect to the fundamental coweights.

To illustrate this, let us look at the almost simple, adjoint groups for which $\Omega$ is a (nontrivial) elementary Abelian 2-group. We will first consider the groups of type $C_n$, where we already studied the simply connected case in detail.

**Theorem 5.2.15** *Let $\mathbb{G}$ be adjoint of type $C_n$. We label the nodes of the extended Dynkin diagram as follows:*

$\widetilde{C}_n$ :




*The finite group $\Omega$ is cyclic of order two and the generator $\rho$ acts on the simple reflections in $W_{af}$ via*

$$\rho s_i \rho = s_{n-i} \text{ for } 0 \leq i \leq n. \tag{5.91}$$

*Let us, as in Section 5.2.2, denote the product $s_i s_{i+1} ... s_j$ by $s_{[i,j]}$ and the $i$-th fundamental coweight by $\omega_i^\vee$.*

*We have $P_0 t_{\omega_i^\vee} P_0 = P_0 s_{[0,i-1]} s_{[0,i-2]} ... s_0 P_0$ for $1 \leq i \leq n-1$, $P_0 t_{\omega_n^\vee} P_0 = P_0 \rho P_0$, and the corresponding $P_0$-double cosets are contained in the Eichler subalgebra, whence the Eichler subalgebra coincides with the full Hecke algebra.*

*Proof.* We have already seen in Section 5.2.2 that the double cosets $P_0 t_{\omega_i^\vee} P_0$, $1 \leq i \leq n-1$, can be described as above and are contained in the Eichler subalgebra. Furthermore if we set $W_{0,n} := \langle s_1, ..., s_{n-1}, \rho \rangle$ the group $I W_{0,n} I$ is maximal compact and since

$$[(W_0 \cap W_{0,n}) \backslash W_{0,n} / (W_0 \cap W_{0,n})] = \{\mathrm{id}, \rho\}, \tag{5.92}$$

the double coset $P_0 \rho P_0$ is contained in the Eichler subalgebra.

It remains to prove that $P_0 \rho P_0 = P_0 t_{\omega_n^\vee} P_0$. To that end define

$$w_n := (s_{[1,n]})^{-1} (s_{[2,n]}) ... (s_{[n-1,n]})^{-1} s_n \in W_0. \tag{5.93}$$

By using the explicit realization of a $C_n$ root system from Lemma 5.2.3 and characterizing $\Omega$ as the subgroup of $\widetilde{W} = W_0 \ltimes X_*(\mathbb{T})$ stabilizing the set

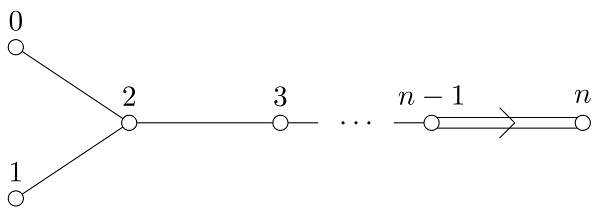
$$C_0 := \{\lambda \in X_*(\mathbb{T}) \otimes \mathbb{R} \mid \langle \lambda, \alpha_i \rangle > 0,\ 1 \leq i \leq n, \text{ and } \langle \lambda, \alpha_0 \rangle < 1\} \tag{5.94}$$

one verifies that $\rho = w_n t_{\omega_n^\vee}$, whence $P_0 t_{\omega_n^\vee} P_0 = P_0 \rho P_0$, which concludes the proof. $\square$

For the groups of type $B_n$ we need to work a little more:

**Theorem 5.2.16** *Let $\mathbb{G}$ be adjoint of type $B_n$. As before we label the nodes of the extended Dynkin diagram as follows:*

$\widetilde{B}_n$ :



*The finite group $\Omega$ is of order two and its generator $\rho$ acts on the simple reflections via the only nontrivial automorphism of the above graph, i.e.:*

$$\rho s_0 \rho = s_1 \textit{ and } \rho s_i \rho = s_i \textit{ for } 2 \leq i \leq n. \tag{5.95}$$

*For ease of notation we denote by $s_{[i,j]}$ either the product $s_i s_{i+1} ... s_j$ or $s_i s_{i-1} ... s_j$ depending on whether $i < j$ or $i > j$. Furthermore - as in the simply connected case - let us denote the element*

$$s_{[2i,1]} s_{[2i+1,2]} s_0 \textit{ for } 2i + 1 \leq n \tag{5.96}$$

*by $v_i$. Then the following identities hold:*

$$\begin{aligned}
&W_0 t(\omega_1^\vee) = W_0 \rho, \\
&W_0 t(\omega_{2i}^\vee) = W_0 s_0 v_1 ... v_{i-1} \textit{ for } 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \textit{ and} \\
&W_0 t(\omega_{2i+1}^\vee) W_0 = W_0 s_0 v_1 ... v_{i-1} s_{[2i,1]} \rho W_0 \textit{ for } 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor.
\end{aligned} \tag{5.97}$$

*Moreover the double cosets with respect to $t_{\omega_i^\vee}$, $1 \leq i \leq n$, are contained in the Eichler subalgebra which hence coincides with the full Hecke algebra.*

*Proof.* We realize the root system $\Phi$ of type $B_n$ in the standard Euclidean space $\mathbb{R}^n$ as

$$\Phi = \{\pm e_i \pm e_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{\pm e_i \mid 1 \leq i \leq n\}. \tag{5.98}$$

We choose the simple system

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, ..., \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\} \tag{5.99}$$

with corresponding highest root $\alpha_0 = e_1 + e_2$ and as usual we realize the dual root sytem in the same vector space via the construction

$$\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha. \tag{5.100}$$

First of all we see that (under this identification) we obtain

$$\omega_i^\vee = \sum_{j=1}^{i} e_j \text{ for } 1 \leq i \leq n, \tag{5.101}$$

in particular $\omega_2^\vee = \alpha_0^\vee$. Thus $W_0 t(\omega_2^\vee) = W_0 t(\alpha_0^\vee) = W_0 s_{\alpha_0} t(\alpha_0^\vee) = W_0 s_0$ and by induction:

$$
\begin{aligned}
W_0 s_0 v_1 ... v_i &= W_0 s_0 v_1 ... v_{i-1} v_i \\
&= W_0 t(\omega_{2i}^\vee) v_i \\
&= W_0 t(\omega_{2i}^\vee) s_{[2i,1]} s_{[2i+1,2]} s_0 \\
&= W_0 t \left( \left( \sum_{j=1}^{2i} e_j \right) s_{[2i,1]} s_{[2i+1,2]} \right) s_{\alpha_0} t(\omega_2^\vee) \\
&= W_0 t \left( \sum_{j=3}^{2i+2} e_j \right) s_{\alpha_0} t(\omega_2^\vee) \\
&= W_0 \left( \sum_{j=1}^{2i+2} e_j \right) \\
&= W_0 t(\omega_{2i+2}^\vee) \text{ for } 2i+2 \leq n.
\end{aligned}
\tag{5.102}
$$

Once more we characterize the finite group $\Omega \leq W_0 \ltimes X_*(\mathbb{T})$ as the stabilizer of the cell

$$
C_0 := \{\lambda \in X_*(\mathbb{T}) \otimes \mathbb{R} \mid \langle \lambda, \alpha_i \rangle > 0,\ 1 \leq i \leq n,\ \text{and } \langle \lambda, \alpha_0 \rangle < 1\}, \tag{5.103}
$$

and thus find $\rho = s_{e_1} t(e_1) = s_{[1,n]} s_{[n-1,1]} t(\omega_1^\vee)$ which is why $W_0 t(\omega_1^\vee) = W_0 \rho$. Furthermore

$$
\begin{aligned}
W_0 s_0 v_1 ... v_{i-1} s_{[2i,0]} s_{[2,2i]} \rho &= W_0 t(\omega_{2i}^\vee) s_{[2i,0]} s_{[2,2i]} s_{[1,n]} s_{[n-1,1]} t(\omega_1^\vee) \\
&= W_0 t \left( \sum_{j=2}^{2i+1} e_j \right) s_{\alpha_0} t(\alpha_0) s_{[2,2i]} s_{[1,n]} s_{[n-1,1]} t(\omega_1^\vee) \\
&= W_0 t \left( \sum_{j=2}^{2i+1} e_j \right) s_{[2,2i]} s_{[1,n]} s_{[n-1,1]} t(\omega_1^\vee) \\
&= W_0 t \left( \sum_{j=2}^{2i+1} e_j \right) s_{e_1} t(e_1) \\
&= W_0 t(\omega_{2i+1}^\vee) \text{ for } 2i+1 \leq n.
\end{aligned}
\tag{5.104}
$$

And so, if we take the double coset with respect to $W_0$, we obtain

$$
\begin{aligned}
W_0 t(\omega_{2i+1}^\vee) W_0 &= W_0 s_0 v_1 ... v_{i-1} s_{[2i,0]} s_{[2,2i]} \rho W_0 \\
&= W_0 s_0 v_1 ... v_{i-1} s_{[2i,1]} \rho W_0.
\end{aligned}
\tag{5.105}
$$

This proves the first assertion and it remains to prove that all of these double cosets actually lie in the Eichler subalgebra.

Let us denote by $W_i,\ 2 \le i \le n$, the group generated by $\rho$ and all the simple reflections except for $s_i$,

$$W_i := \langle \rho, s_0, s_1, ..., \hat{s_i}, ..., s_n \rangle, \tag{5.106}$$

and by $W_{01}$ the group generated by $\rho$ and $s_2, ..., s_n$. By construction the groups $P_i := IW_iI,\ 2 \le i \le n$ and $P_{01} := IW_{01}I$ are maximal compact subgroups of $\mathbb{G}(k_\mathfrak{p})$ containing $I$. To prove our assertion we will show that the double cosets of $W_i$ with respect to $W_i \cap W_0$ are represented by the representatives for the $W_0$-double cosets of $\mathrm{id}, t(\omega_1^\vee), ..., t(\omega_i^\vee)$ for $2 \le i \le n$ which we have chosen above. Since $[(W_0 \cap W_{01})\backslash W_{01}/(W_0 \cap W_{01})] = \{\mathrm{id}, \rho\}$, we can then iteratively obtain $P_0 t(\omega_i^\vee) P_0$ from the Eichler elements $\nu(P_0, P_{01})$ and $\nu(P_0, P_j),\ 2 \le j \le i$.

First of all note that indeed

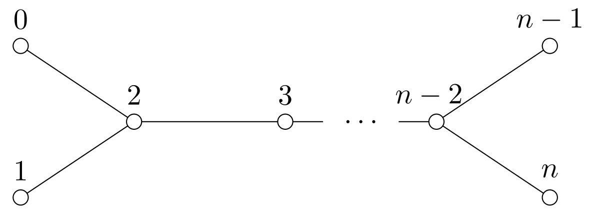
$$\begin{aligned}
&[(W_0 \cap W_{01})\backslash W_0/(W_0 \cap W_{01})] = \{\mathrm{id}, \rho\},\\
&[(W_0 \cap W_2)\backslash W_0/(W_0 \cap W_2)] = \{\mathrm{id}, \rho, s_0\}, \text{ and}\\
&[(W_0 \cap W_3)\backslash W_0/(W_0 \cap W_3)] = \{\mathrm{id}, \rho, s_0, s_0 s_2 s_1 \rho\}.
\end{aligned} \tag{5.107}$$

Moreover our representatives for $t(\omega_j^\vee)$, $1 \le j \le i$, are actually contained in $W_i$ and define distinct $(W_0 \cap W_i)$-double cosets (since they already define distinct $W_0$-double cosets in $\widetilde{W}$). Thus we are done if we can show that $|W_i /\!\!/ (W_0 \cap W_i)| = i + 1$ for $4 \le i \le n$.

To that end first note that there is an obvious bijection between $W_i /\!\!/ (W_0 \cap W_i)$ and $(D_i \rtimes C_2) /\!\!/ A_{i-1}$ where (in suggestive notation) $D_i$ denotes a Coxeter group of type $D_i$ and $A_{i-1}$ either one of the two standard parabolic subgroups of type $A_{i-1}$. The double cosets in $(D_i \rtimes C_2) /\!\!/ A_{i-1}$ fall into two categories. The first one contains the double cosets that have a representative in $D_i$ (in which case the whole coset is contained in $D_i$) and is thus in natural bijection with $D_i /\!\!/ A_{i-1}$. The other one contains the remaining cosets and is in natural bijection with $A_{i-1}\backslash D_i / A_{i-1}^\rho$ (where $\rho$ again generates the $C_2$). By once more using the explicit isomorphism $D_i \cong V \rtimes S_i$, where $V \subset \mathbb{F}_2^i$ is the subspace orthogonal on the all-ones vector, we verify that $|D_i /\!\!/ A_{i-1}| = \left\lfloor \frac{i}{2} \right\rfloor + 1$ and $|A_{i-1}\backslash D_i / A_{i-1}^\rho| = \left\lfloor \frac{i-1}{2} \right\rfloor + 1$, whence $|W_i /\!\!/ (W_0 \cap W_i)| = i + 1$ which concludes our proof. $\square$

**Theorem 5.2.17** *Let $\mathbb{G}$ be adjoint of type $D_n$ with $n \ge 4$ even. We label the nodes of the extended Dynkin diagram as follows:*

$\widetilde{D}_n$ :



*The finite group $\Omega$ is isomorphic to $C_2 \times C_2$ and generated by two elements $\rho_1$ and $\rho_2$ acting as follows. The element $\rho_1$ acts on the simple reflections by interchanging $s_0$ with $s_1$, and $s_{n-1}$ with $s_n$ whereas $\rho_2$ acts via $\rho_2 s_i \rho_2 = s_{n-i}$ for $0 \leq i \leq n$.*

*The Eichler subalgebra contains the double cosets with respect to all fundamental coweights and thus coincides with the full Hecke algebra.*

*Proof.* We realize the root system $\Phi$ of type $D_n$ in the standard Euclidean space $\mathbb{R}^n$ as

$$\Phi = \{\pm e_i \pm e_j \mid 1 \leq i, j \leq n, i \neq j\}. \tag{5.108}$$

We choose the simple system

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, ..., \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\} \tag{5.109}$$

with corresponding highest root $\alpha_0 = e_1 + e_2$ and as usual we realize the dual root sytem in the same vector space via the construction

$$\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha. \tag{5.110}$$

First of all we see that (under this identification) we obtain

$$\omega_i^\vee = \sum_{j=1}^{i} e_j \text{ for } 1 \leq i \leq n-2, \; \omega_{n-1}^\vee = \frac{1}{2}\left(\sum_{j=1}^{n-1} e_j - e_n\right) \text{ and } \omega_n^\vee = \frac{1}{2}\sum_{j=1}^{n} e_j. \tag{5.111}$$

In particular $\omega_2^\vee = \alpha_0^\vee$. We define the symbols $s_{[i,j]}$ and $v_i$ as in the $B_n$-case.

Thus $W_0 t(\omega_2^\vee) = W_0 t(\alpha_0^\vee) = W_0 s_{\alpha_0} t(\alpha_0^\vee) = W_0 s_0$ and by induction:

$$\begin{aligned}
W_0 s_0 v_1 ... v_i &= W_0 s_0 v_1 ... v_{i-1} v_i \\
&= W_0 t(\omega_{2i}^\vee) v_i \\
&= W_0 t(\omega_{2i}^\vee) s_{[2i,1]} s_{[2i+1,2]} s_0 \\
&= W_0 t \left( \left( \sum_{j=1}^{2i} e_j \right) s_{[2i,1]} s_{[2i+1,2]} \right) s_{\alpha_0} t(\omega_2^\vee) \\
&= W_0 t \left( \sum_{j=3}^{2i+2} e_j \right) s_{\alpha_0} t(\omega_2^\vee) \\
&= W_0 \left( \sum_{j=1}^{2i+2} e_j \right) \\
&= W_0 t(\omega_{2i+2}^\vee) \text{ for } 2i+2 \leq n-2.
\end{aligned} \tag{5.112}$$

Furthermore let us define for $1 \leq i \leq n-1$ the element $\sigma_i := s_n^{s_{[n-2,i]} s_{[n-1,i+1]}} s_i$ which acts as $s_{e_i + e_{i+1}} s_{e_i - e_{i+1}}$ on $\mathbb{R}^n$.

One checks that $\rho_1 = w_1 t(\omega_1^\vee)$ where $w_1 = s_2 s_{\alpha_0} s_1 s_2 \sigma_3 \sigma_4 ... \sigma_{n-1}$, and in particular $W_0 \rho_1 W_0 = W_0 t(\omega_1^\vee) W_0$. Moreover we have

$$W_0 s_0 v_1 ... v_{i-1} s_{[1,2i]} w_1 \rho_1 = W_0 t \left( \sum_{j=2}^{2i+1} e_j \right) t(e_1) = W_0 t(\omega_{2i+1}^\vee) \text{ for } 2i+1 \leq n-2. \tag{5.113}$$

As for $\rho_1$ one checks that $\rho_2 = s_{e_1+e_n} s_{e_2+e_{n-1}} ... s_{e_l+e_{l+1}} t(\omega_n^\vee)$ (where $l = \frac{n}{2}$) and thus $\rho_1\rho_2 = w_1 s_{e_1+e_n} s_{e_2+e_{n-1}} ... s_{e_l+e_{l+1}} t(\omega_{n-1}^\vee)$.

Let $W_{0,1} := \langle s_2, ..., s_n, \rho_1 \rangle \leq \widetilde{W}$ then $IW_{0,1}I$ is a maximal compact subgroup of $\mathbb{G}(k_\mathfrak{p})$ and $[(W_{01} \cap W_0)\backslash W_{01}/(W_{01} \cap W_0)] = \{\mathrm{id}, \rho_1\}$ thus the $P_0$-double coset with respect to $t(\omega_1^\vee)$ is contained in the Eichler subalgebra. Analogously we obtain $t(\omega_n^\vee)$ and $t(\omega_{n-1}^\vee)$ from $W_{0,n} = \langle s_1, ..., s_{n-1}, \rho_2 \rangle$ and $W_{0,n-1} = \langle s_1, ..., s_{n-2}, s_n, \rho_1\rho_2 \rangle$.

It remains to prove that the double cosets with respect to $t(\omega_i^\vee)$, $2 \leq i \leq n-2$, lie in the Eichler subalgebra. To that end set $W_i := \langle s_0, ..., \hat{s}_i, ..., s_n, \rho_1 \rangle$ for $i \neq \frac{n}{2}$ and $W_i := \langle s_0, ..., \hat{s}_i, ..., s_n, \rho_1, \rho_2 \rangle$ for $i = \frac{n}{2}$. We will show that the double cosets that appear in the Eichler element with respect to $P_0$ and $IW_iI$ are $P_0, P_0 t(\omega_1^\vee) P_0, ..., P_0 t(\omega_i^\vee) P_0$, and in addition if $i = \frac{n}{2}$ the double cosets $P_0 t(\omega_{n-1}^\vee) P_0$ and $P_0 t(\omega_n^\vee) P_0$.

First we will demonstrate that $W_0 t(\omega_i^\vee) W_0$ has a representative that is contained in $W_i$. If $i = 2j$ is even we have the representative $s_0 v_1 ... v_{j-1}$ which is already an element of $W_i$. For odd $i = 2j+1$ we compute:

$$\begin{aligned}
W_0 s_0 v_1 ... v_{j-1} s_{[1,2j]} w_1 \rho_1 W_0 &= W_0 t \left( \sum_{l=2}^{2j+1} e_l \right) s_2 s_{\alpha_0} s_1 s_2 \rho_1 W_0 \\
&= W_0 t \left( \sum_{l=2}^{2j+1} e_l \right) \rho_1 W_0 \\
&= W_0 s_0 v_1 ... v_{j-1} s_{[1,2j]} \rho_1 W_0.
\end{aligned} \tag{5.114}$$

Hence $s_0 v_1 ... v_{j-1} s_{[1,2j]} \rho_1$ is the desired representative. As in the $B_n$-case it remains to prove $|W_i /\!\!/ (W_i \cap W_0)| = i+1$ for $2 \leq i \leq n-2$ and $i \neq \frac{n}{2}$, and $|W_i /\!\!/ (W_i \cap W_0)| = i+3$ for $i = \frac{n}{2}$.

For $2 \leq i \leq n-2$ and $i \neq \frac{n}{2}$ we have (in suggestive notation):

$$\begin{aligned}
|W_i /\!\!/ (W_0 \cap W_i)| &= |((D_i \times D_{n-i}) \rtimes \langle \rho_1 \rangle) /\!\!/ (A_{i-1} \times D_{n-i})| \\
&= |(D_i \times D_{n-i}) /\!\!/ (A_{i-1} \times D_{n-i})| \\
&\quad + |(A_{i-1} \times D_{n-i})^{\rho_1} \backslash (D_i \times D_{n-i}) / (A_{i-1} \times D_{n-i})| \\
&= |D_i /\!\!/ A_{i-1}| + |A_{i-1}^{\rho_1} \backslash D_i / A_{i-1}| \\
&= i+1.
\end{aligned} \tag{5.115}$$

The last equality was already considered at the end of the proof of the $B_n$-case.

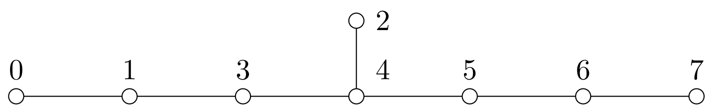
Finally for $i = \frac{n}{2}$ we obtain:

$$\begin{aligned}
|W_i /\!\!/ (W_0 \cap W_i)| &= |((D_i \times D_i) \rtimes \Omega) /\!\!/ (A_{i-1} \times D_i)| \\
&= \sum_{\rho \in \Omega} |(A_{i-1} \times D_i)^{\rho} \backslash (D_i \times D_i)/(A_{i-1} \times D_i)| \\
&= |D_i /\!\!/ A_{i-1}| + |A_{i-1}^{\rho_1} \backslash D_i / A_{i-1}| \\
&\quad + |(D_i \times A_{i-1}^{\rho_2}) \backslash (D_i \times D_i)/(A_{i-1} \times D_i)| \\
&\quad + |(D_i \times A_{i-1}^{\rho_1 \rho_2}) \backslash (D_i \times D_i)/(A_{i-1} \times D_i)| \\
&= (i+1) + 1 + 1 = i + 3.
\end{aligned} \tag{5.116}$$

This concludes the proof. □

**Theorem 5.2.18** *Let $\mathbb{G}$ be adjoint of type $E_7$. We label the nodes of the extended Dynkin diagram as follows:*

$\widetilde{E}_7$ :




*The finite group $\Omega$ is a cyclic group of order 2, generated by the element $\rho$ which acts on the simple reflections as follows:*

$$s_0^{\rho} = s_7, s_1^{\rho} = s_6, s_2^{\rho} = s_2, s_3^{\rho} = s_5, s_4^{\rho} = s_4. \tag{5.117}$$

*The Eichler subalgebra is a polynomial ring in 4 indeterminates, given by the double cosets with respect to the reflections*

$$t(\omega_1^{\vee}), t(\omega_2^{\vee}), t(\omega_6^{\vee}) \text{ and } t(\omega_7^{\vee}). \tag{5.118}$$

*Proof.* By writing down the maximal compact subgroups explicitly we see that the double cosets appearing in the Eichler elements are those already found in Theorem 5.2.12 and additionally those corresponding to the elements $\rho$ and $s_0 s_1 s_3 s_4 s_5 s_6 s_7 \rho$. We compute

$$\begin{aligned}
&P_0 \rho P_0 = P_0 t(\omega_7^{\vee}) P_0, P_0 s_0 P_0 = P_0 t(\omega_1^{\vee}) P_0, P_0 s_0 s_1 s_3 s_4 s_2 s_5 s_4 s_3 s_1 s_0 P_0 = t(\omega_6^{\vee}) \\
&P_0 s_0 s_1 s_3 s_4 s_5 s_6 s_7 \rho P_0 = P_0 t(\omega_2^{\vee}) P_0 \text{ and } P_0 w_7 P_0 = P_0 t(2\omega_7^{\vee}) P_0,
\end{aligned} \tag{5.119}$$

where $w_7$ is chosen as in Theorem 5.2.12. Since $P_0 t(2\omega_7^{\vee}) P_0$ is already contained in the subalgebra generated by $P_0 t(\omega_7^{\vee}) P_0$, the assertion follows. □

# 6. Computational Framework

In this chapter we want to establish the results necessary to make our algorithms applicable to certain examples. Our implementions deal with algebraic modular forms for groups of type $G_2$ and (simply connected) groups of type $C_n$. Thus we briefly describe the structure of these groups (mainly over local fields) and explain how to construct compact and integral forms.

## 6.1. Algebraic Modular Forms for $G_2$

Here we want to review some facts about linear algebraic groups of type $G_2$ which we need in order to compute their algebraic modular forms. We start by looking at split forms of these groups and their structure over local fields (in particular the structure of the Euclidean building and the maximal compact subgroups). Afterwards we briefly concern ourselves with the construction of compact forms of $G_2$.

### 6.1.1. Split Groups of Type $G_2$

To describe the split group $G_2$ we follow the exposition in [CNP98]. Let $k$ be a field of characteristic not 2 and $V_0 := \langle e_1, ..., e_7 \rangle$ a 7-dimensional $k$-vector space. The split Cayley algebra $\mathbb{O} = \mathbb{O}(k) = \mathbb{O}_s(k)$ ($s$ for split) is as a vector space equal to $\langle 1, e_1, ..., e_7 \rangle \cong V_0 \oplus k$ with multiplication given by the following table:

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $e_1$ | $e_2$ | $-e_3$ | $\frac{1}{2}(1-e_4)$ |
| $e_2$ | $0$ | $0$ | $e_1$ | $-e_2$ | $0$ | $\frac{1}{2}(1+e_4)$ | $-e_5$ |
| $e_3$ | $0$ | $-e_1$ | $0$ | $-e_3$ | $\frac{1}{2}(1+e_4)$ | $0$ | $e_6$ |
| $e_4$ | $-e_1$ | $e_2$ | $e_3$ | $1$ | $-e_5$ | $-e_6$ | $e_7$ |
| $e_5$ | $-e_2$ | $0$ | $\frac{1}{2}(1-e_4)$ | $e_5$ | $0$ | $e_7$ | $0$ |
| $e_6$ | $e_3$ | $\frac{1}{2}(1-e_4)$ | $0$ | $e_6$ | $-e_7$ | $0$ | $0$ |
| $e_7$ | $\frac{1}{2}(1+e_4)$ | $e_5$ | $-e_6$ | $-e_7$ | $0$ | $0$ | $0$ |

It is easily seen that this algebra is neither associative nor commutative.

Thinking of the algebra automorphism of $\mathbb{O}$ as elements of $\mathrm{GL}_8(k)$ allows for the following definiton.

**Definition 6.1.1** *The linear algebraic group $G_2$ (over $k$) is given by*

$$G_2(A) := \mathrm{Aut}_{A\text{-}\mathsf{alg}}(A \otimes_k \mathbb{O}(k)), \tag{6.1}$$

*for every commutative $k$-algebra $A$.*

This is in fact a linear algebraic group as it is the stabilizer of the multiplication in the general linear group, where we can think of the multiplication as an element of $\mathbb{O}^* \otimes \mathbb{O}^* \otimes \mathbb{O}$.

We state the following result on the structure of $G_2$ without proof.

**Lemma 6.1.2 ([Spr98, Thm. 17.4.4])** *$G_2$ is connected as a linear algebraic group.*

The algebra $\mathbb{O}$ carries an involution $\bar{\ }$ acting trivially on $k$ and as the negative identity on $V_0$, and we have $x\bar{x} = \bar{x}x \in k$ for all $x \in \mathbb{O}$. The action of $G_2$ on $\mathbb{O}$ fixes the subspaces $k$ and $V_0$ and thus commutes with the involution $\bar{\ }$. In particular this action preserves the quadratic form $V_0 \to k, x \mapsto x\bar{x}$ which has Gram matrix

$$\Psi := \begin{pmatrix} &&&&&&1\\ &&&&&1&\\ &&&&1&&\\ &&&2&&&\\ &&1&&&&\\ &1&&&&&\\ 1&&&&&& \end{pmatrix} \tag{6.2}$$

with respect to the basis $e_1, ..., e_7$. This observation yields an embedding of $G_2$ into the (split) special orthogonal group of degree 7 (defined by $\Psi$) and moreover the Lie algebra of $G_2$ naturally embeds into that of $\mathrm{SO}_7$ which we identify with the following vector space:

$$\mathfrak{g}_2 := \mathrm{Lie}(G_2) \subset \mathrm{Lie}(\mathrm{SO}_7) = \mathfrak{so}_7 = \left\{ x \in k^{7\times 7} \mid x\Psi + \Psi x^{tr} = 0 \right\}. \tag{6.3}$$

Following [CNP98, 4.2] we find the following basis for $\mathfrak{g}_2$: The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_2$ is of dimension 2, generated by the elements

$$h_1 := \mathrm{diag}(0, 1, -1, 0, 1, -1, 0) \text{ and } h_2 := \mathrm{diag}(-1, -1, 0, 0, 0, 1, 1). \tag{6.4}$$

In particular we see that $G_2$ is of rank 2 and that

$$T := G_2(k) \cap \mathrm{diag}((k^*)^7) = \left\{ \mathrm{diag}(y^{-1}, xy^{-1}, x^{-1}, 1, x, x^{-1}y, y) \mid x, y \in k^* \right\} \tag{6.5}$$

is a maximal torus in $G_2$. We have $X_*(T) = \langle \eta_1, \eta_2 \rangle_{\mathbb{Z}}$ where

$$\eta_1(x) = \mathrm{diag}(1, x, x^{-1}, 1, x, x^{-1}, 1) \text{ and } \eta_2(x) = \mathrm{diag}(x^{-1}, x^{-1}, 1, , 1, 1, x, x). \tag{6.6}$$

The two simple roots $\alpha_1$ and $\alpha_2$ (with respect to the adjoint action of either $T$ or $\mathfrak{h}$ on $\mathfrak{g}_2$) take values 2 and $-1$ on $h_1$, and $-1$ and 0 on $h_2$. One possible choice for the root vectors is $g_{\alpha_1} = -E_{2,3} + E_{5,6}$ and $g_{\alpha_2} = E_{1,2} - E_{3,4} + 2E_{4,5} - E_{6,7}$. The other positive roots are $\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2$ and $2\alpha_1 + 3\alpha_2$; one finds their respective root vectors as successive Lie products of the root vectors of $\alpha_1$ and $\alpha_2$. Analogously we see that $g_{-2\alpha_1 - 3\alpha_2} = E_{6,1} - E_{7,2}$ and find the other root vectors for the negative roots again by forming appropriate Lie products.

In conclusion we set

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}, \tag{6.7}$$

and $\Phi := \Phi^+ \sqcup -\Phi^+$ to find that $\mathfrak{g}_2 = \langle h_1, h_2, g_\gamma \mid \gamma \in \Phi \rangle$ as a vector space.

Returning to the torus defined above we set $\chi_1, \chi_2 \in X^*(T)$ the basis of the space of characters which is dual to $\eta_1, \eta_2$ under the usual pairing, i.e. we have

$$\chi_1(t) = t_{5,5} \text{ and } \chi_2(t) = t_{7,7}. \tag{6.8}$$

Under this definition we have $\alpha_1 = 2\chi_1 - \chi_2$ and $\alpha_2 = -\chi_1$, and we easily see that $\langle \Phi \rangle = X^*(T)$ so $G_2$ is an adjoint group. On the other hand we have the fundamental cocharacters $\omega_1 = -\eta_2$ and $\omega_2 = -\eta_1 - 2\eta_2$ which fulfill $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$. As a linear combination of the fundamental cocharacters we compute the coroots as

$$\alpha_1^\vee = 2\omega_1 - \omega_2 \text{ and } \alpha_2^\vee = -3\omega_1 + 2\omega_2. \tag{6.9}$$

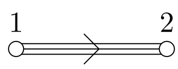
Hence

$$\langle \alpha_1^\vee, \alpha_2^\vee \rangle = \langle \omega_1, \omega_2 \rangle = \langle \Phi \rangle^\# = (X^*(T))^\# = X_*(T), \tag{6.10}$$

so $G_2$ is simply connected as well.

Lastly we note that the ordinary Dynkin diagram of $G_2$ takes the form

$G_2$ :



### 6.1.2. The Group $G_2$ over Local Fields

Here we want to study the (split) groups of type $G_2$ over local ground fields. To that end let $F$ be a local field of characteristic 0 with ring of integers $\mathcal{O}_F$, uniformizer $\pi$ and $\mathcal{O}_F / \pi \mathcal{O}_F \cong \mathbb{F}_q$ of characteristic $p$. Otherwise we carry over the notation from the last subsection.

We want to describe a realization of the affine building for $G_2(F)$, the (extended) affine Weyl group and the maximal parahorics. To describe the building we follow the description in [GY03] though the results are essentially equivalent to those obtained in [CNP98].

The vertices of the affine building $\mathcal{B}$ of $G_2(F)$ correspond to certain $\mathcal{O}_F$-orders in $\mathbb{O}(F)$ which fall into three categories as follows:

By $\mathcal{V}_1$ we denote the set of maximal $\mathcal{O}_F$-orders in $\mathbb{O}(F)$; for an element $L$ of $\mathcal{V}_1$ we say $L$ is an order of type or label 1. A representative of $\mathcal{V}_1$ is given by

$$L_1 := \langle e_1, ..., e_7, \frac{1}{2}(1+e_4)\rangle_{\mathcal{O}_F}. \tag{6.11}$$

The set $\mathcal{V}_2$ consists of all $\mathcal{O}_F$-orders $L$ in $\mathbb{O}(F)$ with the property that $\pi L^{\#}$ is an ideal of $L$ with $(\pi L^{\#})^2 \subset \pi L$, where $L^{\#}$ denotes the dual lattice with respect to the quadratic form defined before. A representative of $\mathcal{V}_1$ is given by

$$L_2 := \langle \pi e_1, \pi e_2, e_3, e_4, ..., e_7, \frac{1}{2}(1+e_4)\rangle_{\mathcal{O}_F}, \tag{6.12}$$

and we call the elements of $\mathcal{V}_2$ orders of type or label 2.

The set $\mathcal{V}_3$ consists of all $\mathcal{O}_F$-orders $L$ in $\mathbb{O}(F)$ with the following properties: $L \subset L^{\#} \subset \pi^{-1}L$ and $M := \pi\left(L^{\#}\right)^2 + L$ is a self dual lattice. A representative of $\mathcal{V}_3$ is given by

$$L_3 := \langle \pi e_1, \pi e_2, \pi e_3, e_4, ..., e_7, \frac{1}{2}(1+e_4)\rangle_{\mathcal{O}_F}. \tag{6.13}$$

and we call the elements of $\mathcal{V}_3$ orders of type or label 3.

Now set $\mathcal{V} := \mathcal{V}_1 \sqcup \mathcal{V}_2 \sqcup \mathcal{V}_3$, then $\mathcal{V}$ is precisely the set of vertices of $\mathcal{B}$. The adjacency relation in $\mathcal{B}$ is given by inclusion, i.e. chambers in $\mathcal{B}$ are in bijection with triples $(L_1', L_2', L_3')$ with $L_i \in \mathcal{V}_i$ and $L_1' \supset L_2' \supset L_3'$. In particular the triple $(L_1, L_2, L_3)$ constitutes a chamber and its stabilizer in $G_2(F)$ is the Iwahori subgroup $I$ which is (following [SFW67]) generated by

$$I = \langle h_\gamma(t), \exp(a g_\gamma), \exp(\pi a g_{-\gamma}) \mid t \in \mathcal{O}_F^*, a \in \mathcal{O}_F, \gamma \in \Phi^+\rangle, \tag{6.14}$$

where for $\gamma \in \Phi$ and $t \in \mathcal{O}_F^*$ we set $w_\gamma(t) = \exp(t g_\gamma)\exp(-t^{-1} g_{-\gamma})\exp(t g_\gamma)$ and $h_\gamma(t) = w_\gamma(t) w_\gamma(1)$.

Keeping this notation and employing [SFW67, Lemma 19] we can find (representatives of) the generators of the affine Weyl group $W_{af}$ which coincides with $\widetilde{W}$ since $G_2$ is simply connected. We find that the finite Weyl group is generated by the following elements (written as permutations in suggestive notation):

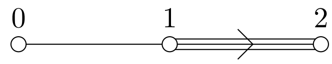
$$\begin{aligned}
&W_0 = \langle s_1, s_2\rangle, \text{ where} \\
&\quad s_1 = w_{\alpha_1}(1) = (e_2, -e_3, -e_2, e_3)(e_5, e_6, -e_5, -e_6) \text{ and} \\
&\quad s_2 = w_{\alpha_2}(1) = (e_1, e_2, -e_1, -e_2)(e_3, -e_5)(e_4, -e_4)(e_6, -e_7, -e_6, e_7).
\end{aligned} \tag{6.15}$$

The affine Weyl group is generated by $s_1, s_2$ and the element

$$s_0 := w_{-2\alpha_1-3\alpha_2}(\pi) = (e_1, \frac{1}{\pi}e_6, -e_1, -\frac{1}{\pi}e_6)(e_2, -\frac{1}{\pi}e_7, -e_2, \frac{1}{\pi}e_7). \tag{6.16}$$

We find the extended affine Dynkin diagram of $G_2$ to be

$\widetilde{G}_2$ :




For $0 \leq i \leq 2$ we now set $W_i := \langle s_j \mid j \neq i \rangle \leq \widetilde{W}$ and $P_i := IW_iI \leq G_2(F)$. Then the $P_i$ form a system of representatives for the maximal parahoric subgroups and we see

$$P_0 = \mathrm{Stab}_{G_2(F)}(L_1), P_1 = \mathrm{Stab}_{G_2(F)}(L_2), P_2 = \mathrm{Stab}_{G_2(F)}(L_3). \tag{6.17}$$

Hence the maximal parahorics are indeed stabilizers of vertices in $\mathcal{B}$ as one would expect. The only hyperspecial maximal compact subgroups among $P_0, P_1, P_2$ is $P_0$ so the hyperspecials in $G_2(F)$ are precisely the stabilizers of maximal $\mathcal{O}_F$-orders in $\mathbb{O}(F)$.

### 6.1.3. Compact Forms of $G_2$

Let $k$ be a totally real number field. If we want to study algebraic modular forms for $G_2$ over $k$ we need a compact form of this group. While we will later see that for (simply connected) groups of type $C_n$ there is a plethora of options on how to construct such forms, it turns out that for $G_2$ the situation is much more restricted. Following [TBM66, Table II] there are only two forms of $G_2$ over $k$; one is the split form we have already seen, the other one we want to construct now.

Let $V_0' = \langle e_1', ..., e_7' \rangle$ be a $k$-vector space. The 8-dimensional vector space $k \oplus V_0' = \langle 1, e_1', ..., e_7' \rangle$ becomes the definite octonion algebra $\mathbb{O}_d(k)$ via the following rule to define multiplication:

For all $1 \leq r \leq 7$ the linear map $\langle 1, e_r, e_{r+1}, e_{r+3} \rangle \mapsto \left(\frac{-1,-1}{k}\right)$ given by

$$1 \mapsto 1, e_r \mapsto i, e_{r+1} \mapsto j, e_{r+3} \mapsto ij, \tag{6.18}$$

is an isomorphism of $k$-algebras (where we take the indeces modulo 7 if necessary).

As before this allows us to define a linear algebraic group $\mathbb{G}$ over $k$ via

$$\mathbb{G}(A) := \mathrm{Aut}_{A\text{-}\mathsf{alg}}(A \otimes_k \mathbb{O}_d(k)), \tag{6.19}$$

for all commutative $k$-algebras $A$.

As in the split case, the algebra $\mathbb{O}_d(k)$ carries an involution $^{-}$ defined to be trivial on 1 and the negative identity on $V_0'$. Moreover $\mathbb{G}$ again fixes the decomposition $\mathbb{O}_d(k) = k \oplus V_0'$ and thus we have a $\mathbb{G}(k)$-invariant quadratic form $x \mapsto x\bar{x} \in k$. With repsect to the standard basis the Gram matrix of this quadratic form is just the identity matrix which implies that the form is totally positive definite whence $\mathbb{G}(k_\infty)$ is in fact compact.

To see how $\mathbb{G}$ behaves at the completions at the finite primes of $k$ one could once again check the table in [TBM66] to see that there is only one form of $G_2$ over $p$-adic fields and thus $\mathbb{G}(k_\mathfrak{p})$ is necessarily split. On the other hand we want to be able to perform actual computations for the group $\mathbb{G}$ which means we need this result in a more explicit manner.

To do this it suffices to find an algebra isomorphism $\mathbb{O}_d(k_\mathfrak{p}) \to \mathbb{O}_s(k_\mathfrak{p})$ and we will give this isomorphism via a base change matrix expressing $(e_1, ..., e_7, 1)$ in terms of $(e_1', ..., e_7', 1)$.

Let us first assume that $2 \notin \mathfrak{p}$, i.e. $\mathrm{char}(\mathcal{O}_k/\mathfrak{p}) \neq 2$. In this case there exist $a \in \mathcal{O}_\mathfrak{p}^\times, b \in \mathcal{O}_\mathfrak{p}$ such that $a^2 + b^2 + 1 = 0$ by a simple counting argument. A possible base change matrix (row-wise on the imaginary part only) is then given by

$$\begin{pmatrix} 1 & a & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & -b & 1 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2a} & \frac{b}{2a} \\ -\frac{b}{a} & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2a} & \frac{b}{2a} \\ 0 & 0 & 0 & 0 & -\frac{1}{4a} & -\frac{b}{4a^2} & \frac{1}{4a^2} \\ \frac{1}{4a^2} & -\frac{1}{4a} & 0 & \frac{b}{4a^2} & 0 & 0 & 0 \end{pmatrix}. \tag{6.20}$$

Let us now consider the case where $2 \in \mathfrak{p}$. Then there exists $a \in \mathcal{O}_\mathfrak{p}^\times$ such that $a^2 + 7 = 0$ and we find a possible base change matrix (on the imaginary part)

$$\begin{pmatrix} 1 & 1 & 1 & 2 & a & 0 & 0 \\ 1 & 1 & -1 & \frac{12a-2}{11} & \frac{a-24}{11} & -\frac{4a+14}{11} & \frac{2a-4}{11} \\ -\frac{a+2}{8} & \frac{a+2}{8} & -\frac{a+2}{8} & -\frac{a+3}{4} & -\frac{2a-5}{8} & -\frac{a-2}{4} & -\frac{1}{2} \\ -\frac{a-2}{22} & -\frac{6a-12}{11} & \frac{5a+12}{11} & -\frac{2a-59}{22} & \frac{24a+7}{22} & -1 & \frac{1}{2} \\ -\frac{13a+18}{88} & \frac{13a+18}{88} & \frac{13a+18}{88} & \frac{129-191a}{484} & \frac{102a+621}{968} & \frac{14a-171}{-242} & \frac{8a-357}{-484} \\ 0 & \frac{4a+25}{32} & \frac{4a+5}{32} & \frac{79a+150}{352} & \frac{45a-123}{176} & -\frac{37a-30}{176} & \frac{7a+30}{-352} \\ \frac{468a+1319}{-7744} & \frac{852a-5653}{7744} & \frac{468a+835}{-7744} & \frac{1255-1029a}{3872} & \frac{155a+4200}{7744} & \frac{100a+163}{352} & \frac{2a+7}{44} \end{pmatrix} \tag{6.21}$$

We will omit a proof here that these are in fact suitable base changes since this can rather easily be done with a computer.

Seeing that we constructed the affine building of $G_2$ via orders in $\mathbb{O}$ it is natural to consider integral forms coming from $\mathcal{O}_k$-lattices in $\mathbb{O}_d(k)$ or more restrictively from $\mathcal{O}_k$-orders in $\mathbb{O}_d(k)$. In particular we will be interested in orders $L \leq \mathbb{O}_d(k)$ such that $L_\mathfrak{p}$ is a vertex of the affine building of $\mathbb{G}(k_\mathfrak{p})$ for all $\mathfrak{p}$. Necessarily $L$ will be a maximal order at all but finitely many primes. On the other hand we see by looking at the base change matrices above that the standard order $\langle 1, e_1', ..., e_7' \rangle_{\mathcal{O}_k}$ is already a maximal order at all primes $\mathfrak{p}$ with $2 \notin \mathfrak{p}$. Thus there exists an order $L$ such that $L$ is maximal at all primes $\mathfrak{p}$. (Equivalently we could have just taken $L$ to be any globally maximal order, since being a maximal order is a local property).

One such order $\mathfrak{M}$ can be obtained in the following way: For $1 \leq r \leq 7$ we define $\tau_r := e_1 + e_r + e_{r+1} + e_{r+3}$ where we again take the indeces modulo 7 wherever needed. Then $\mathfrak{M}$ is generated as a lattice by

$$\mathfrak{M} = \left\langle 1, e_i, \frac{1}{2}\tau_i \mid 1 \leq i \leq 7 \right\rangle_{\mathcal{O}_k}. \tag{6.22}$$

It is noteworthy that for $k = \mathbb{Q}$ the order $\mathfrak{M}$ is as a lattice - and up to scaling of the quadratic form - isometric to the $E_8$ root lattice.

Furthermore there is a mass formula for the genus of $\mathfrak{M}$ (so for the genus of maximal orders in $\mathbb{O}(k)$) which we obtain by employing Theorem 2.3.2:

$$\text{mass}(\text{genus}(\mathfrak{M})) = 2^{-2[k:\mathbb{Q}]}\zeta_k(-1)\zeta_k(-5), \tag{6.23}$$

where $\zeta_k$ is the Dedekind zeta function of $k$.

For a lattice $L$ which is not a maximal order one can now compute the mass of its genus via the usual idea of Eichler's 2.3.1. We want to make this explicit for maximal integral forms, i.e. in the case where $L_\mathfrak{p}$ is a vertex in the building of $\mathbb{G}(k_\mathfrak{p})$ for all primes $\mathfrak{p}$. Since it does not matter which representative of a genus we consider we may assume that $L_\mathfrak{p}$ and $\mathfrak{M}_\mathfrak{p}$ belong to a common chamber or equivalently that their stabilizers (in $\mathbb{G}(k_\mathfrak{p})$) contain a common Iwahori subgroup. In this case the factors for the local correction are easily computed via the formula in Corollary 4.2.4. Now let $\mathcal{P}_2$ be the set of all primes $\mathfrak{p}$ such that $L_\mathfrak{p}$ is of type 2 and $\mathcal{P}_3$ be the set of all primes $\mathfrak{p}$ such that $L_\mathfrak{p}$ is of type 3. Then we compute

$$\text{mass}(\text{genus}(L)) = 2^{-2N}\zeta_k(-1)\zeta_k(-5)\left(\prod_{\mathfrak{p}\in\mathcal{P}_2} N(\mathfrak{p})^4 + N(\mathfrak{p})^2 + 1\right)\left(\prod_{\mathfrak{p}\in\mathcal{P}_3} N(\mathfrak{p})^3 + 1\right), \tag{6.24}$$

where $N = [k : \mathbb{Q}]$ and $N(\mathfrak{p})$ denotes the usual norm or equivalently the cardinality of $\mathcal{O}_k/\mathfrak{p}$.

To conclude this section we briefly want to explain how one can compute the (global) stabilizer of a lattice $L \leq \mathbb{O}_d(k)$. Since $\mathbb{G}(k)$ fixes the totally positive definite quadratic form $x \mapsto x\bar{x}$, we start off by computing the stabilizer $S$ of $L$ in $\mathrm{SO}_8$ (with respect to this form) which can be done by employing the Plesken-Souvignier algorithm (cf. [PS97]). Since $S$ is a finite group it remains to compute the intersection $S \cap \mathbb{G}(k)$ which can in principle be done naively by checking for each element of $S$ if it defines an algebra automorphism of $\mathbb{O}_d$. Unfortunately this is far to slow in pratice since $S$ can already be quite large. To circumvent this problem we note that $\mathbb{G}(k)$ is the distinguished subgroup of $\mathrm{SO}_8(k)$ fixing the multiplication (thought of as an element of $\mathbb{O}^* \otimes_k \mathbb{O}^* \otimes_k \mathbb{O})$) or equivalently fixing its Lie algebra. Hence we can find the intersection $\mathbb{G}(k) \cap S$ by starting with $S$ and computing the stabilizer of one of these two objects by employing the base-and-strong-generators algorithm.

## 6.2. Algebraic Modular Forms for Symplectic Groups

In this section we consider algebraic modular forms for compact forms of symplectic groups. We are particularly interested in the forms for $\mathrm{Sp}_6$ since these have a connection to $G_2$ that will become more clear in the following chapters. However, there is no need to restrict ourselves to this case since the theory works uniformly in all degrees. We will start by briefly stating some facts about split forms of symplectic groups, their Euclidean buildings and maximal compact subgroups over local fields. Afterwards we take a look at compact forms of symplectic groups over totally real number fields and describe how to compute algebraic modular forms in this setting.

### 6.2.1. Split Symplectic Groups

Let $k$ be a field of characteristic not 2, $m$ a natural number and $V$ a $2m$-dimensional $k$-vector space endowed with a nondegenerate alternating form $h$. I.e. $h : V \times V \to k$ is $k$-bilinear with $h(v, v) = 0$ for all $v \in V$. There is a basis $(e_1, ..., e_m, f_1, ..., f_m)$ such that the $(e_i, f_i), 1 \leq i \leq m$, form pairwise orthogonal hyperbolic pairs whence the gram matrix of $h$ takes the form

$$J := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \tag{6.25}$$

with respect to this basis.

**Definition 6.2.1** *The symplectic group of degree $2m$ over $k$ is the group of isometries of $(V, h)$,*

$$\begin{aligned} \mathrm{Sp}_{2m}(k) &:= \{g \in \mathrm{GL}(V) \mid h(vg, wg) = h(v, w) \ \forall\ v, w \in V\} \\ &\cong \{g \in \mathrm{GL}_{2m}(k) \mid gJg^{tr} = J\}. \end{aligned} \tag{6.26}$$

*Via this isomorphism we will consider $\mathrm{Sp}_{2m}$ as a linear algebraic group over $k$.*

For simplicity we merely cite the following result on the structure of $\mathrm{Sp}_{2m}$:

**Lemma 6.2.2 ([Spr98, 17.2.4])** $\mathrm{Sp}_{2m}$ *is connected as an algebraic group.*

Now let $T$ be the set of diagonal matrices in $\mathrm{Sp}_{2m}$, then

$$T = \left\{\mathrm{diag}(a_1, ..., a_m, a_1^{-1}, ..., a_m^{-1}) \mid a_1, ..., a_m \in k^*\right\} \tag{6.27}$$

is a (split) maximal torus in $\mathrm{Sp}_{2m}$. We set $\chi_i : T \to k^*,\ t \mapsto t_{i,i}$ for $1 \le i \le m$ and $\eta_i : k^* \to T,\ a \mapsto \eta_i(a)$ with $\chi_j(\eta_i(a)) = a^{\delta_{i,j}}$ again for $1 \le i, j \le m$. Then we have

$$X^*(T) = \langle \chi_1, ..., \chi_m\rangle_{\mathbb{Z}} \text{ and } X_*(T) = \langle \eta_1, ..., \eta_m\rangle_{\mathbb{Z}}. \tag{6.28}$$

The Lie algebra of $\mathrm{Sp}_{2m}$ is, by a simple computation,

$$\mathrm{Lie}(\mathrm{Sp}_{2m}) = \left\{A \in k^{2m\times 2m} \mid AJ + JA^{tr} = 0\right\} = \mathfrak{sp}_{2m}. \tag{6.29}$$

In particular $\mathrm{Lie}(\mathrm{Sp}_{2m})$ is simple and hence $\mathrm{Sp}_{2m}$ is almost simple.

The torus $T$ acts trivially on the (Cartan) subalgebra $\mathfrak{h} \le \mathfrak{sp}_{2n}(k)$ spanned by the elements $E_{i,i} - E_{m+i,m+i}$, $1 \le i \le m$. The roots in $X^*(T)$ are

$$\Phi = \{\pm\chi_i \pm \chi_j \mid 1 \le i \ne j \le m\} \sqcup \{2\chi_i \mid 1 \le i \le m\} \tag{6.30}$$

with corresponding root spaces (for $1 \le i < j \le m$)

$$\begin{aligned}
\mathfrak{sp}_{2n}(k)_{\chi_i-\chi_j} &= \langle E_{j,i} - E_{m+i,m+j}\rangle,\\
\mathfrak{sp}_{2n}(k)_{-\chi_i-\chi_j} &= \langle E_{i,m+j} + E_{j,m+i}\rangle,\\
\mathfrak{sp}_{2n}(k)_{+\chi_i+\chi_j} &= \langle E_{m+i,j} + E_{m+j,i}\rangle,\\
\mathfrak{sp}_{2n}(k)_{2\chi_i} &= \langle E_{m+i,i}\rangle,\\
\mathfrak{sp}_{2n}(k)_{-2\chi_i} &= \langle E_{i,m+i}\rangle.
\end{aligned} \tag{6.31}$$

We will prove exemplarily for the roots $2\chi_i$ that this is really the root space; the other computations are done in the same manner. To that end let $t \in T$, then $\left(t^{-1}E_{m+i,i}t\right)_{l,k} = 0$ for $m + i \ne l$ or $i \ne k$, while $\left(t^{-1}E_{m+i,i}t\right)_{m+i,i} = t^{-1}_{m+i,m+i} \cdot t_{i,i} = \chi_i(t) \cdot \chi_i(t) = (2\chi_i)(t)$.

A simple dimension argument shows that $\mathfrak{sp}_{2n}(k)$ is (as a $k$-vector space) the direct sum of $\mathfrak{h}$ and these root spaces so that $\Phi$ is in fact the full set of roots. Since the sum of the coefficients of all the roots has even parity we see that $\langle\Phi\rangle_{\mathbb{Z}} \lneq X^*(T)$ is of index 2. Hence $\mathrm{Sp}_{2m}$ is not of adjoint type.

The coroots corresponding to $\Phi$ are given as

$$\begin{aligned}
(\chi_i - \chi_j)^\vee &= \eta_i - \eta_j \in X_*(T),\\
(\chi_i + \chi_j)^\vee &= \eta_i + \eta_j \in X_*(T),\\
(2\chi_i)^\vee &= \eta_i \in X_*(T).
\end{aligned} \tag{6.32}$$

In particular we see that $\langle \Phi^+ \rangle = X_*(T)$ whence $\mathrm{Sp}_{2m}$ is simply connected.

A natural choice of the positive roots is

$$\{\chi_i - \chi_j \mid i < j\} \sqcup \{\chi_i + \chi_j \mid i \neq j\} \sqcup \{2\chi_i\} \tag{6.33}$$

with corresponding simple system

$$\chi_1 - \chi_2, \chi_2 - \chi_3, ..., \chi_{m-1} - \chi_m, 2\chi_m \tag{6.34}$$

and Coxeter-Dynkin diagram

$C_n :$


## 6.2.2. Symplectic Groups over Local Fields

We now want to study the groups defined in the last subsection in the situation where the ground field is a local field and take a look at their affine buildings. Since we are primarily interested in applications to symplectic groups over completions of (totally real) algebraic number fields we will assume characteristic 0 here, however, most of this exposition remains valid for completions of algebraic function fields over finite fields.

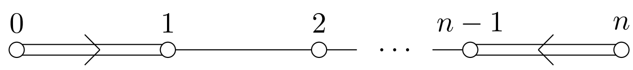
Let $F$ be a local field of characteristic 0 with ring of integers $\mathcal{O}_F$, uniformizer $\pi$ and $\mathcal{O}_F/\pi\mathcal{O}_F \cong \mathbb{F}_q$ of characteristic $p$. As before $m$ is a positive integer and we consider the group $\mathrm{Sp}_{2m}(F)$, the symplectic group with respect to the standard alternating product on $F^{2m}$. In the last section we have already seen that $\mathrm{Sp}_{2m}$ is an almost simple, simply connected, linear algebraic group of type $C_n$. These groups were previously the objects of interest in 5.2.2 so we can make use of those results.

We recall that (since $\mathrm{Sp}_{2m}$ is simply connected) the extended Weyl group $\widetilde{W}$ of $\mathrm{Sp}_{2m}$ over $F$ (with respect to $T$) is just the affine Weyl group $W_{af}$ of the $C_n$ root system. Thus we have

$$X_*(T) \rtimes W_0 = \widetilde{W} = W_{af} = \langle \Phi^\vee \rangle \rtimes W_0 = \langle s_0, s_1, ..., s_n \rangle \tag{6.35}$$

is a Coxeter group with involutive generators $s_0, ..., s_n$ where $s_1 = s_{\alpha_1}, ..., s_n = s_{\alpha_n}$ generate the finite Weyl group $W_0$ and $s_0 = t_{-\alpha_0^\vee} s_{\alpha_0}$. The extended Dynkin diagram takes the form

$\widetilde{C}_n :$



For the sake of completeness we will give (representatives for) the generators of the affine Weyl group in coordinates.

The generators of the finite Weyl group (with respect to the diagonal torus) are represented by the following elements of $\mathrm{Sp}_{2m}(F)$ written as permutations in suggestive notation:

$$\begin{aligned} s_{\alpha_i} &= (e_i, e_{i+1})(f_i, f_{i+1}) \quad \text{for } 1 \leq i \leq n-1, \\ s_{\alpha_n} &= (e_n, f_n, -e_n, -f_n). \end{aligned} \tag{6.36}$$

The additional involution $s_0$ is represented by

$$s_0 = t_{-\alpha_0^\vee} s_{\alpha_0} = \left( e_1, -\frac{1}{\pi} f_1, -e_1, \frac{1}{\pi} f_1 \right). \tag{6.37}$$

To construct a realization of the affine building of $\mathrm{Sp}_{2m}(F)$ as a simplicial complex we follow the construction in [AN02] via lattice chains:

**Definition 6.2.3 ([AN02, Section 3,4])** *1. An $\mathcal{O}_F$-submodule $L$ of $V := F^{2m}$ is called a (full) lattice if it is $\mathcal{O}_F$-free and contains an $F$-basis for $V$.*

2. *For a lattice $L$ we define the dual lattice $L^\#$ via*

$$L^\# = \{ x \in V \mid h(x, L) \subset \mathcal{O}_F \}. \tag{6.38}$$

3. *A chain of lattices*

$$C : ... \subset L_1 \subset L_2 \subset L_3 \subset ... \tag{6.39}$$

*is called admissible if it is closed under scalar multiplication with integral powers of $\pi$.*

4. *A chain of lattices $C$ is called $\#$-admissible if it is admissible and closed under taking duals.*

The affine building $\mathcal{B}$ of $\mathrm{Sp}_{2m}(F)$ can now be realized as the partially ordered (by inclusion) set of $\#$-admissible lattice chains in $V$. We will omit the proof that this is in fact a (thick) building and focus on understanding its structure a little bit better.

To that end we first note that a minimal lattice chain $C$ (i.e. a vertex in $\mathcal{B}$) is of the form $\left\{ \pi^i L \mid i \in \mathbb{Z} \right\} \cup \left\{ \pi^i L^\# \mid i \in \mathbb{Z} \right\}$ where $L$ is any representative of $C$. Furthermore there is a unique lattice $L$ in $C$ such that $\pi L^\# \subset L \subset L^\#$ and we can thus identify the vertices in $\mathcal{B}$ with the set of lattices having this property.

The maximal tori in $\mathrm{Sp}_{2m}$ are in bijection with the hyperbolic frames in $V$, i.e. the collections

$$\langle e_1'\rangle, ..., \langle e_m'\rangle, \langle f_1'\rangle, ..., \langle f_m'\rangle \tag{6.40}$$

where $h(e_i, e_j) = h(f_i, f_j) = 0$ for all $i, j$ and $h(e_i, f_j) \neq 0$ if and only if $i = j$. The apartment corresponding to such a hyperbolic frame (or equivalently the torus which consists of all diagonal elements with respect to the basis $(e_1', ..., e_m', f_1', ..., f_m')$ is then the set $\Sigma$ of lattice chains $C$ such that every lattice in $C$ has a basis consisting of $\pi$-power multiples of the $e_i', f_j'$.

Now set

$$L_0 := \langle e_1, ..., e_m, f_1, ..., f_m\rangle \text{ and } L_i := \langle \pi e_1, ..., \pi e_i, e_{i+1}, ..., e_m, f_1, ..., f_m\rangle. \tag{6.41}$$

Then all the $L_i$ satisfy $\pi L_i^{\#} \subset L_i \subset L_i^{\#}$ and we have a chain

$$... \supset L_0^{\#} = L_0 \supset L_1 \supset ... \supset L_m = \pi L_m^{\#} \supset \pi L_{m-1}^{\#} \supset ... \supset \pi L_0^{\#} = \pi L_0 \supset ... \tag{6.42}$$

The index of a lattice of this chain in its predecessor is $\pi$, hence this is a maximal lattice chain, i.e. a chamber of $\mathcal{B}$. The stabilizer of this chain (which we will call the standard chamber) is the Iwahori subgroup $I \leq \mathrm{Sp}_{2m}(F)$ and consists of all elements of $\mathrm{Sp}_{2m}(\mathcal{O}_F)$ that are upper triangular modulo $\pi\mathcal{O}_F$.

Since $\mathrm{Sp}_{2m}$ is simply connected, there are $m+1$ conjugacy classes of maximal compact parahoric subgroups and each of these classes is represented by a group of the form $P_i := IW_iI,\ 0 \leq i \leq m$, where $W_i < \widetilde{W}$ is the subgroup of the affine Weyl group generated by all simple reflections except $s_i$,

$$W_i = \langle s_0, ..., \widehat{s_i}, ..., s_m\rangle < \widetilde{W}. \tag{6.43}$$

Each maximal compact parahoric subgroup stabilizes a vertex in $\mathcal{B}$ and under our choices it is easy to see that $P_i = \mathrm{Stab}_{\mathrm{Sp}_{2m}(F)}(L_i)$. The hyperspecial maximal compact subgroups are precisely those with the property that the Dynkin diagram of $W_i$ is again of type $C_m$. Thus we have two conjugacy classes of hyperspecial maximal compact subgroups represented by $P_0$ and $P_m$. These are precisely the stabilizers of minimal lattice chains $C$ represented by a lattice $L$ which takes one of the maximal spots in $\pi L^{\#} \subset L \subset L^{\#}$. In other words $C$ is represented by a lattice that is a $\pi$-power multiple of its dual (this is usually called a $\pi$-modular lattice).

### 6.2.3. Compact Forms of Symplectic Groups

Here we want to describe a method to construct compact forms of symplectic groups over totally real number fields which are well-suited for computations. To that end let $k$ denote a totally real number field and let $H = \left(\frac{a,b}{k}\right)$ be a quaternion algebra over $k$ which is ramified at all infinite places of $k$, i.e. $a$

and $b$ are totally negative. By $^-$ we will denote the usual involution on $H$, i.e. for $\gamma = x + iy + jz + ijw \in H$ we set $\overline{\gamma} = x - iy - jz - ijw$. We choose a natural number $m$ and consider the $m$-dimensional $H$-left vector space $V = H^m$ together with the Hermitian form

$$h : V \times V \to H,\ h((x_1, ..., x_m), (y_1, ..., y_m)) = \sum_{i=1}^{m} x_i \overline{y_i}. \tag{6.44}$$

The form $h$ is obviously totally positive definite and hence in particular nondegenerate. Using $h$ we now define a linear algebraic group $\mathbb{G}$ over $k$ via

$$\begin{aligned} \mathbb{G}(A) &= \{g \in \mathrm{GL}_m(A \otimes_k H) \mid h(gx, gy) = h(x, y)\ \forall\ x, y \in A \otimes_k V\} \\ &= \left\{g \in \mathrm{GL}_m(A \otimes_k H) \mid g\overline{g}^{tr} = I_m\right\} \end{aligned} \tag{6.45}$$

for every commutative $k$-algebra $A$. In this way $\mathbb{G}(A)$ is just the unitary group of degree $m$ over $A \otimes_k H$.

Since $h$ is positive definite it is easy to see the following:

**Proposition 6.2.4** $\mathbb{G}(k_\infty)$ *is compact.*

We now want to show that $\mathbb{G}$ is in fact a form of the symplectic group $\mathrm{Sp}_{2m}$ which is an easy consequence of the following lemma.

**Lemma 6.2.5** *The $k \leq F$ be a field such that $H$ is split over $F$, which means $F \otimes_k H \cong F^{2\times 2}$. Then*

$$\mathbb{G}(F) \cong_F \mathrm{Sp}_{2m}(F) = \left\{g \in \mathrm{GL}_{2m}(F) \mid gJg^tr = J\right\} \text{ where } J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}. \tag{6.46}$$

*Proof.* We set $H_F := F \otimes_k H \cong F^{2\times 2}$. By assumption there are primitive orthogonal idempotents $e_1, e_2 = (1 - e_1)$ and it is easily seen that necessarily $e_2 = \overline{e_1}$. These idempotents yield a decomposition $V_F = F \otimes_k V = V_1 \oplus V_2$ as $F$-vector spaces with $V_i = e_i V_F, i = 1, 2$.

Now $V_1$ is $2m$-dimensional (over $F$) and we compute

$$h(e_1 v, e_1 w) = e_1 h(v, w)\overline{e_1} = e_1 h(v, w) e_2. \tag{6.47}$$

The $F$-space $e_1 H_F e_2$ has dimension 1 and may thus be identified with with $F$ which yields an $F$-bilinear form $h' : V_1 \times V_1 \to F$. Since $h(v, v)$ is central in $H_F$ for all $v \in V_F$, the form $h'$ is alternating and moreover $h'$ is nondegenerate since this is true for $h$. In conclusion $V_1$ is a $2m$-dimensional symplectic $F$-space.

We will now show that $\phi : \mathbb{G}(F) \to \mathrm{Sp}((V_1, h')),\ g \mapsto g_{|V_1}$ is an isomorphism. First note that $\phi$ is well-defined since $V_1 = e_1 V$ is stable under right multiplication by elements of $\mathbb{G}(F)$ and $h'(e_1 vg, e_1 wg) = h'(e_1 v, e_1 w)$ for all $v, w \in V$

since $h'$ is defined via $h$. Since $e_1$ and $e_2$ are idempotents in $H_F \cong F^{2\times 2}$ there is an involution $\sigma \in H_F$ with $\sigma e_1 \sigma = e_2$. Using this involution we define $\phi' : \mathrm{Sp}(V_1) \to \mathbb{G}(F)$ as follows:

$$g \mapsto g' \text{ where } vg' = \underbrace{(e_1 v)}_{\in V_1} g + \sigma(\underbrace{(e_1\sigma v)}_{\in V_1} g). \tag{6.48}$$

We compute

$$\begin{aligned} h(vg', wg') &= h((e_1v)g + \sigma((e_1\sigma v)g), (e_1w)g + \sigma((e_1\sigma w)g)) \\ &= h((e_1v)g, (e_1w)g) + h((e_1v)g, \sigma((e_1\sigma w)g)) \\ &\quad + h(\sigma((e_1\sigma v)g), (e_1w)g) + h(\sigma((e_1\sigma v)g), \sigma((e_1\sigma w)g)) \\ &= h'((e_1v)g, (e_1w)g) + h'((e_1v)g, ((e_1\sigma w)g))\overline{\sigma} \\ &\quad + \sigma h'(((e_1\sigma v)g), (e_1w)g) + \sigma h'(((e_1\sigma v)g), ((e_1\sigma w)g))\overline{\sigma} \\ &= h(v, w). \end{aligned} \tag{6.49}$$

Hence $\phi'$ is well-defined and since $\phi$ and $\phi'$ are mutually inverse this proves the assertion (the fact that this is an isomorphism of algebraic groups can be seen by choosing coordinates and explicitly writing down the above construction).□

Having now seen that the group $\mathbb{G}$ is in fact a compact form of $\mathrm{Sp}_{2m}$, we turn to the construction of integral forms for $\mathbb{G}$. To that end we choose a maximal $\mathcal{O}_k$-order $\mathcal{O}_H \leq H$ and consider an $\mathcal{O}_H$-lattice $L \leq V$. This enables us to define a group scheme $\mathbb{G}_L$ over $\mathcal{O}_k$ in the usual way via

$$\mathbb{G}_L(\mathcal{O}_F) = \mathrm{Stab}_{\mathbb{G}(F)}(\mathcal{O}_F \otimes_{\mathcal{O}_k} L) \tag{6.50}$$

for $F$ any completion or finite extension of $k$.

Since for local fields $F$ the algebra $F^{2\times 2}$ admits only one maximal order, namely $\mathcal{O}_F^{2\times 2}$, up to conjugation, we have $\mathcal{O}_\mathfrak{p} \otimes_{\mathcal{O}_k} \mathcal{O}_H^m \cong (\mathcal{O}_p^{2\times 2})^m$ for all $\mathfrak{p} \nmid \mathrm{disc}(H)$ which means that $\mathbb{G}_{\mathcal{O}_H^m}(\mathcal{O}_\mathfrak{p})$ is a hyperspecial maximal compact subgroup for all $\mathfrak{p} \nmid \mathrm{disc}(H)$. In particular we see that $\mathbb{G}$ only admits models (in the sense of Definition 2.2.6) if $[k : \mathbb{Q}]$ is even and $D$ is the unique quaternion algebra ramified at all infinite and no finite places. In this case, however, $\mathbb{G}$ automatically admits infinitely many models since at each prime we have the choice between two conjugacy classes of hyperspecial maximal compact subgroups.

**Definition 6.2.6** *We will call the $\mathbb{G}$-genus of $\mathcal{O}_H^m$ (or equivalently of $\mathbb{G}_{\mathcal{O}_H^m}$) the principal genus of $\mathbb{G}$.*

The mass of the principal genus can be computed via a mass formula of Gan and Yu which we will simply cite here without much further commentary.

**Theorem 6.2.7 ([GY00, 11.2])** *Let $d_k$ be the discriminant of $\mathcal{O}_k$, $d := [k : \mathbb{Q}]$ the degree of $k$ over $\mathbb{Q}$, $\zeta_k$ the Dedekind zeta function of $k$, and $S_H$ the* *(possibly empty) set of finite ramified places of $H$. Then*

$$\mathrm{mass}(\mathrm{genus}(\mathcal{O}^m)) = d_k^{m(m+1)/2} \cdot \left( \prod_{j=1}^{m} \left( \zeta_k(2j) \left( \frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \cdot \prod_{\nu \in S_H} (q_\nu^j + (-1)^j) \right) \right). \tag{6.51}$$

*Using the transformation*

$$\zeta_k(s) = d_k^{1/2-s} \pi^{ds-d/2} \Gamma\left(\frac{s}{2}\right)^{-d} \Gamma\left(\frac{1-s}{2}\right)^d \zeta_k(1-s) \tag{6.52}$$

*we can get rid of the irrational terms in the formula and compute*

$$\mathrm{mass}(\mathrm{genus}(\mathcal{O}^m)) = \prod_{j=1}^{m} \left( d_k^{m+1-2j} \cdot (-1)^{jd} \cdot 2^{-d} \cdot \zeta_k(1-2j) \cdot \prod_{\nu \in S_H} (q_\nu^j + (-1)^j) \right). \tag{6.53}$$

For a lattice $L$ which is not in the genus of $\mathcal{O}^m$ one uses the formula above together with Eichler's method (2.3.1) to obtain the mass of its genus.

If we want to compute algebraic modular forms for $\mathbb{G}$ we need to be able to compute stabilizers of lattices (in $\mathbb{G}(k)$) as well as check lattices for being $\mathbb{G}(k)$-isomorphic. Since these two tasks basically amount to the same thing we will only describe a way to construct stabilizers:

By choice of a basis we identify $V$ with the $4m$-dimensional $k$-vectorspace $k^{4m}$ and $L$ with the corresponding $\mathcal{O}_k$-lattice in $k^{4m}$. The simple algebra $H^{m\times m}$ embeds into $k^{4m\times 4m}$ in this way and we choose a $k$-basis $C$ for its centralizer in $k^{4m\times 4m}$. By the double centralizer theorem an element $x \in k^{4m\times 4m}$ lies in $H^{m\times m}$ if and only if $xc = cx$ for all $c \in C$.

Now let $\Psi$ be the gram matrix of the $k$-bilinear form $h$ on $k^{4m}$ with respect to the basis chosen above, then $\mathbb{G}(k)$ is the group of all matrices (under the above identification) $g \in \mathrm{GL}_{4m}(k)$ with $g\Psi g^{tr} = \Psi$ and $g \in H^{4m\times 4m}$. Equivalently, by the double centralizer argument, we have $g \in \mathbb{G}(k)$ if and only if $gc\Psi g^{tr} = c\Psi$ for all $c \in C$. Hence the stabilizer of $L$ is

$$\mathrm{Stab}_{\mathbb{G}(k)}(L) = \left\{ g \in \mathrm{GL}_{4m}(k) \mid gc\Psi g^{tr} = c\Psi \ \forall\, c \in C,\ Lg = L \right\}. \tag{6.54}$$

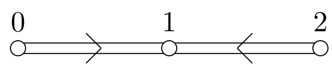
If $k = \mathbb{Q}$ this is precisely the kind of computation which can be handled by the Plesken-Souvignier algorithm (cf. [PS97]). If $k$ is a proper extension of $\mathbb{Q}$ we can just apply the same kind of argument again, this time choosing a basis for $k$ over $\mathbb{Q}$ and thus embedding the algebra $H^{m\times m}$ into $\mathbb{Q}^{4dm\times 4dm}$, where $d = [k : \mathbb{Q}]$, and then applying the Plesken-Souvignier algorithm.

We conclude this section by briefly looking at the dependency of our definitions on the choice of the maximal order $\mathcal{O}_H$. Since a maximal order in $H$ is in general not even unique up to conjugacy in $H^\times$, all the computational results might in principle change when considering $\mathcal{O}'_H$- instead of $\mathcal{O}_H$-lattices. However, for two maximal orders $\mathcal{O}_H$ and $\mathcal{O}'_H$ in $H$ there is always an $\mathcal{O}_k$-lattice $\mathfrak{a} \subset H$ with left order $\mathcal{O}_H$ and right order $\mathcal{O}'_H$ (take $\mathfrak{a} = \mathcal{O}_H\mathcal{O}'_H$ for instance). Thus the map $L \to \mathfrak{a}L$ (where $L \subset H^m$) yields a bijection between $\mathcal{O}_H$- and $\mathcal{O}'_H$-lattices in $H^m$. Moreover this bijection preserves the integral form defined by a lattice. In particular stabilizers, genera, isomorphisms, and the action of Hecke operators remain unchanged. Hence our choice of a maximal order has no essential influence on the results obtained.

### 6.2.4. Eichler's Method in the Nonhyperspecial Case

As we have already seen in Subsection 5.2.2 the local Hecke algebra of a split symplectic group with respect to a hyperspecial maximal compact subgroup is generated by the (local) Eichler elements with respect to the other two maximal compact subgroups such that the intersection of all three is precisely an Iwahori subgroup. For the symplectic group of degree 4, two out of the three conjugacy classes of maximal parahorics are in fact hyperspecial so their theory is included in what we already know. Here we want to take a look at the remaining conjugacy class and we will see that the Eichler elements with respect to certain other parahorics still generate the local Hecke algebra. However, these other parahorics we have to consider are not necessarily maximal in this case.

To that end let $F$ be a local field of characteristic 0 with residue class field order $q$ and let $\mathrm{Sp}_4$ be the split symplectic group of degree 4 over $F$. The extended affine Dynkin diagram has the form:

$\widetilde{C}_2$ : 


Let $W_1 := \langle s_0, s_2\rangle \leq \widetilde{W}$ and $P_1 := IW_1I$ where $I$ is an Iwahori subgroup containing the integral part of the torus with respect to which we defined the Weyl group. We are interested in the Hecke algebra $H_{P_1} = H(\mathrm{Sp}_4(F), P_1)$ and we already checked in Example 3.2.20 that $H_{P_1}$ is generated by the characteristic functions on the double cosets $P_1s_1K_1$, $P_1s_1s_0s_1P_1$, and $P_1s_1s_2s_1P_1$. It is now our aim to find suitable additional parahoric subgroups such that the Eichler elements with respect to these also generate $H_{P_1}$.

**Lemma 6.2.8** *The Hecke algebra $H_{P_1}$ is not generated by the Eichler elements with respect to the two other parahoric subgroups containing $I$, i.e. $P_0 := IW_0I$ and $P_2 := IW_2I$, where $W_0 = \langle s_1, s_2\rangle$ and $W_2 = \langle s_0, s_1\rangle$. These Eichler elements are $\nu(P_1, P_0) = (q+1) + \mathbb{1}_{P_1s_1P_1} + \mathbb{1}_{P_1s_1s_2s_1P_1}$ and $\nu(P_1, P_2) = (q+1) + \mathbb{1}_{P_1s_1P_1} + \mathbb{1}_{P_1s_1s_0s_1P_1}$, respectively.*

*Proof.* We have $W_0 \cap W_1 = \langle s_2 \rangle$ and $[W_0 \cap W_1 \backslash W_0] = \{1, s_1, s_1 s_2 s_1\}$. Furthermore we compute

$$W_1^{W_1 s_1} = \{1\} \text{ and } W_1^{W_1 s_1 s_2 s_1} = \langle s_2 \rangle = W_1^{W_1 s_1 s_2 s_1} \cap W_0. \tag{6.55}$$

Hence by Theorem 5.1.9 we see that the Eichler element with respect to $K_1$ and $K_0$ takes the form

$$\nu(P_1, P_0) = (q+1) + \mathbb{1}_{P_1 s_1 P_1} + \mathbb{1}_{P_1 s_1 s_2 s_1 P_1}. \tag{6.56}$$

Performing an analogous computation for $P_2$ yields:

$$\nu(P_1, P_2) = (q+1) + \mathbb{1}_{P_1 s_1 P_1} + \mathbb{1}_{P_1 s_1 s_0 s_1 P_1}. \tag{6.57}$$

Thus the Eichler elements with respect to the maximal parahoric subgroups (containing the same Iwahori subgroup as $P_1$) are in this case not enough to obtain a generating system for the whole Hecke algebra $H_{P_1}$. □

While the two Eichler elements considered above are not enough to generate the whole Hecke algebra this can be rectified by considering the Eichler element with respect to the nonmaximal parahoric subgroup $P_{0,2} := P_0 \cap P_2 = I W_{0,2} I$.

**Theorem 6.2.9** *$H_{P_1}$ is generated by the Eichler elements $\nu(P_1, P_0), \nu(P_1, P_2)$ and $\nu(P_1, P_{0.2})$.*

*Proof.* We have $W_{0,2} \cap W_1 = \{1\}$ whence

$$\nu(P_1, P_{0,2}) = (q+1) + \mathbb{1}_{P_1 s_1 P_1}. \tag{6.58}$$

□

We now want to lose a few words on how to actually perform the necessary computations. To that end let $k$ be a totally real number field and $H$ a totally definite quaternion algebra over $k$. As noted before $\mathbb{G}$, the unitary group of degree 2 over $H$, yields a compact form of $\mathrm{Sp}_4$ which is split over $k_{\mathfrak{p}}$ whenever $H$ is. We want to apply our above considerations to the study of algebraic modular forms of arbitrary weight $V$ and level $K_1$ given by an integral form $\mathbb{G}_L$ of $\mathbb{G}$, where $L$ is some $\mathcal{O}_H$-lattice in $H^2$ and $\mathcal{O}_H$ is a $\mathcal{O}_k$-maximal order in $H$. Let $\mathfrak{p}$ be a finite prime of $\mathcal{O}_k$ where $H$ (and thus $\mathbb{G}$) is split and assume that $\mathbb{G}_L(\mathcal{O}_{\mathfrak{p}})$ is maximal parahoric but not hyperspecial. We then may identify $K_{1,\wp} = \mathbb{G}_L(\mathcal{O}_{\mathfrak{p}})$ with $P_1$ in our above notation for $F = k_{\mathfrak{p}}$.

If we are interested in the action of the local Hecke algebra $H_{K_{1,\mathfrak{p}}}$ on $M(V, K_1)$ and want to use the Eichler method we need to be able to compute certain intertwining operators $T(K_1, K')$ where $K'$ only differs from $K$ at $\mathfrak{p}$ and $K'_{\mathfrak{p}}$ runs through the parahoric subgroups $P_0, P_2$ and $P_{0,2}$ from above. Without loss of generality we may assume that $L_{\mathfrak{p}}$ corresponds to the lattice $\langle \mathfrak{p} e_1, e_2, f_1, f_2 \rangle$ in the standard symplectic space $k_{\mathfrak{p}}^4$ and one easily constructs lattices $L_0$ and

$L_2$ in $\mathcal{O}_H^2$ corresponding to $\langle e_1, e_2, f_1, f_2 \rangle$ and $\langle \mathfrak{p}e_1, \mathfrak{p}e_2, f_1, f_2 \rangle$ and coinciding with $L$ away from $\mathfrak{p}$. These lattices then define integral forms $\mathbb{G}_{L_0}$ and $\mathbb{G}_{L_2}$, respectively, from which we obtain the maximal parahoric subgroups $P_0$ and $P_2$ at $\mathfrak{p}$.

However, there is no obvious way to construct a lattice $L_{0,2}$ yielding the parahoric subgroup $P_{0,2}$ from above at $\mathfrak{p}$. To get around this problem one can consider the lattice $L_0 \oplus L_2$ in the representation $H^2 \oplus H^2$ and the so-defined integral form. Equivalently one can consider the stabilizer of the pair $(L_0, L_2)$ which highlights the fact that we are (at least at $\mathfrak{p}$) dealing with the stabilizer of a 2-simplex of the local building.

# 7. Lifting of Algebraic Modular Forms

This chapter is concerned with the lifting of algebraic modular forms. While we will not actually construct lifts we will gather some evidence for the existence of a lifting process of modular forms of $G_2$ to modular forms of $\mathrm{Sp}_6$ corresponding to the embedding $G_2(\mathbb{C}) \hookrightarrow \mathrm{SO}_7(\mathbb{C})$.

## 7.1. The Satake Homomorphism

In this section $F$ again denotes a local field of characteristic 0 with ring of integers $\mathcal{O}_F$, uniformizer $\pi$ and $\mathcal{O}_F/\pi\mathcal{O}_F \cong \mathbb{F}_q$. Most of this introductory exposition is taken from Benedict Gross's article [Gro98]. At times we add certain details, namely from [Hai00] and [Rap00].

Let $\mathbb{G}$ be a connected semisimple linear algebraic group defined and split over $F$. The Satake homomorphism establishes an isomorphism between (a scalar extension of) the Hecke algebra of $\mathbb{G}(F)$ with respect to a hyperspecial maximal compact subgroup and the representation ring (or ring of formal characters) of the complex dual of $\mathbb{G}$. This isomorphism in turn yields a correspondence between Hecke eigenforms and semisimple conjugacy classes in the complex dual.

Let us first start with some notation: As usual let $\underline{T}$ be a maximal torus of $\mathbb{G}$ contained in a fixed Borel subgroup $\underline{B}$ both of which we assume to be defined over $\mathcal{O}_F$. We set $W_0 := N_{\mathbb{G}(F)}(\underline{T}(F))/\underline{T}(F)$ the usual finite Weyl group of $\mathbb{G}$ (with respect to $\underline{T}$) and denote as before by $X^* := X^*(\underline{T})$ and $X_* := X_*(\underline{T})$ the characters and cocharacters of $\underline{T}$, respectively. The former contains the set $\Phi$ of roots. The choice of $\underline{B}$ determines a set of positive roots $\Phi^+$ satisfying $\Phi = \Phi^+ \sqcup -\Phi^+$ and we denote the corresponding system of simple roots by $\Delta$.

Furthermore our choice of $\underline{B}$ (and hence of our system of positive roots) yields a positive Weyl chamber, classically denoted by $P^+$:

$$\begin{aligned} P^+ :&= \{\lambda \in X_* \ : \ \langle \lambda, \alpha \rangle \geq 0 \ \forall \ \alpha \in \Phi^+\} \\ &= \{\lambda \in X_* \ : \ \langle \lambda, \alpha \rangle \geq 0 \ \forall \ \alpha \in \Delta\}. \end{aligned} \tag{7.1}$$

We define the half-character $\rho \in \frac{1}{2}X^*$ by $2\rho = \sum_{\alpha \in \Phi^+} \alpha$ and note that for each $\lambda \in P^+$ the half-integer $\langle \lambda, \rho \rangle$ is non-negative.

Lastly we will need a certain partial ordering on $P^+$ which is defined by $\mu \succeq \lambda$ if and only if $\mu - \lambda$ can be written as a (possibly empty) sum of positive coroots.

### 7.1.1. The Complex Dual

Here we want to give a very short overview over the necessary details of the theory of the complex dual. It is by no means a complete introduction but rather only provides all definitions and facts which are needed for the following computations.

Consider the root datum associated with $\mathbb{G}$, that is the tuple $(X^*, \Phi, X_*, \Phi^\vee)$. Then the tuple $(X_*, \Phi^\vee, X^*, \Phi)$ is again a root datum called the *dual root datum* and there is a unique (up to isomorphism) connected semisimple group $\widehat{\mathbb{G}}$ over $\mathbb{C}$ having the dual root datum as its root datum. This group is called the complex dual of $\mathbb{G}$.

Fixing a maximal torus $\widehat{T}$ in a Borel subgroup $\widehat{B}$ of $\widehat{\mathbb{G}}$ we get an isomorphism $X^*(\widehat{T}) \cong X_*(T)$ taking the positive roots with respect to $\widehat{B}$ to the positive coroots with respect to $B$. Under this identification the elements $\lambda \in P^+ \subset X^*(\widehat{T})$ are in natural bijection with the irreducible representations $V_\lambda$ of $\widehat{\mathbb{G}}$, where the character $\lambda$ is the highest weight for $\widehat{T}$ on $V_\lambda$.

Now let $\chi_\lambda$ be the character $\operatorname{Trace}(V_\lambda)$, which we view as an element of the ring $\mathbb{Z}[X^*(\widehat{T})]$. Then $\chi_\lambda$ actually already lies in the subring

$$\mathfrak{R}(\widehat{G}) := \mathbb{Z}[X^*(\widehat{T})]^{W_0} \tag{7.2}$$

of invariants of the Weyl group.

Since we want to compute in the ring $\mathfrak{R}(\widehat{G})$ we will write the group $X^*(\widehat{T})$ multiplicatively by identifying the element $\lambda$ with the (standard) abstract symbol $e^\lambda$.

In this notation the character $\chi_\lambda$ can be computed as follows.

**Theorem 7.1.1 (Weyl's character formula, [GW09, Thm. 7.1.1])** *For $\lambda \in P^+$ the following holds:*

$$\chi_\lambda = e^{-\rho^\vee} \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})^{-1} \sum_{w\in W_0} (-1)^{\ell(w)} e^{(\lambda+\rho^\vee)w} = \sum_{w\in W_0} e^{\lambda w} \prod_{\alpha\in\Phi^+} (1-e^{-\alpha w})^{-1}, \tag{7.3}$$

*where we denote by $\rho^\vee$ half of the sum over the positive coroots.*

*Proof.* Strictly speaking only the first identity is known as Weyl's character formula so we prove the second identity. We have

$$e^{\rho^\vee} \prod_{\alpha\in\Phi^+} (1-e^{-\alpha}) = \prod_{\alpha\in\Phi^+} (e^{\alpha/2} - e^{-\alpha/2}). \tag{7.4}$$

The simple reflection $s_\alpha$ maps $\alpha$ to $-\alpha$ and permutes the other positive roots. Thus

$$e^{\rho^\vee w} \prod_{\alpha\in\Phi^+}(1-e^{-\alpha w}) = (-1)^{\ell(w)} \prod_{\alpha\in\Phi^+}(e^{\alpha/2}-e^{-\alpha/2}) = (-1)^{\ell(w)} e^{\rho^\vee} \prod_{\alpha\in\Phi^+}(1-e^{-\alpha}). \tag{7.5}$$

Plugging in the resulting expression for $\prod_{\alpha\in\Phi^+}(1-e^{-\alpha w})^{-1}$ into the right-hand side of the formula in our assertion concludes the proof. □

This description allows us to do explicit calculations in the ring $\mathfrak{R}(\widehat{G})$ for example if we want to decompose a given character as a sum of irreducibles.

### 7.1.2. The Satake Transform

Let us now, in addition to the notation from before, denote a hyperspecial maximal compact subgroup of $\mathbb{G}(F)$ by $K$. As in the previous chapters we will consider the ring $H_K := H(\mathbb{G}(F), K)$; the Hecke algebra of $\mathbb{G}(F)$ (here over $\mathbb{Z}$) with respect to $K$ which is the ring of all compactly supported $K$-bi-invariant functions on $\mathbb{G}(F)$ with values in $\mathbb{Z}$ and multiplication given by convolution. We have already seen that $H_K$ is commutative (though this will again follow from what we are about to see) and that $H_K$ can be considered as a condensation of the Iwahori-Hecke algebra of the extended affine Weyl group. Here we want to add one more viewpoint:

We start by reminding us that $\mathbb{G}(F)$ admits the following convenient decomposition into double cosets with respect to $K$:

$$\mathbb{G}(F) = \bigsqcup_{\lambda\in P^+} K\lambda(\pi)K. \tag{7.6}$$

Since the characteristic functions on the double cosets with respect to $K$ form a basis of $H$ this allows us to simplify some of our computations since we only need to consider double cosets of this form.

The main result of this section (which we state without proof) is the following:

**Theorem 7.1.2 ([Gro98, Prop. 3.6])** *There is a ring isomorphism*

$$\mathcal{S} : H_K \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \to \mathfrak{R}(\widehat{G}) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]. \tag{7.7}$$

*If $\rho$ is already an element of $X^*(T)$ we obtain a ring isomorphism*

$$\mathcal{S} : H_K \otimes \mathbb{Z}[q^{-1}] \to \mathfrak{R}(\widehat{G}) \otimes \mathbb{Z}[q^{-1}]. \tag{7.8}$$

The homomorphism $\mathcal{S}$ from the previous theorem is called the Satake homomorphism (or Satake transform).

The Satake homomorphism can be explicitly given via a certain integral. However, for our purposes the following description is far more accessible.

**Theorem 7.1.3 (Macdonald's formula,[Kat82, Thm. 2.4])** *For $\lambda \in P^+$ the Satake transform can be computed as*

$$\mathcal{S}(\mathbb{1}_{K\lambda(\pi)K}) = \frac{q^{\langle\lambda,\rho\rangle}}{(W_0)_\lambda(q^{-1})} \sum_{w\in W_0} e^{\lambda w} \prod_{\alpha\in\Phi^+} \frac{1-q^{-1}e^{-\alpha w}}{1-e^{-\alpha w}}.$$

*Here $(W_0)_\lambda$ denotes the stabilizer of $\lambda$ in the Weyl group (which is always a special subgroup) and $(W_0)_\lambda(q) = \sum_{w\in(W_0)_\lambda} q^{\ell(w)}$ is the Poincaré polynomial of $(W_0)_\lambda$.*

Macdonald's formula allows us to explicitly construct the images of the canonical basis of $H_K$ under the Satake transform as elements of $\mathfrak{R}(\widehat{G})$. For most purposes though it is more desirable to have these images written as a linear combination of the irreducible characters $\chi_\lambda$, $\lambda \in P^+$. Fortunately this can be done easily since the only characters appearing in $\mathcal{S}(\mathbb{1}_{K\lambda(\pi)K})$ are the $\chi_\mu$ with $\mu \preceq \lambda$ (cf. [FH91, pg. 202 ff]) and there are only finitely many of these.

## 7.2. Satake Parameters

In the last section we established the Satake homomorphism, which was an isomorphism (after scalar extension with $\mathbb{C}$)

$$\mathcal{S} : H_K \otimes \mathbb{C} \to \mathbb{Z}[X^*(\widehat{T})]^{W_0} \otimes \mathbb{C}$$

where $K$ was a hyperspecial maximal compact subgroup of $\mathbb{G}(F)$ and $\widehat{T} \subset \widehat{\mathbb{G}}(\mathbb{C})$ a maximal torus in the complex dual of $\mathbb{G}$.

In particular this means that characters $\chi : H_K \otimes \mathbb{C} \to \mathbb{C}$ can be considered as characters of $\mathbb{Z}[X^*(\widehat{T})]^{W_0} \otimes \mathbb{C}$ via $\mathcal{S}$. These characters have a convenient description as follows.

**Remark 7.2.1 ([Gro98, Sect. 6])** *The characters of $\mathbb{Z}[X^*(\widehat{T})]^{W_0} \otimes \mathbb{C}$ are in bijection with the conjugacy classes of semisimple elements in $\widehat{\mathbb{G}}(\mathbb{C})$. More precisely for such a conjugacy class $C$ we define the character*

$$\omega_C : \mathbb{Z}[X^*(\widehat{T})]^{W_0} \otimes \mathbb{C} \to \mathbb{C},\ \sum a_\chi \chi \mapsto \sum a_\chi \chi(c_0), \tag{7.9}$$

*where $c_0 \in \widehat{T} \cap C$. This is independent of the choice of $c_0$ since $\sum a_\chi \chi$ is $W_0$-invariant.*

This remark together with the Satake isomorphism inspires the following definition:

**Definition 7.2.2** *Let $V$ be any representation of $H_K \otimes \mathbb{C}$ and $0 \neq f \in V$ a (simultaneous) eigenform (i.e. a 1-dimensional subrepresentation). Then $f$ defines a character $\eta_f$ on $H_K \otimes \mathbb{C}$ via $f\theta = \eta_f(\theta)f$, which we will consider as a character of $\mathbb{Z}[X^*(\widehat{T})]^{W_0} \otimes \mathbb{C}$ via the Satake homomorphism. By the above remark there is a semisimple conjugacy class $s(f)$ of $\widehat{\mathbb{G}}(\mathbb{C})$ such that $\omega_{s(f)} = \eta_f : \mathbb{Z}[X^*(\widehat{T})]^{W_0} \otimes \mathbb{C} \to \mathbb{C}$. This conjugacy class is called the* Satake parameter *of $f$.*

## 7.3. Lifting of Modular Forms

Let $k$ be a number field and $\mathbb{G}_1, \mathbb{G}_2$ be two connected semisimple linear algebraic groups defined over $k$. We consider the Hecke algebra $H_1 := H(\mathbb{G}(\hat{k}), K_1)$ where $K_1 = \prod_{\mathfrak{p}} K_{1,\mathfrak{p}} \subset \mathbb{G}_1(\hat{k})$ is an open and compact subgroup. If $f \in M(V, K_1)$ is a Hecke eigenform (for some representation $V$) we get a collection of Satake parameters $\{s_{\mathfrak{p}}(f) \ : \ \mathbb{G} \text{ split at } \mathfrak{p},\ K_{1,\mathfrak{p}} \text{ hyperspecial}\}$ by considering the natural embeddings of the local Hecke algebras into the global one.

Let us now assume we have a morphism $\rho : \widehat{\mathbb{G}_1}(\mathbb{C}) \to \widehat{\mathbb{G}_2}(\mathbb{C})$. Then we also get a collection of semisimple conjugacy classes $\{\rho(s_{\mathfrak{p}}(f)) \ : \ \mathfrak{p}\}$ by applying $\rho$ element-wise to the collection above. It is a natural question to ask whether there is an open, compact subgroup $K_2 \subset \mathbb{G}_2(\hat{k})$ and an Eigenform $f'$ for the Hecke algebra of $\mathbb{G}_2$ with respect to $K_2$ such that this new collection (or at least this new collection up to a finite set of exceptions) arises as the collection of Satake parameters for $f'$. If such an $f'$ exists it is called a lift of $f$ (via $\rho$).

For certain groups $\mathbb{G}_1$ and $\mathbb{G}_2$ there are proofs that lifts always exist but in general this is still an open problem. However, the Langlands functoriality conjectures state that the answer to this question should be positive if $\mathbb{G}_2$ is quasi-split (cf. [AG91]).

In this section we want to provide some evidence for the correctness of the conjecture in one special case.

### 7.3.1. The Embedding of $G_2$ into $\mathrm{SO}_7$

In [LP02] Joshua Lansky and David Pollack discuss lifts of modular forms for $\mathrm{PGL}_2$ (or $\mathrm{PGL}_2 \times \mathrm{PGL}_2$) to $G_2$ via the possible embeddings $\mathrm{SL}_2(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$

(or $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$, respectively). Here we want to consider lifts of modular forms for $G_2$ to another algebraic group.

The group $G_2(\mathbb{C})$ respects a (non-degenerate) inner product on the imaginary complex octonions (since $G_2(\mathbb{C})$ commutes with octonionic conjugation), which form a representation of degree 7. Hence $G_2(\mathbb{C})$ embeds naturally into $\mathrm{SO}_7(\mathbb{C})$, the special orthogonal group of degree 7. The special orthogonal group $\mathrm{SO}_7$ is an almost simple, adjoint group of type $B_3$, which means it is the complex dual of the almost simple, simply connected groups of type $C_3$ which are the forms of the symplectic group of degree 6. Therefore one should hope to be able to lift modular forms for $G_2$ to modular forms for $\mathrm{Sp}_6$.

Let $F$, $\mathcal{O}_F$ and $\pi$ be as in the first section of this chapter.

The Satake transform for $G_2$ is one of the explicit examples in [Gro98] so we merely state the result.

**Lemma 7.3.1 ([Gro98, §5])** *Let $\eta_1, \eta_2$ denote the two fundamental coweights of $G_2$ and $V_1, V_2$ the corresponding representations of $G_2(\mathbb{C})$ with characters $\chi_{\eta_1}$ and $\chi_{\eta_2}$. The spherical Hecke algebra of $G_2(F)$ is generated by $b_1$ and $b_2$, the characteristic functions of the double cosets of $\eta_1(\pi)$ and $\eta_2(\pi)$, and the Satake homomorphism is given as follows:*

$$\begin{aligned} q^3\chi_{\eta_1} &= \mathcal{S}(b_1) + 1, \\ q^5\chi_{\eta_2} &= \mathcal{S}(b_2) + \mathcal{S}(b_1) + q^4 + 1. \end{aligned} \tag{7.10}$$

Additionally, the following identities hold:

$$\bigwedge^2 V_{\eta_1} = V_{\eta_1} \oplus V_{\eta_2},\ V_{\eta_1} \otimes V_{\eta_1} = \left(\bigwedge^3 V_{\eta_1}\right) \oplus V_{\eta_2} \tag{7.11}$$

Now we need to compute the Satake transform for the group $\mathrm{Sp}_6$. As mentioned before $\mathrm{Sp}_6$ is simply connected of type $C_3$ and so there are three simple roots $\Delta = \{\alpha_1, \alpha_2, \alpha_3\} \subset X^*(T)$. Corresponding to these three roots there are the three fundamental coweights $\lambda_1, \lambda_2, \lambda_3 \in X_*(T) \otimes \mathbb{Q} (= X^*(\widehat{T}) \otimes \mathbb{Q})$ such that $\langle \lambda_i, \alpha_j \rangle = \delta_{i,j}$. We have $\lambda_1, \lambda_2 \in X_*(T)$, $\lambda_3 \notin X_*(T)$ and $2\lambda_3 \in X_*(T)$ as well as $X_*(T) = \langle \lambda_1, \lambda_2, 2\lambda_3 \rangle_{\mathbb{Z}}$. In particular

$$P^+ = \{\lambda \in X_*(T) \mid \langle \lambda, \alpha_i \rangle \geq 0,\ 1 \leq i \leq 3\} \tag{7.12}$$

is the "integral cone" on $\lambda_1, \lambda_2$ and $2\lambda_3$.

As before, the finite-dimensional irreducible representations of $\mathrm{SO}_7(\mathbb{C})$ are in bijection with $P^+$; we set $V_\lambda$ the representation corresponding to $\lambda$ in this sense and $\chi_\lambda$ the character of this representation. In this notation $V_{\lambda_1}$ is the 7-dimensional standard representation. Using Weyl's character formula 7.1.1 and a suitable computer algebra system one easily verifies the following identities:

$$V_{\lambda_2} = \bigwedge^2 V_{\lambda_1},\ V_{2\lambda_3} = \bigwedge^3 V_{\lambda_1}. \tag{7.13}$$

**Lemma 7.3.2** *The spherical Hecke algebra of* $\mathrm{Sp}_6(F)$ *is generated by the characteristic functions of the double cosets of* $\lambda_1(\pi), \lambda_2(\pi)$ *and* $(2\lambda_3)(\pi)$*, and we will denote these generators by* $c_1, c_2$ *and* $c_3$*, respectively.*

*We set* $2\rho := \sum_{\alpha\in\Phi^+} \alpha \in X^*(T)$ *as before. The following holds:*

$$\begin{aligned} \mathcal{S}(c_1) &= q^3\chi_{\lambda_1} - 1, \\ \mathcal{S}(c_2) &= q^5\chi_{\lambda_2} - q^3\chi_{\lambda_1} - (q^4 + q^2) \\ \mathcal{S}(c_3) &= q^6\chi_{2\lambda_3} - q^5\chi_{\lambda_2} - q^5\chi_{\lambda_1} + q^2. \end{aligned} \tag{7.14}$$

*Proof.* $\underline{\mathcal{S}(c_1)}$: In the partial ordering on $P^+$ we have $0 \prec \lambda_1$ and 0 is the only element smaller than $\lambda_1$. The pairing with $\rho$ yields $\langle\lambda_1, \rho\rangle = 3$ and $(W_0)_{\lambda_1}$ is a Coxeter group of type $B_2$ so the length generating polynomial evaluated at $q^{-1}$ is $(W_0)_{\lambda_1}(q^{-1}) = 1 + 2q^{-1} + 2q^{-2} + 2q^{-3} + q^{-4}$. Using Weyl's character formula 7.1.1 and Macdonald's formula 7.1.3 one computes (e.g. with Magma ([BCP97]))

$$\mathcal{S}(c_1) = q^3\chi_{\lambda_1} - 1. \tag{7.15}$$

$\underline{\mathcal{S}(c_2)}$: We have $0 \prec \lambda_1 \prec \lambda_2$, $\langle\lambda_2, \rho\rangle = 5$ and $(W_0)_{\lambda_2}(q^{-1}) = (A_1 + A_1)(q^{-1}) = 1 + 2q^{-1} + q^{-2}$. Hence we compute

$$\mathcal{S}(c_2) = q^5\chi_{\lambda_2} - q^3\chi_{\lambda_1} - (q^4 + q^2). \tag{7.16}$$

$\underline{\mathcal{S}(c_3)}$: We have $0 \prec \lambda_1 \prec \lambda_2 \prec 2\lambda_3$, $\langle 2\lambda_3, \rho\rangle = 6$ and $(W_0)_{\lambda_2}(q^{-1}) = (A_2)(q^{-1}) = 1 + 2q^{-1} + 2q^{-2} + q^{-3}$. Hence we compute

$$\mathcal{S}(c_3) = q^6\chi_{2\lambda_3} - q^5\chi_{\lambda_2} - q^5\chi_{\lambda_1} + q^2. \tag{7.17}$$

□

The explicit knowledge of the Satake homomorphism now allows us to determine what lifts of modular froms from $G_2$ to $\mathrm{Sp}_6$ should look like.

**Lemma 7.3.3** *Now let* $f$ *be an eigenform for the Hecke algebra of* $G_2(F)$ *with* $fb_1 = af$, $fb_2 = bf$ *and corresponding Satake parameter* $s(f)$*. We denote the embedding* $G_2(\mathbb{C}) \hookrightarrow \mathrm{SO}_7(\mathbb{C})$ *by* $\phi$*.*

*If there is a lift* $f'$ *of* $f$ *with Satake parameter* $s(f') = \phi(s(f))$ *the generators* $c_1, c_2$ *and* $c_3$ *of the spherical Hecke algebra of* $\mathrm{Sp}_6$ *have the eigenvalues* $a, b + q^2a$ *and* $a^2 - (2q^2 + q - 1)a - (q + 1)b - (q^5 + q^4 + q^2 + q)$ *on* $f'$*:*

*Proof.* We compute the following.

$$\begin{aligned} \mathcal{S}(c_1)(s(f')) &= q^3\chi_{\lambda_1}(s(f')) - 1 \\ &= q^3\chi_{\eta_1}(s(f)) - 1 \\ &= \mathcal{S}(b_1)(s(f)) \\ &= a. \end{aligned} \tag{7.18}$$

$$\begin{aligned}
\mathcal{S}(c_2)(s(f')) &= q^5\chi_{\lambda_2}(s(f')) - q^3\chi_{\lambda_1}(s(f')) - q^4 - q^2 \\
&= q^5(\wedge^2\chi_{\lambda_1}(s(f'))) - a - 1 - q^4 - q^2 \\
&= q^5(\chi_{\eta_1}(s(f)) + \chi_{\eta_2}(s(f))) - a - 1 - q^4 - q^2 \\
&= a + b + 1 + q^4 + q^2(a+1) - a - 1 - q^4 - q^2 \\
&= b + q^2 a.
\end{aligned} \tag{7.19}$$

And in an analogous fashion:

$$\begin{aligned}
\mathcal{S}(c_3)(s(f')) &= q^6\chi_{2\lambda_3}(s(f')) - q^5\chi_{\lambda_2}(s(f')) - q^5\chi_{\lambda_1}(s(f')) + q^2 \\
&= q^6 \wedge^3 \chi_{\lambda_1}(s(f')) - (a + b + 1 + q^4 + q^2 a + q^2) - q^2(a+1) + q^2 \\
&= q^6 \wedge^3 \chi_{\eta_1}(s(f)) - a - b - 1 - q^4 - 2q^2 a - q^2 \\
&= q^6(\chi_{\eta_1}(s(f))^2 - \chi_{\eta_2}(s(f))) - a - b - 1 - q^4 - 2q^2 a - q^2 \\
&= (a+1)^2 - q(a + b + 1 + q^4) - a - b - 1 - q^4 - 2q^2 a - q^2 \\
&= a^2 - (2q^2 + q - 1)a - (q+1)b - (q^5 + q^4 + q^2 + q).
\end{aligned} \tag{7.20}$$

□

In the above proof we have extensively used the fact that $\chi_{\lambda_1}(\phi(s(f))) = \chi_{\eta_1}(s(f))$ as well as the identities on the representations for $G_2$ and $\mathrm{SO}_7$.

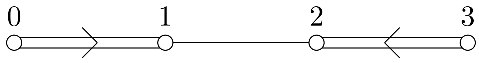
The following lemma provides a plausibility check for our computations.

**Lemma 7.3.4** *The trivial representation of the Hecke algebra of $G_2$ lifts to the trivial representation of the Hecke algebra of* $\mathrm{Sp}_6$.

*Proof.* Under the trivial representation each double coset acts simply by multiplying with its degree (i.e. the number of right cosets contained in it). Thus we should be able to predict these numbers for $c_1, c_2$ and $c_3$, provided we know them for $b_1$ and $b_2$. As already computed, the number of single cosets in $b_1$ is $q^6+q^5+q^4+q^3+q^2+q$ and the number of single cosets in the double coset $b_2$ is $q^{10}+q^9+...+q^5$. Evaluating the formulas from Lemma 7.3.3 above at these two values we expect $c_1$ to decompose into $q^6+q^5+q^4+q^3+q^2+q$ single cosets, $c_2$ into $q^2(q^6+..+q)+q^{10}+q^9+...+q^5 = q^3(1+q+2q^2+2q^3+2q^4+2q^5+q^6+q^7)$ single cosets, and $c_3$ into $q^6(1+q+q^2+2q^3+q^4+q^5+q^6)$ single cosets.

On the other hand we know from Lansky and Pollack ([LP02]), or equivalently from the formulas in Chapter 4, that $K\lambda(\pi)K$, $\lambda \in P^+$, decomposes into $q^{\ell(\sigma_0(\lambda))} \cdot \sum_{w \in [W_0{}^{\lambda}\backslash W_0]} q^{\ell(w)}$ right-cosets, where $\sigma_0(\lambda)$ is the element of shortest length in the double coset $W_0 t_\lambda W_0$.

We compute these values for $\mathrm{Sp}_6$: As previously noted, the (extended) affine Weyl group of $\mathrm{Sp}_6$ is just the affine Weyl group of type $\widetilde{C_3}$ with Dynkin diagram

$\widetilde{C}_3$ : 

and generators $s_i = s_{\alpha_i}, 1 \leq i \leq 3$, corresponding to the simple roots (these generate $W_0$) and $s_0 = s_{\alpha_0} t_{-\alpha_0^\vee}$.

A reduced expression for $t_{\lambda_1} = t_{\alpha_0^\vee}$ is given by $s_0 s_1 s_2 s_3 s_2 s_1$, $\sigma_0(\lambda_1) = s_0$ and the stabilizer is ${W_0}^{\lambda_1} = \langle s_2, s_3 \rangle$, whence

$$[{W_0}^{\lambda_1} \backslash W_0] = \{1, s_1, s_1 s_2, s_1 s_2 s_3, s_1 s_2 s_3 s_2, s_1 s_2 s_3 s_2 s_1\}. \tag{7.21}$$

Hence we can conclude that $c_1$ indeed decomposes into

$$q \sum_{i=0}^{5} q^i = q^6 + q^5 + ... + q \tag{7.22}$$

single cosets.

A reduced expression for $t_{\lambda_2}$ is given by $s_0 s_1 s_0 s_2 s_1 s_3 s_2 s_3 s_1 s_2$, $\sigma_0(\lambda_2) = s_0 s_1 s_0$ and the stabilizer is ${W_0}^{\lambda_2} = \langle s_1, s_3 \rangle$, so

$$\begin{aligned}[{W_0}^{\lambda_2} \backslash W_0] = \{&1, s_2, s_2 s_1, s_2 s_3, s_2 s_1 s_3, s_2 s_3 s_2, s_2 s_1 s_3 s_2, s_2 s_3 s_2 s_1, \\ &s_2 s_1 s_3 s_2 s_1, s_2 s_1 s_3 s_2 s_3, s_2 s_1 s_3 s_2 s_1 s_3, s_2 s_1 s_3 s_2 s_1 s_3 s_2\}.\end{aligned} \tag{7.23}$$

Hence we can conclude that $c_2$ indeed decomposes into

$$q^3(1 + q + 2q^2 + 2q^3 + 2q^4 + 2q^5 + q^6 + q^7) \tag{7.24}$$

single cosets as predicted.

A reduced expression for $t_{2\lambda_3}$ is given by $s_0 s_1 s_0 s_2 s_1 s_0 s_3 s_2 s_3 s_1 s_2 s_3$, $\sigma_0(2\lambda_3) = s_0 s_1 s_0 s_2 s_1 s_0$ and the stabilizer is ${W_0}^{2\lambda_3} = \langle s_1, s_2 \rangle$, so

$$\begin{aligned}[{W_0}^{2\lambda_3} \backslash W_0] = \{&1, s_3, s_3 s_2, s_3 s_2 s_1, s_3 s_2 s_3, s_3 s_2 s_1 s_3, \\ &s_3 s_2 s_1 s_3 s_2, s_3 s_2 s_1 s_3 s_2 s_3\}.\end{aligned} \tag{7.25}$$

Hence we can conclude that $c_3$ indeed decomposes into $q^6(1 + q + q^2 + 2q^3 + q^4 + q^5 + q^6)$ single cosets as predicted. This completes the proof. $\square$

Some nontrivial modular forms for $\mathrm{Sp}_6$ which appear to be lifts can be found in the chapter on computational results.

# 8. Applications to $S$-arithmetic Groups

So far our algorithms for determining the combinatorial structure of the affine building of an algebraic group were used to obtain the action of certain Hecke operators on the space of algebraic modular forms. In this chapter we want to present another application, namely studying the structure of $S$-arithmetic subgroups of a linear algebraic group.

## 8.1. Group Homology

The first application in which we are interested, is constructing a free resolution for an $S$-arithmetic subgroup.

As before let $k$ be a totally real number field and $\mathbb{G}$ a connected, semisimple, linear algebraic group defined over $k$ with $\mathbb{G}(k_\infty)$ compact. We denote by $\mathcal{P}$ the set of finite primes of $k$ and by $\hat{k}$ the finite adeles of $k$. Let $S \subset \mathcal{P}$ be some finite set of finite primes and fix the following notation:

$$\begin{aligned} k_S &:= \prod_{\mathfrak{p} \in S} k_{\mathfrak{p}} \\ \mathcal{O}_S &= \{x \in k \mid x \in \mathcal{O}_{\mathfrak{p}} \text{ for all } \mathfrak{p} \notin S\} \\ \hat{k}^S &:= \prod_{\mathfrak{p} \notin S} \mathcal{O}_{\mathfrak{p}}. \end{aligned} \tag{8.1}$$

Note that $k_S \times \hat{k}^S$ can be thought of as a subring of $\hat{k}$.

Let us assume that $\mathbb{G}$ is split at all $\mathfrak{p} \in S$ and choose some integral form $\underline{G} = \mathbb{G}_L$ of $\mathbb{G}$. The group $\underline{G}(\mathcal{O}_S)$ can be identified with a subgroup of $\mathbb{G}(\hat{k})$,

$$\begin{aligned} \underline{G}(\mathcal{O}_S) &= \{g = (g_{\mathfrak{p}})_{\mathfrak{p} \in \mathcal{P}} \mid g \in \mathbb{G}(k), g_{\mathfrak{p}} \in \underline{G}(\mathcal{O}_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \notin S\} \\ &= \mathbb{G}(k) \cap \underline{G}(k_S \times \hat{k}^S), \end{aligned} \tag{8.2}$$

and thus consists of all elements of $\mathbb{G}(k)$ fixing $L$ away from $S$. In particular this group does not depend on $L_{\mathfrak{p}}, \mathfrak{p} \in S$, and moreover two integral forms that coincide with $\underline{G}$ away from $S$ are ($\mathbb{G}(k)$-)isomorphic if and only if they are $\underline{G}(\mathcal{O}_S)$-isomorphic.

For $\mathfrak{p} \in \mathcal{P}$ we denote by $\mathfrak{B}(\mathfrak{p}) := \mathfrak{B}(\mathbb{G}, k_\mathfrak{p})$ the affine building of $\mathbb{G}(k_\mathfrak{p})$ and we define

$$\mathfrak{B}(S) := \prod_{\mathfrak{p} \in S} \mathfrak{B}(\mathfrak{p}), \tag{8.3}$$

the product space of the affine buildings at the places belonging to $S$. Since $\mathbb{G}$ is semisimple, $\mathfrak{B}(S)$ is a product of polysimplicial complexes and thus itself a polysimplicial complex. Each $\mathbb{G}(k_\mathfrak{p})$ admits only finitely many orbits on the facets of $\mathfrak{B}(\mathfrak{p})$, whence the same is true for $\mathbb{G}(k_S)$ and its action on $\mathfrak{B}(S)$. We choose a system of representatives of these orbits, say $\mathfrak{F}'$. For each $f \in \mathfrak{F}'$ we choose an integral form $\underline{G}_f$ that coincides with $\underline{G}$ away from $S$ and such that $\prod_{\mathfrak{p} \in \mathcal{P}} \underline{G}_f(\mathcal{O}_\mathfrak{p})$ is the stabilizer of $f$. This is in fact always possible as follows: The stabilizers of vertices in $\mathfrak{B}(S)$ are maximal compact subgroups of $\mathbb{G}(k_S)$ and thus stabilizers of lattices. Hence by adjusting $L$ at the $\mathfrak{p} \in S$ we can make sure to obtain the stabilizers of vertices. Since facets are just collections of vertices, we get the remaining integral forms as intersections of those integral forms that stabilize vertices.

Via the canonical embedding $\underline{G}(k_S \times \hat{k}^S) \hookrightarrow \mathbb{G}(\hat{k})$ we obtain an action of $\underline{G}(k_S \times \hat{k}^S)$ on the set of integral forms of $\mathbb{G}$ keeping the subset of integral forms, that coincide with $\underline{G}$ away from $S$, stable. The orbits of $\underline{G}(k_S \times \hat{k}^S)$ on this set are the same as those of $\mathbb{G}(\hat{k})$ by the observation above.

We identify each facet with its stabilizer and thus get an action of $\underline{G}(k_S \times \hat{k}^S)$ on the polysimplicial complex $\mathfrak{B}(S)$ with finitely many orbits on the cells (facets) in each dimension. By our identification of cells with integral forms each $\underline{G}(k_S \times \hat{k}^S)$-orbit decomposes into finitely many $\underline{G}(\mathcal{O}_S)$ orbits (since otherwise the genus of the corresponding integral form would consist of infinitely many isomorphism classes). Hence we have an action of $\underline{G}(\mathcal{O}_S)$ on a finite-dimensional polysimplicial complex with finitely many orbits in each dimension and moreover each stabilizer (in $\underline{G}(\mathcal{O}_S)$) of a cell is finite.

We now want to use this information to construct a free resolution of $\mathbb{Z}$ as a $\underline{G}(\mathcal{O}_S)$-module. To that end let $\mathfrak{B}(S)_i$ be the set of faces of dimension $i$ in $\mathfrak{B}(S)$. Each $\mathfrak{B}(S)_i$ is a $\underline{G}(\mathcal{O}_S)$-set (with finitely many orbits) and thus gives rise to a finite-rank $\mathbb{Z}\left[\underline{G}(\mathcal{O}_S)\right]$-module. Let us choose for each $i$ a system $\mathfrak{F}_i$ of representatives of $\mathfrak{B}(S)_i$ modulo the action of $\underline{G}(\mathcal{O}_S)$ and denote by $S_f$ the stabilizer of $f \in \mathfrak{B}(S)_i$ in $\underline{G}(\mathcal{O}_S)$. We endow each representative $f$ with an orientation and denote by $\chi_f : S_f \to \{\pm 1\}$ the corresponding character of $S_f$ (i.e. $\chi_f(s) = 1$ if and only if $s$ fixes the orientation of $f$). Note that $\chi_f \equiv 1$ for all $f$ if $\mathbb{G}$ is simply connected.

Now we can identify the $\underline{G}(\mathcal{O}_S)$-module with underlying set $\mathfrak{B}(S)_i$ with

$$M_i := \bigoplus_{f \in \mathfrak{F}_i} \mathbb{Z}\left[\underline{G}(\mathcal{O}_S)\right] \otimes_{\mathbb{Z}S_f} \mathbb{Z}^{\chi_f}, \tag{8.4}$$

where $\mathbb{Z}^{\chi_f}$ denotes the $S_f$-module $\mathbb{Z}$ with action given by $\chi_f$. The cellular chain map $\partial : \mathbb{Z}\mathfrak{B}(S)_i \to \mathbb{Z}\mathfrak{B}(S)_{i-1}$ of $\mathfrak{B}(S)$ gives rise to $\underline{G}(\mathcal{O}_S)$-module homomorphisms $M_i \to M_{i-1}$ which we will again call $\partial$. This makes

$$M : ...0 \to 0 \to M_d \to M_{d-1} \to ... \to M_1 \to 0 \tag{8.5}$$

into an acyclical chain complex of $\underline{G}(\mathcal{O}_S)$ modules (where $d$ denotes the dimension of $\mathfrak{B}(S)$). Since $\mathfrak{B}(S)$ is contractible (as a product of contractible spaces), we have $H_0(M) \cong \mathbb{Z}$. However, this is not a free resolution of $\mathbb{Z}$ as a $\underline{G}(\mathcal{O}_S)$-module since the $M_i$ are not free (as long as at least one of the stabilizers is nontrivial).

Nevertheless we can use the chain complex to construct a free resolution by employing the following result, the original idea of which is due to Charles T. C. Wall (cf. [Wal61]) and which was first used by Graham Ellis et al (cf. [EHS06]) in the presented form.

**Theorem 8.1.1 ([DSES11, Lemma 4])** *Let $H$ be a group and $\{A_{p,q} \mid p, q \geq 0\}$ a bigraded family of free $\mathbb{Z}H$-modules with module homomorphisms $d_0 : A_{p,q} \to A_{p,q-1}$ , such that $(A_{p,*}, d_0)$ becomes an acyclic chain complex for every $p$. We set $C_p := H_0(A_{p,*})$ and assume the existence of homomorphisms $\partial : C_p \to C_{p-1}$ such that $(C_*, \partial)$ again is a $\mathbb{Z}H$-chain complex. Then we have:*

1. *There are $\mathbb{Z}H$-homomorphisms $d_k : A_{p,q} \to A_{p-k,q+k-1}$ for $k \geq 1, p > k$ with the property:*

$$d = d_0 + d_1 + d_2 + ... : R_n := \bigoplus_{p+q=n} A_{p,q} \to R_{n-1} = \bigoplus_{p+q=n-1} A_{p,q}$$

   *is the boundary map of a chain complex $R_*$ of free $\mathbb{Z}H$-modules. We call $d$ a perturbation.*

2. *The canonical chain maps $\phi_p : A_{p,*} \to H_0(A_{p,*})$ yield a chain map $\phi_* : R_* \to C_*$, which induces a homology isomorphism.*

3. *Assume there are $\mathbb{Z}$-module homomorphisms $h_0 : A_{p,q} \to A_{p,q+1}$, such that $d_0 h_0 d_0(x) = d_0(x)$ for all $x \in A_{p,q+1}$ (a so-called contraction homotopy). Then we can construct $d_k$ by first lifting $\partial$ to $d_1 : A_{p,0} \to A_{p-1,0}$ set recursively $d_k = -h_0(\sum_{i=1}^{k} d_i d_{k-i})$ on the free generators of $A_{p,q}$.*

In our situation the group $H$ from the above theorem is $\underline{G}(\mathcal{O}_S)$ and the chain complex $(C_*, \partial)$ is given by $(M_*, \partial)$. For each $f \in \mathfrak{F}_i$ we choose a $\mathbb{Z}[S_f]$-free resolution of the integers. Forming first the tensor product with $\mathbb{Z}^{\chi_f}$ and then tensoring with $\mathbb{Z}[\underline{G}(\mathcal{O}_S)]$ we obtain a free resolution of $\mathbb{Z}[\underline{G}(\mathcal{O}_S)] \otimes_{\mathbb{Z}S_f} \mathbb{Z}^{\chi_f}$. We take the direct sum over all $f \in \mathfrak{F}_i$ and thus get a $\mathbb{Z}[\underline{G}(\mathcal{O}_S)]$-free resolution of $M_i$, which we take as the chain complex $(A_{i,*}, d_0)$ in the above theorem. Note that the implemented algorithm for constructing a free resolution for a finite group are also capable of constructing the contraction homotopy we need to turn the above theorem into an algorithm.

Sadly this algorithm is only applicable if the combinatorial structure of $\mathfrak{B}(S)$ is sufficiently nice and the stabilizers do not get too big. However, if we are in a situation where this is not the case we can still obtain at least the rational homology (so ignoring the torsion) by eliminating the cells whose stabilizers do not act orientation preservingly, tensoring our chain complex with $\mathbb{Q}$, and then taking the homology (cf. [EVGS13]).

**Example 8.1.2** *A combinatorially rather simple example is provided by the model of the compact form of $G_2$ over the rationals: We set $S = \{2\mathbb{Z}\}$ and $\underline{G}$ the (up to isomorphism) unique model of $G_2$ over the integers. The group $G_2(\mathbb{Q}_2)$ has one orbit on the chambers, three orbits on the edges, and three orbits on the vertices of $\mathfrak{B}(S) = \mathfrak{B}(G_2, \mathbb{Q}_2)$. One easily checks (e.g. by employing the mass formula) that $\underline{G}(\mathbb{Z}_S) = \underline{G}\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$ is already transitive on all of these orbits. We choose representatives $v_1, v_2, v_3$ of the vertices such that $|S_{v_1}| = 12096, |S_{v_2}| = 576, |S_{v_3}| = 1344$ and $(v_1, v_2, v_3)$ forms a chamber. The orbits on the edges are represented by $(v_1, v_2), (v_2, v_3)$ and $(v_3, v_1)$ (all with stabilizer order $192$) and we choose the orientation such that (in this notation) we have $\partial(v_i, v_j) = v_i - v_j$. The orbit on the chambers is represented by $(v_1, v_2, v_3)$ with stabilizer order $64$ and $\partial(v_1, v_2, v_3) = (v_1, v_2) + (v_2, v_3) + (v_3, v_1)$. Since $G_2$ is simply connected, the action of the stabilizers on the orientation is trivial.*

*The implemented algorithms in GAP are not powerful enough to handle resolutions for finite groups of order $12096$ in large degrees. However, we were able to verify using HAP ([Ell13]) that $H_1(\underline{G}(\mathbb{Z}_S)) \cong \underline{G}(\mathbb{Z}_S)/\underline{G}(\mathbb{Z}_S)'$ is trivial. Furthermore $\underline{G}(\mathbb{Z}_S)$ has trivial rational homology (except of course at $0$) which we see by applying the above observation.*

**Example 8.1.3** 1. *Let $H = \left(\frac{-2,-5}{\mathbb{Q}}\right)$ be the quaternion algebra over $\mathbb{Q}$ which is ramified at $5$ and infinity. We choose a maximal order $\mathcal{O}$ in $H$ and denote by $\underline{G}$ the integral form of $U_2(H)$ defined by $\mathcal{O}^2$. Again we set $S = \{2\mathbb{Z}\}$ and are interested in the group $\underline{G}(\mathbb{Z}_S) = \underline{G}(\mathbb{Z}\left[\frac{1}{2}\right])$.*

*There are three genera of integral forms corresponding to vertices in the affine building $\mathfrak{B}_2$ of $U_2(H)$ at $2$, each of which decomposes into $2$ isomorphism classes. This means there are $6$ orbits of vertices under the action of $\underline{G}(\mathbb{Z}_S)$. The edges of $\mathfrak{B}_2$ come in three genera (corresponding to the three possible pairs of genera of vertices) which decompose into $3$, $5$, and $3$ orbits under the action of $\underline{G}(\mathbb{Z}_S)$, respectively. Finally, there is one genus of chambers which decomposes into $6$ orbits.*

*The largest appearing stabilizer (fixing a hyperspecial vertex) is of order $240$, which still makes it difficult to compute a free resolution to a high degree from this information. However, we can verify the following identities:*

$$\begin{aligned} &H_1(\underline{G}(\mathbb{Z}_S), \mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z}, \\ &H_2(\underline{G}(\mathbb{Z}_S), \mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} \textit{ and} \\ &H_n(\underline{G}(\mathbb{Z}_S), \mathbb{Q}) \cong \{1\} \textit{ for all } n \geq 1. \end{aligned} \tag{8.6}$$

2. *We now take* $H = \left(\frac{-1,-7}{\mathbb{Q}}\right)$ *the quaternion algebra over* $\mathbb{Q}$ *ramified at* 7 *and infinity. Let us otherwise keep the notation from the first example. The vertices, edges and chambers in this case decompose into* 8, 24 *and* 16 *orbits, respectively (so quite a bit more than above). However, the largest occuring stabilizer has order* 48 *(again fixing a hyperspecial point), whence we can construct the free resolution to a higher degree obtaining the following results:*

$$\begin{aligned}
&H_1(\underline{G}(\mathbb{Z}_S),\mathbb{Z}) \cong \mathbb{Z}/8\mathbb{Z},\\
&H_2(\underline{G}(\mathbb{Z}_S),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},\\
&H_3(\underline{G}(\mathbb{Z}_S),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/48\mathbb{Z},\\
&H_4(\underline{G}(\mathbb{Z}_S),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z},\\
&H_5(\underline{G}(\mathbb{Z}_S),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z},\\
&H_6(\underline{G}(\mathbb{Z}_S),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},\\
&H_n(\underline{G}(\mathbb{Z}_S),\mathbb{Q}) \cong \{1\} \textit{ for all } n \geq 1.
\end{aligned} \tag{8.7}$$

## 8.2. Generators and Relations

An alternative application of our algorithms to $S$-arithmetic groups is the computation of a presentation, or, to be more precise, of a generating system together with defining relations in these generators. To some extent one could think of this as a solved problem in light of the last section, since a free resolution actually gives rise to a presentation. However, we have already seen that the algorithms for computing a resolution are not always applicable and thus we want to give an independent description of how to obtain generators and relations.

All of the notation from the last section remains valid.

The method we will use is known under the name Bass-Serre theory, but since we might have cell stabilizers which do not preserve the orientation we cannot strictly follow the original articles [Ser77] and [Bas93]. This problem is dealt with in [Bro84] and we will essentially repeat the synopsis found in [BCNS15]. The theory is applicable since $\mathfrak{B}_S$ is contractible and hence simply connected. Note that in the following construction we can skip all steps dealing with orientation reversing stabilizers if we are working with a simply connected group.

Denote by $\mathcal{V}$ the set of vertices of $\mathfrak{B}_S$ (cells of dimension 0) and by $\mathcal{E}$ the set of edges of $\mathfrak{B}_S$ (cells of dimension 1), where we have already fixed an orientation for each $e \in \mathcal{E}$, i.e. $e$ comes with two vertices $o(e)$ and $t(e)$ in $\mathcal{V}$, the origin and target of $e$, respectively. Note that we can think of $(\mathcal{V}, \mathcal{E})$ as a (directed) graph. For $e \in \mathcal{E}$ we set $\overline{e}$ the same edge with reversed orientation (i.e. $o(\overline{e}) = t(e)$ and

$t(\overline{e}) = o(e)$). We say that the orientation of $e$ is preserved under the action of $\underline{G}(\mathcal{O}_S)$ if there is no $g \in \underline{G}(\mathcal{O}_S)$ such that $eg = \overline{e}$; otherwise we say $e$ is reversed by the action. Let us assume that we have chosen the orientation for the cells in $\mathcal{E}$ in a way that $o(eg) = o(e)g$ and $t(eg) = t(e)g$ for all $g \in \underline{G}(\mathcal{O}_S)$ whenever the orientation of $e$ is preserved by the action of $\underline{G}(\mathcal{O}_S)$. Obviously this is always possible.

We decompose $\mathcal{E} = \mathcal{E}^+ \sqcup \mathcal{E}^-$ where $\mathcal{E}^+$ is precisely the set of edges whose orientation is preserved under the action of $\underline{G}(\mathcal{O}_S)$. By

$$\underline{G}(\mathcal{O}_S)_e = \underline{G}(\mathcal{O}_S)_{o(e)} \cap \underline{G}(\mathcal{O}_S)_{t(e)} \tag{8.8}$$

we denote the stabilizer of $e$ together with its orientation and by $\underline{G}(\mathcal{O}_S)_{e,\overline{e}} = \underline{G}(\mathcal{O}_S)_{\{o(e),t(e)\}}$ the stabilizer of $e$ ignoring the orientation. Clearly we have $[\underline{G}(\mathcal{O}_S)_{e,\overline{e}} : \underline{G}(\mathcal{O}_S)_e]$ equal to 1 or 2 according to $e \in \mathcal{E}^+$ or $e \in \mathcal{E}^-$.

We fix a tree $\mathcal{T}$ in our graph such that the vertices $\mathcal{V}_\mathcal{T}$ of $\mathcal{T}$ form a system of representatives of $\mathcal{V}/\underline{G}(\mathcal{O}_S)$ and such that each edge of $\mathcal{T}$ is in $\mathcal{E}^+$. Choose systems $E^+, E^-$ of representatives of $\mathcal{E}^+/\underline{G}(\mathcal{O}_S)$ and $\mathcal{E}^-/\underline{G}(\mathcal{O}_S)$ such that $o(e) \in \mathcal{V}_\mathcal{T}$ for all $e \in E^+ \cup E^-$. For any $e \in E^+$ we choose an element $g_e \in \underline{G}(\mathcal{O}_S)$ such that $t(e)g_e^{-1} \in \mathcal{V}_\mathcal{T}$ (with the convention $g_e = 1$ if $t(e) \in \mathcal{V}_\mathcal{T}$). For any $e \in E^-$ we choose $g_e \in \underline{G}(\mathcal{O}_S)_{e,\overline{e}} - \underline{G}(\mathcal{O}_S)_e$.

Finally we choose a system of representatives $\mathcal{F}$ of the dimension 2-cells of $\mathfrak{B}_S$ modulo the action of $\underline{G}(\mathcal{O}_S)$ and fix for any $f \in \mathcal{F}$ a sequence $(e_1, ..., e_m)$ of edges with the following properties:

- The 1-cells in the boundary of $f$ are exactly $\{e_1, ..., e_m\}$.
- $o(e_1) \in \mathcal{V}_\mathcal{T}$.
- $o(e_{i+1}) = t(e_i)$ for $1 \leq i \leq m-1$ and $t(e_m) = o(e_1)$.
- $e_{i+1} \neq \overline{e_i}$ for all $1 \leq i \leq m-1$ and $e_1 \neq \overline{e_m}$.

For each $1 \leq i \leq m$ we fix an element $g_i$ in the following way: Find the edge $e \in E^+ \cup E^-$ in the same orbit as $e_i$ and an element $g \in \underline{G}(\mathcal{O}_S)$ such $eg = e_i$ (this is in general not unique). Then set $g_i$ the conjugate of $g_e$ by $g$. In particular this means we get $t(e_i)g_i g_{i-1} ... g_1 \in \mathcal{V}_\mathcal{T}$ and thus $g_m ... g_1 \in \mathbb{G}(\mathcal{O}_S)_{o(e_1)}$. We call $(g_1, ..., g_m)$ the cycle associated to $f$.

**Theorem 8.2.1 ([Bro84, Thm. 1])** *The group $\underline{G}(\mathcal{O}_S)$ is generated by the $\underline{G}(\mathcal{O}_S)_v$, $v \in \mathcal{V}_\mathcal{T}$, and the elements $g_e$, $e \in E^+ \cup E^-$, subject to the following relations:*

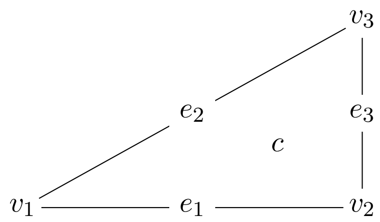
1. *The multiplication table of the groups $\underline{G}(\mathcal{O}_S)_v$, $v \in \mathcal{V}_\mathcal{T}$.*
2. *$g_e = 1$ if $e$ is an edge of $\mathcal{V}_\mathcal{T}$.*

3. $g_e \cdot g \cdot g_e^{-1} \in \underline{G}(\mathcal{O}_S)_{t(e)g_e^{-1}}$ *for all* $e \in E^+$ *and* $g \in \underline{G}(\mathcal{O}_S)_e \subset \underline{G}(\mathcal{O}_S)_{o(e)}$.

4. $g_e \cdot g \cdot g_e \in \underline{G}(\mathcal{O}_S)_{o(e)}$ *for all* $e \in E^-$ *and* $g \in \underline{G}(\mathcal{O}_S)_e \subset \underline{G}(\mathcal{O}_S)_{o(e)}$.

5. $g_m \cdot ... \cdot g_1 \in \underline{G}(\mathcal{O}_S)_{o(e_1)}$ *for any cycle* $(g_1, ..., g_m)$ *associated to an element of* $\mathcal{F}$ *as above.*

**Example 8.2.2** *We return to the situation of Example 8.1.2 and describe how to obtain a presentation for* $\underline{G}(\mathbb{Z}_S)$. *We choose a chamber* $c$ *in* $\mathfrak{B}_2$ *with vertices* $v_1, v_2, v_3$ *and edges* $e_1 = (v_1, v_2), e_2 = (v_1, v_3)$ *and* $e_3 = (v_2, v_3)$. *Since* $G_2$ *is simply connected we need not actually choose an orientation.*




*As the edges for the tree* $\mathcal{T}$ *we choose* $e_1$ *and* $e_2$ *and as the set* $E^+$ *we take* $\{e_1, e_2, e_3\}$. *By our convention we necessarily have* $g_{e_i} = 1$ *for* $i = 1, 2, 3$. *Finally we have* $\mathcal{F} = \{c\}$ *with attached cycle* $(1, 1, 1)$. *Thus the only relations we need to consider are those of 1. and 3. in Theorem 8.2.1.*

*In conclusion we see that* $\underline{G}(\mathbb{Z}_S)$ *is generated by the three stabilizers of vertices subject only to the relations in these finite groups and the relations coming from the fact that they intersect nontrivially. After simplifying the presentation we obtain a presentation of* $\underline{G}(\mathbb{Z}_S)$ *on two generators of order* $7$ *(lying in* $\underline{G}(\mathbb{Z}_S)_{e_2}$*) and order* $3$ *(lying in* $\underline{G}(\mathbb{Z}_S)_{v_2}$*), respectively, with too many relations to be printed here. However, we can use this presentation to verify our earlier result* $H_1(\underline{G}(\mathbb{Z}_S)) \cong \underline{G}(\mathbb{Z}_S)/\underline{G}(\mathbb{Z}_S)' \cong \{1\}$.

# 9. Computational Results

We now want to present some of the results we obtained using our implementation of the described algorithms. Obviously we cannot cover all of our computations here due to space constraints so we aim instead for giving an overview of the capabilities of the written programs, focussing primarily on algebraic modular forms for symplectic groups.

## 9.1. Reliability

First we briefly want to discuss the reliability of our implementation.

Luckily the theory of algebraic modular forms offers a variety of rather strong plausibility checks for our results.

When we enumerate a set of representatives of lattices in a given genus we have the mass formula which postulates that the inverses of the stabilizer orders should add up to a certain (precomputed) rational number. Moreover we often obtain several systems of representatives for the same genus from distinct computations which yields an additional check.
Furthermore two Hecke operators which are supported at distinct primes (or at the same primes $\mathfrak{p}$, where $K_{\mathfrak{p}}$ is hyperspecial) necessarily have to commute. This is a particularly strong check since the representing matrices may have - depending on the dimension of the space - several hundred entries, so the probability that two such matrices commute by chance is essentially zero. Moreover the Hecke operators we compute have a prescribed adjoint with respect to the Peterson scalar product (most of them ought to be self-adjoint).
Finally we can look at the eigenvalues of a Hecke operator that is only supported at a single prime $\mathfrak{p}$ and we know that these should be (algebraic) integers away from $\mathfrak{p}$. This is no condition whatsoever if we are dealing with the trivial representation (since the representing matrices have only integral entries in this case anyway) but it offers an additional check in the case of nontrivial weight.

Our results passed all of these checks in several hundred sample computations we performed which should be seen as a strong indicator for the validity of our computations.

## 9.2. Algebraic Modular Forms for $G_2$

### 9.2.1. Genera of Maximal Integral Forms

While it is certainly not the main task, computing a system of representatives in a given genus of maximal integral forms is still an important step in our algorithms. Therefore we want to take the time to present some of the results we obtained.

Table 9.1 lists all 26 genera of maximal integral forms of the compact form of $G_2$ over $\mathbb{Q}$ with mass less than or equal to 1 (in particular this includes all one-class genera of maximal integral forms over $\mathbb{Q}$). The results should be understood as follows: The first column specifies the genus via a list of pairs $(p, i)$ with $p$ a prime and $i \in \{2, 3\}$ specifying the label of the integral form as an element of the local building at $p$. This means we write $(p, 2)$ if the corresponding local Dynkin diagram at $p$ is $A_1 + A_1$ and $(p, 3)$ if it is $A_2$. At all primes which are not listed we take the integral form to be hyperspecial. The second column simply contains the mass of the genus, the third column its class number $h$, and the last column gives the mass decomposition in suggestive notation.

Since our algorithms work over general (totally real) number fields we present the same computation over $\mathbb{Q}\left(\sqrt{5}\right)$. Table 9.2 contains all genera of maximal integral forms of $G_2$ over $\mathbb{Q}\left(\sqrt{5}\right)$ of mass less than or equal to one. It should be read the same way as the table for $\mathbb{Q}$ and we write $\mathfrak{p}_p$ for a prime ideal above $p$. Genera corresponding to Galois conjugate ideals behave essentially the same way so we only consider one representative of each orbit.

### 9.2.2. Hecke Operators

Table 9.3 describes the action of some Hecke operators on certain spaces of algebraic modular forms. While in principle there is not much added difficulty in working with forms of arbitrary weight we focus here on the trivial weight case. The table should be read as follows: The first collumn describes a genus of maximal integral forms of $G_2$ over $\mathbb{Q}$ in the same way as in the last section, the second column contains the dimension of the space $M(\mathrm{triv.}, K)$ where $K$ is a maximal compact subgroup of $G_2(\hat{\mathbb{Q}})$ in the conjugacy class defined by the genus. In the last column we present the eigenvalues of certain Hecke operators, where we denote by $h_p(w)$ the Hecke operator which is supported only at $p$ with coset representative $w$ in the extended affine Weyl group. If there are multiple Hecke operators given for a single space we order the eigenspaces in the same way for all of these. If an eigenvalue is not in $\mathbb{Q}$ we instead write down its minimal polynomial as an element of $\mathbb{Z}[x]$. Again we cannot present every single result we obtained so we simply give some examples of the capibilities of our algorithms.

| Genus | Mass | h | Mass decomposition |
|---|---|---|---|
| $[\,]$ | $\frac{1}{12096}$ | 1 | $\frac{1}{12096}$ |
| $[(2,2)]$ | $\frac{1}{576}$ | 1 | $\frac{1}{576}$ |
| $[(2,3)]$ | $\frac{1}{1344}$ | 1 | $\frac{1}{1344}$ |
| $[(3,2)]$ | $\frac{13}{1728}$ | 2 | $\frac{1}{192}+\frac{1}{432}$ |
| $[(2,3),(3,2)]$ | $\frac{13}{192}$ | 3 | $\frac{1}{24}+\frac{1}{48}+\frac{1}{192}$ |
| $[(3,3)]$ | $\frac{1}{432}$ | 1 | $\frac{1}{432}$ |
| $[(2,2),(3,3)]$ | $\frac{7}{144}$ | 2 | $\frac{1}{24}+\frac{1}{144}$ |
| $[(2,3),(3,3)]$ | $\frac{1}{48}$ | 1 | $\frac{1}{48}$ |
| $[(5,2)]$ | $\frac{31}{576}$ | 3 | $\frac{1}{36}+\frac{1}{48}+\frac{1}{192}$ |
| $[(5,3)]$ | $\frac{1}{96}$ | 1 | $\frac{1}{96}$ |
| $[(2,3),(5,3)]$ | $\frac{3}{32}$ | 2 | $\frac{1}{12}+\frac{1}{96}$ |
| $[(7,3)]$ | $\frac{43}{1512}$ | 2 | $\frac{1}{42}+\frac{1}{216}$ |
| $[(2,2),(3,2)]$ | $\frac{91}{576}$ | 5 | $\frac{1}{16}+\frac{1}{18}+\frac{1}{32}+\frac{1}{144}+\frac{1}{576}$ |
| $[(2,3),(5,2)]$ | $\frac{31}{64}$ | 6 | $\frac{1}{6}+\frac{3}{12}+\frac{1}{16}+\frac{1}{192}$ |
| $[(2,2),(5,3)]$ | $\frac{7}{32}$ | 3 | $\frac{1}{6}+\frac{1}{24}+\frac{1}{96}$ |
| $[(3,2),(5,3)]$ | $\frac{91}{96}$ | 6 | $\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{24}+\frac{1}{32}$ |
| $[(3,3),(5,3)]$ | $\frac{7}{24}$ | 2 | $\frac{1}{6}+\frac{1}{8}$ |
| $[(7,2)]$ | $\frac{817}{4032}$ | 6 | $\frac{1}{12}+\frac{1}{16}+\frac{1}{36}+\frac{1}{48}+\frac{1}{192}+\frac{1}{336}$ |
| $[(2,2),(7,3)]$ | $\frac{43}{72}$ | 4 | $\frac{1}{4}+\frac{2}{6}+\frac{1}{72}$ |
| $[(2,3),(7,3)]$ | $\frac{43}{168}$ | 3 | $\frac{1}{6}+\frac{1}{21}+\frac{1}{24}$ |
| $[(3,3),(7,3)]$ | $\frac{43}{54}$ | 4 | $\frac{1}{2}+\frac{1}{6}+\frac{1}{8}+\frac{1}{216}$ |
| $[(11,3)]$ | $\frac{37}{336}$ | 2 | $\frac{1}{16}+\frac{1}{21}$ |
| $[(2,3),(11,3)]$ | $\frac{111}{112}$ | 5 | $\frac{1}{2}+\frac{1}{3}+\frac{1}{16}+\frac{2}{21}$ |
| $[(13,3)]$ | $\frac{157}{864}$ | 3 | $\frac{1}{6}+\frac{1}{96}+\frac{1}{216}$ |
| $[(17,3)]$ | $\frac{13}{32}$ | 3 | $\frac{1}{3}+\frac{1}{16}+\frac{1}{96}$ |
| $[(19,3)]$ | $\frac{245}{432}$ | 4 | $\frac{1}{3}+\frac{1}{6}+\frac{1}{16}+\frac{1}{216}$ |

Table 9.1.: Genera of maximal integral forms of $G_2$ over the rationals.

| Genus | Mass | h | Mass decomposition |
|---|---|---|---|
| $[\,]$ | $\frac{67}{302400}$ | 2 | $\frac{1}{7200}+\frac{1}{12096}$ |
| $[(\mathfrak{p}_2,2)]$ | $\frac{871}{14400}$ | 6 | $\frac{1}{36}+\frac{2}{96}+\frac{1}{100}+\frac{1}{576}+\frac{1}{7200}$ |
| $[(\mathfrak{p}_2,3)]$ | $\frac{871}{60480}$ | 4 | $\frac{1}{120}+\frac{1}{216}+\frac{1}{1344}+\frac{1}{1440}$ |
| $[(\mathfrak{p}_5,2)]$ | $\frac{2077}{14400}$ | 6 | $\frac{1}{16}+\frac{1}{18}+\frac{1}{50}+\frac{1}{192}+\frac{1}{1200}+\frac{1}{7200}$ |
| $[(\mathfrak{p}_5,3)]$ | $\frac{67}{2400}$ | 3 | $\frac{1}{60}+\frac{1}{96}+\frac{1}{1200}$ |
| $[(\mathfrak{p}_3,3)]$ | $\frac{4891}{30240}$ | 6 | $\frac{1}{12}+\frac{1}{21}+\frac{1}{60}+\frac{1}{96}+\frac{1}{432}+\frac{1}{720}$ |
| $[(\mathfrak{p}_{11},3)]$ | $\frac{2479}{8400}$ | 5 | $\frac{1}{10}+\frac{1}{12}+\frac{1}{16}+\frac{1}{21}+\frac{1}{600}$ |

Table 9.2.: Genera of maximal integral forms of $G_2$ over $\mathbb{Q}\left(\sqrt{5}\right)$.

| Genus | dim | Operator | Eigenvalues |
|---|---|---|---|
| $[(3,2)]$ | 2 | $h_2(s_0)$ | $126, 9$ |
| | | $h_5(s_0)$ | $19530, 810$ |
| | | $h_3(s_1)$ | $48, -4$ |
| | | $h_3(s_1 s_2 s_1)$ | $432, -36$ |
| | | $h_3(s_1 s_2 s_0 s_1)$ | $1296, 48$ |
| $[(5,2)]$ | 2 | $h_2(s_0)$ | $126, -17, 33$ |
| | | $h_3(s_0)$ | $1092, 0, 100$ |
| $[(7,3)]$ | 2 | $h_2(s_0)$ | $126, -3$ |
| | | $h_2(s_0 s_1 s_2 s_1 s_0)$ | $2016, -134$ |
| | | $h_3(s_0)$ | $1092, 60$ |
| | | $h_3(s_0 s_1 s_2 s_1 s_0)$ | $88452, -816$ |
| | | $h_5(s_0)$ | $19530, 438$ |
| $[(7,2)]$ | 6 | $h_2(s_0)$ | $126, -3, -3, -14, x^2-81x+1512$ |
| | | $h_3(s_0)$ | $1092, 60, -4, -48, x^2-208x+2608$ |

Table 9.3.: Some Hecke eigenvalues for $G_2$ over $\mathbb{Q}$.

## 9.3. Algebraic Modular Forms for Symplectic Groups

### 9.3.1. Genera of Maximal Integral Forms

As for $G_2$ we want to present some results regarding the decomposition of genera of maximal integral forms (for compact forms of symplectic groups) into isomorphism classes. Table 9.4 contains the principle genera of compact forms of $\mathrm{Sp}_4$ over $\mathbb{Q}$ which are realized via a (definite) quaternion algebra of discriminant less than or equal to 100. These genera were already studied by Sarah Chisholm in her PhD-thesis ([Chi14]). However the results presented there do not match the predicted mass of the genus which is why we present a corrected version here.

The table should be read as follows: The first column contains the discriminant of the maximal order (we mark the discriminant with a $*$ if the results differ from those in [Chi14]), the second the mass of the principle genus, the third the corresponding class number $h$, and in the last column we give the mass decomposition (in suggestive notation).

Again we do the same as for $G_2$ and present the analogous table over $\mathbb{Q}\left(\sqrt{5}\right)$. Table 9.5 contains all principle genera for compact forms of $\mathrm{Sp}_4$ over $\mathbb{Q}\left(\sqrt{5}\right)$ defined by a definite quaternion algebra with discriminant of norm no greater than 100. The first column contains the factorization of the discriminant into prime ideals. Since the genera for quaternion algebras with Galois conjugate discriminant correspond to one another we only give one representative of each orbit.

In addition we present the analogous table for principle genera of compact forms of $\mathrm{Sp}_{2m}$, $3 \leq m \leq 5$, over $\mathbb{Q}$, defined by a quaternion algebra of discriminant at most 7 (see Table 9.6).

| Discriminant | Mass | h | Mass decomposition |
|---|---|---|---|
| 2 | $\frac{1}{1152}$ | 1 | $\frac{1}{1152}$ |
| 3 | $\frac{1}{288}$ | 1 | $\frac{1}{288}$ |
| 5 | $\frac{13}{720}$ | 2 | $\frac{1}{72}+\frac{1}{240}$ |
| 7 | $\frac{5}{96}$ | 2 | $\frac{1}{32}+\frac{1}{48}$ |
| 11 | $\frac{61}{288}$ | 5 | $\frac{1}{12}+\frac{2}{24}+\frac{1}{32}+\frac{1}{72}$ |
| 13 | $\frac{17}{48}$ | 4 | $\frac{2}{8}+\frac{1}{12}+\frac{1}{48}$ |
| 17 | $\frac{29}{36}$ | 8 | $\frac{1}{4}+\frac{2}{8}+\frac{3}{12}+\frac{1}{24}+\frac{1}{72}$ |
| 19 | $\frac{181}{160}$ | 10 | $\frac{1}{4}+\frac{4}{8}+\frac{1}{10}+\frac{3}{12}+\frac{1}{32}$ |
| 23$*$ | $\frac{583}{288}$ | 16 | $\frac{5}{4}+\frac{3}{8}+\frac{3}{12}+\frac{2}{24}+\frac{1}{32}+\frac{1}{48}+\frac{1}{72}$ |
| 29 | $\frac{2947}{720}$ | 24 | $\frac{1}{2}+\frac{10}{4}+\frac{4}{8}+\frac{1}{10}+\frac{5}{12}+\frac{1}{24}+\frac{1}{48}+\frac{1}{72}$ |
| 30$*$ | $\frac{65}{36}$ | 12 | $\frac{1}{2}+\frac{2}{4}+\frac{2}{6}+\frac{2}{8}+\frac{2}{12}+\frac{1}{36}+\frac{2}{72}$ |
| 31 | $\frac{481}{96}$ | 26 | $\frac{2}{2}+\frac{11}{4}+\frac{7}{8}+\frac{4}{12}+\frac{1}{32}+\frac{1}{48}$ |
| 37 | $\frac{137}{16}$ | 37 | $\frac{5}{2}+\frac{19}{4}+\frac{7}{8}+\frac{5}{12}+\frac{1}{48}$ |
| 41$*$ | $\frac{841}{72}$ | 50 | $\frac{8}{2}+\frac{24}{4}+\frac{7}{8}+\frac{9}{12}+\frac{1}{24}+\frac{1}{72}$ |
| 42$*$ | $\frac{125}{24}$ | 22 | $\frac{6}{2}+\frac{5}{4}+\frac{3}{6}+\frac{2}{8}+\frac{1}{16}+\frac{1}{24}+\frac{2}{32}+\frac{2}{48}$ |
| 43$*$ | $\frac{1295}{96}$ | 55 | $\frac{10}{2}+\frac{26}{4}+\frac{11}{8}+\frac{7}{12}+\frac{1}{32}$ |
| 47$*$ | $\frac{5083}{288}$ | 72 | $\frac{14}{2}+\frac{34}{4}+\frac{10}{8}+\frac{9}{12}+\frac{2}{24}+\frac{1}{32}+\frac{1}{48}+\frac{1}{72}$ |
| 53$*$ | $\frac{3653}{144}$ | 93 | $\frac{23}{2}+\frac{47}{4}+\frac{9}{8}+\frac{11}{12}+\frac{1}{24}+\frac{1}{48}+\frac{1}{72}$ |
| 59$*$ | $\frac{50489}{1440}$ | 125 | $\frac{35}{2}+\frac{58}{4}+\frac{14}{8}+\frac{1}{10}+\frac{13}{12}+\frac{2}{24}+\frac{1}{32}+\frac{1}{72}$ |
| 61$*$ | $\frac{1861}{48}$ | 128 | $\frac{40}{2}+\frac{66}{4}+\frac{12}{8}+\frac{9}{12}+\frac{1}{48}$ |
| 66$*$ | $\frac{1525}{72}$ | 69 | $\frac{31}{2}+\frac{16}{4}+\frac{5}{6}+\frac{2}{8}+\frac{2}{12}+\frac{1}{16}+\frac{5}{24}+\frac{2}{32}$ $+\frac{1}{36}+\frac{2}{48}+\frac{2}{72}$ |
| 67$*$ | $\frac{4939}{96}$ | 166 | $\frac{57}{2}+\frac{79}{4}+\frac{18}{8}+\frac{11}{12}+\frac{1}{32}$ |
| 70$*$ | $\frac{325}{12}$ | 75 | $\frac{41}{2}+\frac{20}{4}+\frac{4}{6}+\frac{5}{8}+\frac{3}{12}+\frac{2}{48}$ |
| 71$*$ | $\frac{17647}{288}$ | 198 | $\frac{70}{2}+\frac{91}{4}+\frac{17}{8}+\frac{15}{12}+\frac{2}{24}+\frac{1}{32}+\frac{1}{48}+\frac{1}{72}$ |
| 73$*$ | $\frac{533}{8}$ | 204 | $\frac{78}{2}+\frac{99}{4}+\frac{15}{8}+\frac{12}{12}$ |
| 78$*$ | $\frac{425}{12}$ | 94 | $\frac{54}{2}+\frac{25}{4}+\frac{8}{6}+\frac{6}{8}+\frac{1}{12}$ |
| 79$*$ | $\frac{40573}{480}$ | 256 | $\frac{103}{2}+\frac{117}{4}+\frac{21}{8}+\frac{1}{10}+\frac{12}{12}+\frac{1}{32}+\frac{1}{48}$ |
| 83$*$ | $\frac{28249}{288}$ | 296 | $\frac{123}{2}+\frac{129}{4}+\frac{21}{8}+\frac{19}{12}+\frac{2}{24}+\frac{1}{32}+\frac{1}{72}$ |
| 89$*$ | $\frac{43571}{360}$ | 352 | $\frac{157}{2}+\frac{154}{4}+\frac{17}{8}+\frac{1}{10}+\frac{21}{12}+\frac{1}{24}+\frac{1}{72}$ |
| 97$*$ | $\frac{941}{6}$ | 436 | $\frac{212}{2}+\frac{188}{4}+\frac{20}{8}+\frac{16}{12}$ |

Table 9.4.: Principle genera of compact forms of $\mathrm{Sp}_4$ over the rationals.

| Discriminant | Mass | h | Mass decomposition |
|---|---|---|---|
| 1 | $\frac{1}{28800}$ | 1 | $\frac{1}{28800}$ |
| $\mathfrak{p}_2\mathfrak{p}_5$ | $\frac{221}{1200}$ | 4 | $\frac{1}{8}+\frac{1}{30}+\frac{1}{48}+\frac{1}{200}$ |
| $\mathfrak{p}_2\mathfrak{p}_3$ | $\frac{697}{600}$ | 12 | $\frac{2}{4}+\frac{3}{8}+\frac{2}{10}+\frac{1}{20}+\frac{1}{60}+\frac{1}{100}+\frac{2}{200}$ |
| $\mathfrak{p}_2\mathfrak{p}_{11}$ | $\frac{1037}{480}$ | 14 | $\frac{1}{2}+\frac{5}{4}+\frac{2}{12}+\frac{2}{16}+\frac{1}{20}+\frac{1}{30}+\frac{1}{32}+\frac{1}{240}$ |
| $\mathfrak{p}_2\mathfrak{p}_{19}$ | $\frac{9231}{800}$ | 52 | $\frac{10}{2}+\frac{21}{4}+\frac{3}{8}+\frac{2}{10}+\frac{4}{12}+\frac{2}{16}+\frac{3}{20}+\frac{1}{32}+\frac{2}{40}$ $+\frac{1}{100}+\frac{2}{200}+\frac{1}{240}$ |
| $\mathfrak{p}_5\mathfrak{p}_3$ | $\frac{533}{225}$ | 15 | $\frac{1}{2}+\frac{5}{4}+\frac{1}{6}+\frac{3}{8}+\frac{1}{30}+\frac{1}{60}+\frac{1}{72}+\frac{1}{120}+\frac{1}{200}$ |
| $\mathfrak{p}_5\mathfrak{p}_{11}$ | $\frac{793}{180}$ | 23 | $\frac{4}{2}+\frac{6}{4}+\frac{4}{8}+\frac{3}{12}+\frac{2}{24}+\frac{1}{36}+\frac{1}{60}+\frac{2}{72}$ |
| $\mathfrak{p}_5\mathfrak{p}_{19}$ | $\frac{2353}{100}$ | 78 | $\frac{29}{2}+\frac{28}{4}+\frac{1}{6}+\frac{10}{8}+\frac{6}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{40}+\frac{1}{200}$ |
| $\mathfrak{p}_3\mathfrak{p}_{11}$ | $\frac{2501}{90}$ | 91 | $\frac{32}{2}+\frac{40}{4}+\frac{1}{6}+\frac{10}{8}+\frac{1}{12}+\frac{4}{20}+\frac{1}{24}+\frac{1}{30}+\frac{1}{72}$ |

Table 9.5.: Principle genera of compact forms of $\mathrm{Sp}_4$ over $\mathbb{Q}\left(\sqrt{5}\right)$.

| $m$ | disc | Mass | $h$ | Mass decomposition |
|---|---|---|---|---|
| 3 | 2 | $\frac{1}{82944}$ | 1 | $\frac{1}{82944}$ |
| | 3 | $\frac{13}{72576}$ | 2 | $\frac{1}{10368}+\frac{1}{12096}$ |
| | 5 | $\frac{403}{90720}$ | 3 | $\frac{1}{336}+\frac{1}{1296}+\frac{1}{1440}$ |
| | 7 | $\frac{95}{2688}$ | 5 | $\frac{1}{48}+\frac{2}{192}+\frac{1}{384}+\frac{1}{672}$ |
| 4 | 2 | $\frac{17}{39813120}$ | 2 | $\frac{1}{3317760}+\frac{1}{7962624}$ |
| | 3 | $\frac{533}{17418240}$ | 3 | $\frac{1}{46080}+\frac{1}{145152}+\frac{1}{497664}$ |
| | 5 | $\frac{126139}{21772800}$ | 8 | $\frac{1}{384}+\frac{1}{720}+\frac{1}{1296}+\frac{1}{2016}+\frac{1}{2304}+\frac{1}{17280}$ $+\frac{1}{31104}+\frac{1}{115200}$ |
| | 7 | $\frac{22819}{129024}$ | 21 | $\frac{1}{24}+\frac{1}{32}+\frac{2}{48}+\frac{1}{64}+\frac{1}{96}+\frac{1}{144}+\frac{1}{192}$ $+\frac{2}{240}+\frac{1}{256}+\frac{1}{288}+\frac{2}{384}+\frac{1}{768}+\frac{1}{1536}$ $+\frac{1}{2304}+\frac{1}{2688}+\frac{1}{4608}+\frac{1}{6144}+\frac{1}{46080}$ |
| 5 | 2 | $\frac{527}{10510663680}$ | 3 | $\frac{1}{27371520}+\frac{1}{79626240}+\frac{1}{955514880}$ |
| | 3 | $\frac{5863}{209018880}$ | 5 | $\frac{1}{61440}+\frac{1}{103680}+\frac{1}{552960}+\frac{1}{3483648}$ $+\frac{1}{29859840}$ |
| | 5 | $\frac{8955869}{130636800}$ | 27 | $\frac{1}{72}+\frac{2}{96}+\frac{2}{144}+\frac{1}{192}+\frac{1}{240}+\frac{1}{384}+\frac{1}{512}$ $+\frac{1}{648}+\frac{1}{1152}+\frac{3}{1440}+\frac{1}{2304}+\frac{2}{3840}+\frac{1}{4320}$ $+\frac{1}{7776}+\frac{1}{13824}+\frac{1}{19440}+\frac{1}{24192}+\frac{1}{51840}$ $+\frac{1}{80640}+\frac{1}{311040}+\frac{1}{691200}+\frac{1}{933120}$ |
| | 7 | $\frac{63916019}{5677056}$ | 228 | Omitted |

Table 9.6.: Principle genera of compact forms of $\mathrm{Sp}_{2m}, 3 \leq m \leq 5$, over $\mathbb{Q}$.

### 9.3.2. Hecke Operators

As for $G_2$ we present some eigenvalues for Hecke operators acting on spaces of algebraic modular forms for symplectic groups. The tables should be read the same way as the ones for $G_2$ except that we specify the genus in the same way as in the last subsection. Again we cannot present all of our results so for $\mathrm{Sp}_4$ we instead focus on a few discriminants and compute the operators for as many primes as feasible. For $\mathrm{Sp}_6$ we will need the results again in the section on lifts of modular forms so we only compute operators at $2, 3$ and $5$ (see Table 9.10). Again we show that our algorithms work over totally real number fields that are not $\mathbb{Q}$ as well, choosing $\mathbb{Q}\left(\sqrt{5}\right)$ and $\mathbb{Q}\left(\zeta_7 + \zeta_7^{-1}\right)$ as examples (see Table 9.11 and 9.12).

In addition we demonstrate that we are able to compute algebraic modular forms of nontrivial weight. To that end we choose the following representation: Via the regular representation of the quaternion algebra $H$ we can embed $U_2(H)$ into $k^{8\times 8}$. While this representation is irreducible (over $k$) it does not admit nontrivial algebraic modular forms since the image of each stabilizer contains $-I_8$ which has no nontrivial fixed points. Thus we take instead the tensor product of this representation with itself. The resulting module is not irreducible, however we can still compute modular forms. It turns out that the trivial module appears with multiplicity 4 and so does another irreducible module that admits algebraic modular forms. In Table 9.13 we present the Hecke eigenvalues that appear in this representation.

| Discriminant | dim | Operator | Eigenvalues |
|---|---|---|---|
| 5 | 2 | $h_2(s_0)$ | $30, -9$ |
| | | $h_2(s_0s_1s_0)$ | $120, 16$ |
| | | $h_3(s_0)$ | $120, 16$ |
| | | $h_3(s_0s_1s_0)$ | $1080, 92$ |
| | | $h_7(s_0)$ | $2800, 96$ |
| | | $h_7(s_0s_1s_0)$ | $137200, 2676$ |
| | | $h_{11}(s_0)$ | $16104, 504$ |
| | | $h_{11}(s_0s_1s_0)$ | $1948584, 19384$ |
| | | $h_{13}(s_0)$ | $30940, -364$ |
| | | $h_{13}(s_0s_1s_0)$ | $5228860, 23452$ |
| | | $h_{17}(s_0)$ | $88740, 756$ |
| | | $h_{17}(s_0s_1s_0)$ | $25645860, 91396$ |
| | | $h_{19}(s_0)$ | $137560, 2360$ |
| | | $h_{19}(s_0s_1s_0)$ | $49659160, 175960$ |
| | | $h_{23}(s_0)$ | $292560, -1344$ |
| | | $h_{23}(s_0s_1s_0)$ | $154764240, 244212$ |
| | | $h_{29}(s_0)$ | $732540, -660$ |
| | | $h_{29}(s_0s_1s_0)$ | $616066140, 666940$ |

Table 9.7.: Hecke eigenvalues for $\mathrm{Sp}_4$ over $\mathbb{Q}$, part 1(3).

| Discriminant | dim | Operator | Eigenvalues |
|---|---|---|---|
| 7 | 2 | $h_2(s_0)$ | $30, 0$ |
| | | $h_2(s_0s_1s_0)$ | $120, 10$ |
| | | $h_3(s_0)$ | $120, 0$ |
| | | $h_3(s_0s_1s_0)$ | $1080, 60$ |
| | | $h_5(s_0)$ | $780, 120$ |
| | | $h_5(s_0s_1s_0)$ | $19500, 1240$ |
| | | $h_{11}(s_0)$ | $16104, 24$ |
| | | $h_{11}(s_0s_1s_0)$ | $1948584, 13624$ |
| | | $h_{13}(s_0)$ | $30940, 560$ |
| | | $h_{13}(s_0s_1s_0)$ | $5228860, 33880$ |
| | | $h_{17}(s_0)$ | $88740, 1260$ |
| | | $h_{17}(s_0s_1s_0)$ | $25645860, 101700$ |
| | | $h_{19}(s_0)$ | $137560, -1840$ |
| | | $h_{19}(s_0s_1s_0)$ | $49659160, 102460$ |
| | | $h_{23}(s_0)$ | $292560, 1680$ |
| | | $h_{23}(s_0s_1s_0)$ | $154764240, 306960$ |
| | | $h_{29}(s_0)$ | $732540, -2460$ |
| | | $h_{29}(s_0s_1s_0)$ | $616066140, 626140$ |

Table 9.8.: Hecke eigenvalues for $\mathrm{Sp}_4$ over $\mathbb{Q}$, part 2(3).

| Discriminant | dim | Operator | Eigenvalues |
|---|---|---|---|
| 11 | 5 | $h_2(s_0)$ | $30, x^2 - 12x + 9, x^2 - 2x - 11$ |
| | | $h_2(s_0s_1s_0)$ | $120, x^2 - 38x + 286, -6, -6$ |
| | | $h_3(s_0)$ | $120, x^2 - 8x - 752, x^2 - 18x + 33$ |
| | | $h_3(s_0s_1s_0)$ | $1080, x^2 - 226x + 11041, x^2 + 24x - 624$ |
| | | $h_5(s_0)$ | $780, x^2 - 60x - 6012, x^2 - 50x + 433$ |
| | | $h_5(s_0s_1s_0)$ | $19500, x^2 - 1634x + 537697,$ |
| | | | $x^2 + 156x - 828$ |
| | | $h_7(s_0)$ | $2800, x^2 - 256x + 13312, x^2 - 56x + 592$ |
| | | $h_7(s_0s_1s_0)$ | $137200, x^2 - 5960x + 8658448,$ |
| | | | $x^2 + 1120x + 310528$ |
| | | $h_{13}(s_0)$ | $30940, x^2 - 1456x + 294784,$ |
| | | | $x^2 - 656x + 88384$ |
| | | $h_{13}(s_0s_1s_0)$ | $5228860, x^2 - 75824x + 1363514944,$ |
| | | | $x^2 - 5384x - 12413936$ |
| | | $h_{17}(s_0)$ | $88740, x^2 + 1656x + 545616,$ |
| | | | $x^2 - 824x + 168016$ |
| | | $h_{17}(s_0s_1s_0)$ | $25645860, x^2 - 139304x + 4839782032,$ |
| | | | $x^2 + 5976x - 1585008$ |
| | | $h_{19}(s_0)$ | $137560, x^2 - 2160x - 3153600, 360, 360$ |
| | | $h_{19}(s_0s_1s_0)$ | $49659160, x^2 - 310032x + 22014421056,$ |
| | | | $x^2 + 4688x - 50492864$ |

Table 9.9.: Hecke eigenvalues for $\mathrm{Sp}_4$ over $\mathbb{Q}$, part 3(3).

| disc | dim | Operator | Eigenvalues |
|---|---|---|---|
| 3 | 2 | $h_2(s_0)$ | $126, 9$ |
| | | $h_2(s_0s_1s_0)$ | $2520, -54$ |
| | | $h_2(s_0s_1s_2s_0s_1s_0)$ | $8640, 216$ |
| | | $h_5(s_0)$ | $19530, 810$ |
| | | $h_5(s_0s_1s_0)$ | $12694500, 39780$ |
| | | $h_5(s_0s_1s_2s_0s_1s_0)$ | $307125000, 491400$ |
| 5 | 3 | $h_2(s_0)$ | $126, 33, -17$ |
| | | $h_2(s_0s_1s_0)$ | $2520, 226, 76$ |
| | | $h_2(s_0s_1s_2s_0s_1s_0)$ | $8640, 456, -44$ |
| | | $h_3(s_0)$ | $1092, 100, 0$ |
| | | $h_3(s_0s_1s_0)$ | $98280, 1064, 364$ |
| | | $h_3(s_0s_1s_2s_0s_1s_0)$ | $816480, 7008, -1792$ |
| 7 | 5 | $h_2(s_0)$ | $126, -3, -14, x^2 - 81x + 1512$ |
| | | $h_2(s_0s_1s_0)$ | $2520, -18, 70, x^2 - 708x + 92484$ |
| | | $h_2(s_0s_1s_2s_0s_1s_0)$ | $8640, 0, -110, x^2 - 1548x + 432864$ |
| | | $h_3(s_0)$ | $1092, -4, -48, x^2 - 208x + 2608$ |
| | | $h_3(s_0s_1s_0)$ | $98280, -276, 780,$ |
| | | | $x^2 - 3444x - 7969824$ |
| | | $h_3(s_0s_1s_2s_0s_1s_0)$ | $816480, 720, -1920,$ |
| | | | $x^2 - 26928x + 131341824$ |
| | | $h_5(s_0)$ | $19530, -138, 610, x^2 - 1440x + 467100$ |
| | | $h_5(s_0s_1s_0)$ | $12694500, -72, 37240$ |
| | | | $x^2 - 54780x - 419479200$ |
| | | $h_5(s_0s_1s_2s_0s_1s_0)$ | $307125000, 2448, 203440,$ |
| | | | $x^2 - 941400x + 201197520000$ |

Table 9.10.: Hecke eigenvalues for $\mathrm{Sp}_6$ over $\mathbb{Q}$.

| disc | dim | Operator | Eigenvalues |
|---|---|---|---|
| $\mathfrak{p}_2 \cdot \mathfrak{p}_5$ | 4 | $h_{\mathfrak{p}_3}(s_0)$ | $7380, 580, 180, -420$ |
| | | $h_{\mathfrak{p}_3}(s_0 s_1 s_0)$ | $597780, 12980, 7380, 4980$ |
| | | $h_{\mathfrak{p}_{11}}(s_0)$ | $16104, -216, 264, 504$ |
| | | $h_{\mathfrak{p}_{11}}(s_0 s_1 s_0)$ | $1948584, 11944, 16104, 19384$ |
| $\mathfrak{p}_2 \cdot \mathfrak{p}_3$ | 12 | $h_{\mathfrak{p}_5}(s_0)$ | $0^{(2)}, 780, 60, 48, 24^{(4)}, -36,$ |
| | | | $x^2 - 84x - 11520$ |
| | | $h_{\mathfrak{p}_5}(s_0 s_1 s_0)$ | $40^{(2)}, 19500, 780, 712, -200^{(4)}, -460,$ |
| | | | $x^2 - 2100x + 770400$ |
| | | $h_{\mathfrak{p}_{11}}(s_0)$ | $144^{(2)}, 16104, 264, 960, (x^2 - 528x + 58176)^{(2)},$ |
| | | | $-216, x^2 + 264x - 35712$ |
| | | $h_{\mathfrak{p}_{11}}(s_0 s_1 s_0)$ | $-2036^{(2)}, 1948584, 16104, 27820,$ |
| | | | $(x^2 - 5048x - 2027504)^{(2)}, 11944,$ |
| | | | $x^2 - 25620x + 161851104$ |

Table 9.11.: Hecke eigenvalues for $\mathrm{Sp}_4$ over $\mathbb{Q}\left(\sqrt{5}\right)$.

| disc | dim | Operator | Eigenvalues |
|---|---|---|---|
| $\mathfrak{p}_2$ | 4 | $h_{\mathfrak{p}_7}(s_0)$ | $2800, 272, -112, -176$ |
| | | $h_{\mathfrak{p}_7}(s_0 s_1 s_0)$ | $137200, 4480, 1792, 1792$ |
| | | $h_{\mathfrak{p}_{13}}(s_0)$ | $30940, -28, 1092, -812$ |
| | | $h_{\mathfrak{p}_{13}}(s_0 s_1 s_0)$ | $5228860, 26236, 43836, 21532$ |
| | | $h_{\mathfrak{p}_3}(s_0)$ | $551880, -5544, 2968, 8568$ |
| | | $h_{\mathfrak{p}_3}(s_0 s_1 s_0)$ | $402320520, 417816, 595352, 812952$ |

Table 9.12.: Hecke eigenvalues for $\mathrm{Sp}_4$ over $\mathbb{Q}\left(\zeta_7 + \zeta_7^{-1}\right)$.

| Discriminant | Operator | Eigenvalues |
| --- | --- | --- |
| 5 | $h_2(s_0)$ | $1$ |
| | $h_2(s_0s_1s_0)$ | $9$ |
| | $h_3(s_0)$ | $\frac{64}{9}$ |
| | $h_3(s_0s_1s_0)$ | $-68$ |
| | $h_7(s_0)$ | $\frac{3504}{49}$ |
| | $h_7(s_0s_1s_0)$ | $\frac{7124}{49}$ |
| | $h_{11}(s_0)$ | $\frac{9784}{121}$ |
| | $h_{11}(s_0s_1s_0)$ | $-\frac{252936}{121}$ |
| | $h_{13}(s_0)$ | $\frac{1348}{13}$ |
| | $h_{13}(s_0s_1s_0)$ | $-\frac{46484}{13}$ |
| 7 | $h_2(s_0)$ | $x^2 - \frac{15}{4}x + \frac{21}{8}$ |
| | $h_2(s_0s_1s_0)$ | $x^2 + 31x + \frac{733}{4}$ |
| | $h_3(s_0)$ | $x^2 - \frac{52}{3}x + \frac{448}{9}$ |
| | $h_3(s_0s_1s_0)$ | $x^2 + \frac{52}{3}x - \frac{5024}{9}$ |
| | $h_5(s_0)$ | $x^2 - \frac{912}{25}x - \frac{156864}{625}$ |
| | $h_5(s_0s_1s_0)$ | $x^2 + \frac{68}{5}x - \frac{119456}{25}$ |
| | $h_{11}(s_0)$ | $x^2 - \frac{2352}{11}x + \frac{153317184}{14641}$ |
| | $h_{11}(s_0s_1s_0)$ | $x^2 + \frac{202336}{121}x - \frac{11644284224}{14641}$ |
| | $h_{13}(s_0)$ | $x^2 - \frac{46984}{169}x + \frac{374507392}{28561}$ |
| | $h_{13}(s_0s_1s_0)$ | $x^2 + \frac{938308}{169}x + \frac{220061134048}{28561}$ |

Table 9.13.: Nontrivial Hecke eigenvalues appearing in the tensor product representation

### 9.3.3. Runtime Comparison

In principle we have two methods at our disposal to compute Hecke operators. On the one hand we can compute the double coset decomposition following [Lan01] and [LP02] and on the other hand we can use the Eichler method outlined in Chapter 5. Here we want to briefly compare their performances in computing the action of the two generators of the hyperspecial Hecke algebra of compact forms of $\mathrm{Sp}_4$. To do this we take the principle genera of compact forms of $\mathrm{Sp}_4$ defined by a quaternion algebra of discriminant $5, 7, 11$ or 13 and compute for $p = 2, 3, 5$ not dividing said discriminant the action of the Hecke operators corresponding to the Weyl group elements $s_0$ and $s_0 s_1 s_0$ at $p$. In Table 9.14 we write down the discriminant, the primes $p$ and the runtimes for the standard and Eichler method. In order to correctly gauge the usefulness of the Eichler method one should bear in mind that at every prime one actually computes (in addition to the information we are primarily after) representatives for two further genera of integral forms and 2 more Hecke operators.

In addition we compared the runtime in one dimension 3 example (discriminant 3 at prime 2) and got a runtime of 335.17 second for the Eichler method and 7114.73 seconds for the standard method.

| Discriminant | Prime | Runtime Standard | Runtime Eichler |
|---|---|---|---|
| 5 | 2 | $39.32s$ | $9.83s$ |
| | 3 | $295.93s$ | $21.93s$ |
| 7 | 2 | $44.32s$ | $7.27s$ |
| | 3 | $213.15s$ | $15.29s$ |
| | 5 | $3295.69s$ | $53.02s$ |
| 11 | 2 | $139.75s$ | $34.60s$ |
| | 3 | $713.32s$ | $69.30s$ |
| | 5 | $11344.21s$ | $187.27s$ |
| 13 | 2 | $81.96s$ | $28.55s$ |
| | 3 | $634.54s$ | $57.80s$ |
| | 5 | $9017.63s$ | $165.29s$ |

Table 9.14.: Runtime comparison of the standard and Eichler method.

## 9.4. Lifts of Modular Forms

In Chapter 7 we computed how Hecke eigenvalues should lift from $G_2$ to $\mathrm{Sp}_6$. More precisely we gave formulas (see equations 7.18-7.20) on how Hecke eigenvalues $a_p, b_p$ at the hyperspecial prime $p$ lift to Hecke eigenvalues $a'_p, b'_p, c'_p$ (again at $p$) for $\mathrm{Sp}_6$. Here we want to present some results that suggest that there are indeed suitable lifts of some forms. For the eigenforms of $G_2$ we draw on the results of [LP02] as they present a few eigenforms more than we did in the previous sections.

Taking an eigenform of the Hecke algebra $H$ coming from a maximal compact subgroup $K$ that is hyperspecial everywhere put at $q$ with eigenvalues $a_p, b_p$ for $p \neq q$ we would expect there to be a lift to an eigenform of a Hecke algebra which comes again from an open compact subgroup that is hyperspecial away from $q$. In Table 9.15 we present all nontrivial eigenforms for $q = 3, 5, 7$ and trivial weight with their eigenvalues for the Hecke algebra of $G_2$ (these results are taken from [LP02, Table V]). We then compute what the eigenvalues of a possible lift should look like and check whether there is a suitable candidate among the modular forms we computed (cf. Table 9.10), realizing all of these eigenvalues simultaneously. Note that we write down "yes"/"no" depending on whether there is a suitable candidate; this does not prove/disprove that there actually exists a lift.

| | | $G_2$-eigenvalues | | $\mathrm{Sp}_6$-eigenvalues | | | |
|---|---|---|---|---|---|---|---|
| $q$ | $p$ | $a$ | $b$ | $a'$ | $b'$ | $c'$ | Lift? |
| 3 | 2 | 9 | 90 | 9 | −54 | 216 | yes |
| | 5 | 810 | | 810 | | | |
| 5 | 2 | −3 | −38 | −3 | −50 | 96 | no |
| | 3 | 28 | −196 | 28 | 56 | 672 | |
| 5 | 2 | 33 | 94 | 33 | 226 | 456 | yes |
| | 3 | 100 | 164 | 100 | 1064 | 7008 | |
| 5 | 2 | −17 | 144 | −17 | 76 | −44 | no |
| | 3 | 0 | 364 | 0 | 364 | −1792 | |
| 7 | 2 | −14 | 126 | −14 | 70 | −110 | yes |
| | 3 | −48 | 1212 | −48 | 780 | −1920 | |
| | 5 | 610 | | 810 | | | |
| 7 | 2 | −3 | −134 | −3 | −146 | 384 | no |
| | 3 | 60 | −816 | 60 | −276 | 5328 | |
| | 5 | 438 | | 438 | | | |
| 7 | 2 | −3 | −6 | −3 | −18 | 0 | yes |
| | 3 | −4 | −240 | −4 | −276 | 720 | |
| | 5 | −138 | | −138 | | | |

Table 9.15.: Evidence for lifts for modular forms of $G_2$ to $\mathrm{Sp}_6$.

# A. Algebraic Affine Groups Schemes

As noted in the beginning of this thesis, we want to give a short introduction into the theory of algebraic affine group schemes. We will only present the theory to the extent needed in this work; in particular we will - wherever necessary - restrict ourselves to algebraic affine group schemes over fields or rings of integers of global or local fields of characteristic zero, since these are the only applications we need.

Let us first establish some basic notation for this chapter. By $R$ we will always denote a commutative ring with unit and by $k$ a field. The category of commutative $R$-algebras will be denoted by $R$-$\mathsf{Alg}$, the category of sets by $\mathsf{Set}$, and the category of groups by $\mathsf{Grp}$. Furthermore $A$ will always be an object of either $R$-$\mathsf{Alg}$ or $k$-$\mathsf{Alg}$, the base ring being clear from context. If not explicitly stated otherwise the definitions and results in this appendix are taken essentially verbatim from [Wat79].

## A.1. Affine Group Schemes

**Definition A.1.1** *Let $\mathbb{F}$ be a functor from $R$-$\mathsf{Alg}$ to $\mathsf{Set}$. Then $\mathbb{F}$ is called representable if there is an $R$-algebra $A$ such that there is a natural correspondence between* $\operatorname{Hom}(A, X)$ *and $\mathbb{F}(X)$ for every $R$-algebra $X$. If this is the case we say $\mathbb{F}$ is represented by $A$. An affine group scheme over $R$ is a representable functor $R$-$\mathsf{Alg} \to \mathsf{Grp}$.*

Since any $R$-algebra $A$ can be defined by generators and relations (i.e. as a quotient of a polynomial ring (in possibly infinitely many variables) over $R$) this means that a representable functor is one given by finding solutions to systems of polynomial equations over $R$ in the algebra $X$.

**Example A.1.2**
- *The functor $\mathbb{G}_m : R$-$\mathsf{Alg} \to \mathsf{Grp}$, $X \mapsto X^\times$ mapping each $R$-algebra to its unit group is an affine group scheme represented by the $R$-algebra $A := R[x, y]/(xy - 1) \cong R[x, \frac{1}{x}]$.*
- *The functor $\mathrm{SL}_2 : R$-$\mathsf{Alg} \to \mathsf{Grp}$, $X \mapsto \mathrm{SL}_2(R)$ mapping each $R$-algebra $X$ to the special linear group of degree $2$ over $X$ is represented by $A = R[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]/(x_{1,1}x_{2,2} - x_{1,2}x_{2,1} - 1)$.*

- *For $n \in \mathbb{N}$ we have the functor $\mu_n$ assigning to each $R$-algebra $A$ the group of $n$th roots of unity in $A$, $\mu_n(A) = \{x \in A \mid x^n = 1\}$.*

- *The functor assigning to each $R$-algebra $A$ its general linear group of degree $n$ is representable since we can identify $\mathrm{GL}_n(A)$ with the set of functions assigning each matrix entry a value under the only condition that the determinant (which is a polynomial in the entries) becomes invertible. This means that $\mathrm{GL}_n$ is represented by*

$$R[x_{1,1}, ..., x_{n,n}, \frac{1}{\det}]. \tag{A.1}$$

**Theorem A.1.3 (Yoneda's lemma)** *Let $\mathbb{E}, \mathbb{F}$ be two $\mathsf{Set}$-valued representable functors on $R$-$\mathsf{Alg}$ represented by $A$ and $B$ respectively. Then the natural transformations $\mathbb{E} \to \mathbb{F}$ correspond to $R$-algebra homomorphisms $B \to A$.*

**Example A.1.4** *Consider the natural transformation $\eta : \mathbb{G}_m \to \mathrm{SL}_2$ with $\eta(X) : \mathbb{G}_m(X) \to \mathrm{SL}_2(X), t \mapsto \mathrm{diag}(t, t^{-1})$. Then $\eta$ corresponds to the homomorphism $R[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]/(x_{1,1}x_{2,2} - x_{1,2}x_{2,1} - 1) \to R[x, \frac{1}{x}]$ with $x_{1,1} \mapsto x, x_{2,2} \mapsto \frac{1}{x}$ and $x_{1,2}, x_{2,1} \mapsto 0$.*

**Corollary A.1.5** *In the setting of Yoneda's lemma, the functor $\mathbb{E} \to \mathbb{F}$ is a natural correspondence if and only if the corresponding homomorphism $B \to A$ is an isomorphism of $R$-algebras.*

So far we have seen that the category of representable functors is equivalent to the opposite category of the category of (commutative) $R$-algebras. Since we want to look at affine group schemes, that is representable functors with additional structure, it is natural to ask if the group properties have a counterpart on the side of algebras. To that end first note that the $R$-algebra $R$ represents the functor mapping each algebra to a singleton. Furthermore if $\mathbb{E}$ and $\mathbb{F}$ are represented by $A$ and $B$, respectively, then $\mathbb{E} \times \mathbb{F} : R\text{-}\mathsf{Alg} \to \mathsf{Set},\ X \mapsto \mathbb{E}(X) \times \mathbb{F}(X)$ is represented by $\mathbb{E} \otimes_R \mathbb{F}$. This is to say that homomorphisms $A \otimes_R B \to X$ correspond to pairs of homomorphisms $A \to X,\ B \to X$ which is essentially the definition of the tensor product.

Now a group is simply a set $G$ together with associative multiplication, a unit element, and (left-)inverses, i.e. a set with three maps

$$\begin{aligned} &\mathrm{mult} : G \times G \to G, \\ &\mathrm{unit} : \{e\} \to G \text{ and} \\ &\mathrm{inv} : G \to G \end{aligned} \tag{A.2}$$

such that the following diagrams commute:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{mult} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mathrm{mult}"'] & G \times G \arrow[d, "\mathrm{mult}"] \\
G \times G \arrow[r, "\mathrm{mult}"] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \times \{e\} \arrow[r, "\mathrm{id} \times \mathrm{unit}"] \arrow[d, "\cong"'] & G \times G \arrow[d, "\mathrm{mult}"] \\
G \arrow[r, "="] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "(\mathrm{id}, \mathrm{inv})"] \arrow[d] & G \times G \arrow[d, "\mathrm{mult}"] \\
\{e\} \arrow[r, "\mathrm{unit}"] & G
\end{tikzcd}$$

For an affine group scheme $\mathbb{F}$ and $R$-algebras $X, Y$ together with an algebra homomorpism $X \to Y$ there is now a corresponding group homomorphism $\mathbb{F}(X) \to \mathbb{F}(Y)$, which means a map making the following diagram commutative:

$$\begin{tikzcd}
\mathbb{F}(X) \times \mathbb{F}(X) \arrow[r, "\mathrm{mult}"] \arrow[d] & \mathbb{F}(X) \arrow[d] \\
\mathbb{F}(Y) \times \mathbb{F}(Y) \arrow[r, "\mathrm{mult}"] & \mathbb{F}(Y)
\end{tikzcd}$$

In other words the map mult : $\mathbb{F} \times \mathbb{F} \to \mathbb{F}$ (and similarly unit and inv) is a natural transformation. If $\mathbb{F}$ is represented by $A$, then by Yoneda's lemma these natural transformations need to correspond to certain morphisms in the category of $R$-algebras, leading us to the following definition.

**Definition A.1.6** *An $R$-Hopf algebra is an $R$-algebra $A$ together with specified homomorphisms*

$$\begin{aligned}
&\Delta : A \to A \otimes_R A, \quad \textit{(comultiplication)} \\
&\epsilon : A \to R, \quad \textit{(counit or augmentation)} \\
&S : A \to A, \quad \textit{(coinverse or antipode)}
\end{aligned} \tag{A.3}$$

*such that the following diagrams become commutative:*

$$
\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes \mathrm{id}"'] \\
A \otimes A \arrow[u, "\mathrm{id} \otimes \Delta"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \otimes R & A \otimes A \arrow[l, "\mathrm{id} \otimes \epsilon"'] \\
A \arrow[u, "\cong"] & A \arrow[l, "="'] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A & A \otimes A \arrow[l, "(\mathrm{id}, S)"'] \\
R \arrow[u] & A \arrow[l, "\epsilon"'] \arrow[u, "\Delta"']
\end{tikzcd}
$$

**Theorem A.1.7** *The category of affine group schemes over the ring $R$ is anti-equivalent to the category of Hopf algebras over $R$.*

**Remark A.1.8** *The essence of the above theorem is that every statement about groups (or group schemes) has a corresponding statement about Hopf algebras and vice versa. This makes it possible to translate results between the two structures whenever one deems it necessary.*

Suppose there is a ring homomorphism $R \to R'$, then any $R'$-algebra $A'$ becomes an $R$-algebra via $R \to R' \to A'$ and $R'$-algebra homomorphisms are simply $R$-algebra homomorphisms in this sense. In particular a functor $\mathbb{F}$ from $R$-$\mathsf{Alg}$ to $\mathsf{Set}$ can be evaluated on $A'$ and we denote the resulting functor $R'$-$\mathsf{Alg} \to \mathsf{Set}$ by $\mathbb{F}_{R'}$ (or simply $\mathbb{F}$ again if there is no risk of confusion) and call it the restriction of $\mathbb{F}$ to $R'$-$\mathsf{Alg}$. If $\mathbb{F}$ is represented by $A$, then $\mathbb{F}_{R'}$ is represented by $A \otimes_R R'$, since we know that $\mathrm{Hom}_R(A, X) \cong \mathrm{Hom}_{R'}(A \otimes_R R', X)$ for any $R'$-algebra $X$.

If $R$ is a discrete valuation ring with maximal ideal $\pi$, there are two fields naturally associated with $R$, namely its residue class field $R/\pi$ and its field of fractions $\mathrm{Quot}(R)$. If $\mathbb{F} : R$-$\mathsf{Alg} \to \mathsf{Set}$ is a representable functor, we call the corresponding restricted functors $\mathbb{F}_{R/\pi}$ and $\mathbb{F}_{\mathsf{Quot}(R)}$ the special and the generic fiber of $\mathbb{F}$, respectively. This nomenclature borrows from the fact that $\mathrm{spec}(R)$ has exactly two points, the generic point 0 and the special point $\pi$.

**Definition A.1.9** *Let $\mathbb{G}$ and $\mathbb{H}$ be affine group schemes over $R$. A morphism $\mathbb{G} \to \mathbb{H}$ is a natural transformation such that all the corresponding maps $\mathbb{G}(X) \to \mathbb{H}(X)$ for $X \in R$-$\mathsf{Alg}$ are group homomorphisms.*

If $\mathbb{G}$ and $\mathbb{H}$ are represented by $A$ and $B$ respectively then, as one would expect by the Yoneda lemma, a natural transformation $\mathbb{G} \to \mathbb{H}$ is a homomorphism if and only if the corresponding algebra homomorphism $B \to A$ preserves the Hopf algebra structure. Since on the level of groups a map is a homomorphism if it preserves multiplication it is enough to check that $\Delta$ is preserved on the level of Hopf algebras.

**Definition A.1.10** *A homomorphism $\psi : \mathbb{H}' \to \mathbb{G}$ is called a closed embedding if the corresponding map $A \to B'$ (where $A$ represents $\mathbb{G}$ and $B'$ represents $\mathbb{H}'$) is surjective. In this situation $\psi$ is an isomorphism onto a closed subgroup $\mathbb{H}$ of $\mathbb{G}$ represented by an algebra $B$ which is a quotient of $A$ and isomorphic to $B'$.*

If we think of $\mathbb{G}$ as finding solutions to certain polynomial equations this means that $\mathbb{H}$ is defined by those equations defining $\mathbb{G}$ and possibly some additional ones. In general not every ideal $I$ of $A$ will again yield an affine group scheme. Those that do are called Hopf ideals and are characterized by the properties

$$\Delta(I) \subset A \otimes I + I \otimes A, \ S(I) \subset I, \ \epsilon(I) = 0. \tag{A.4}$$

A trivial example of a Hopf ideal is the kernel of $\epsilon$, the so-called augmentation ideal, corresponding to the trivial subgroup.

**Definition A.1.11** *Let $\mathbb{G}$ be an affine group scheme over $R$. We call $\mathbb{G}$ algebraic if its representing algebra is finitely generated.*

Over fields algebraic affine group schemes are precisely characterized by being embeddable into the schemes of type $\mathrm{GL}_n$ for some $n$ (these are obviously algebraic) as stated by the following theorem.

**Theorem A.1.12** *Assume $R$ to be a field and let $\mathbb{G}$ be an algebraic affine group scheme over $R$. Then there is an $n \in \mathbb{N}$ such that there exists a closed embedding $\mathbb{G} \to \mathrm{GL}_n$.*

## A.2. Chevalley-Demazure Group Schemes

We want to take the time to briefly introduce the most basic definitions from the theory of Chevalley-Demazure group schemes.

Let $\mathcal{L}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathcal{H}$, root system $\Phi$ and simple system $\Delta = \{\alpha_1, ..., \alpha_r\}$ corresponding to a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$. Since the Killing form, $(-,-)$, is nondegenerate we have an element $h'_\gamma \in \mathcal{H}$ for each $\gamma \in \mathcal{H}^*$ such that $(h, h'_\gamma) = \gamma(h)$ for all $h \in \mathcal{H}$ and we set $(\gamma, \mu) := (h'_\gamma, h'_\mu)$ for $\gamma, \mu \in \mathcal{H}^*$. As usual we denote by $\langle \alpha, \beta \rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)}$ for $\alpha, \beta \in \Phi$ (the Cartan integers).

**Lemma A.2.1 ([SFW67, La. 1])** *For $\alpha \in \Phi$ denote $h_\alpha = \frac{2}{(\alpha,\alpha)} h'_\alpha$ and $h_i = h_{\alpha_i}$. Then every $h_\alpha$ is an integral linear combination of the $h_i$, $1 \leq i \leq r$.*

For a root $\alpha \in \Phi$ we denote by $\mathcal{L}_\alpha$ the (1-dimensional) root space, i.e. the subspace of $\mathcal{L}$ on which $\mathcal{H}$ acts via $\alpha$, and we choose a nonzero $x_\alpha \in \mathcal{L}_\alpha$. For $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$ we denote by $N_{\alpha,\beta}$ the multiplication constant given by $[x_\alpha, x_\beta] = N_{\alpha,\beta} x_{\alpha+\beta}$. If $\alpha + \beta$ is not a root we set $N_{\alpha,\beta} = 0$.

**Lemma A.2.2 ([SFW67, La. 2])** *The $x_\alpha$ can be chosen in a way such that the following relations hold:*

- $[x_\alpha, x_{-\alpha}] = h_\alpha$ *for all* $\alpha \in \Phi$.
- *For* $\alpha, \beta \in \Phi$ *with* $\beta \notin \{\pm\alpha\}$ *we have*
$$N_{\alpha,\beta}^2 = q(r+1)\frac{|\alpha+\beta|^2}{|\beta|^2} \tag{A.5}$$
*where* $\beta - r\alpha, ..., \beta + q\alpha$ *is the* $\alpha$*-string through* $\beta$ *and* $|\mu|^2 = (\mu, \mu)$.

*Furthermore if* $\alpha, \beta, \alpha + \beta$ *are roots we have (in the same notation as above)*
$$q(r+1)\frac{|\alpha+\beta|^2}{|\beta|^2} = (r+1)^2. \tag{A.6}$$

In conclusion we obtain a basis $h_i,\ 1 \leq i \leq r, x_\alpha,\ \alpha \in \Phi$, of $\mathcal{L}$ such that the multiplication constants are as follows:

- $[h_i, h_j] = 0$ for all $i, j$.
- $[h_i, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha$ for all $\alpha \in \Phi$, $1 \leq i \leq r$.
- $[x_\alpha, x_{-\alpha}] = h_\alpha =$ some integral linear combination of the $h_i$.
- $[x_\alpha, x_\beta] = 0$ for all $\alpha, \beta \in \Phi$ with $\beta \notin \{\pm\alpha\}$ and $\alpha + \beta \notin \Phi$.
- $[x_\alpha, x_\beta] = \pm(r+1)x_{\alpha+\beta}$ for $\alpha, \beta, \alpha + \beta \in \Phi$ (with $r$ as above).

A basis of this sort is called a Chevalley basis of $\mathcal{L}$ and is unique up to sign changes and automorphisms of $\mathcal{L}$. The ($\mathbb{Z}$-)lattice generated by a Chevalley basis is called a Chevalley lattice.

In particular we see that all multiplication constants are integers and we obtain a $\mathbb{Z}$-structure of $\mathcal{L}$ in the following sense: We set $\mathcal{L}_\mathbb{Z} := \langle h_i, x_\alpha \mid 1 \leq i \leq r, \alpha \in \Phi \rangle_\mathbb{Z}$. Then $\mathcal{L}_\mathbb{Z}$ has the structure of a Lie algebra over $\mathbb{Z}$ and $\mathcal{L}_\mathbb{Z} \otimes \mathbb{C} \cong \mathcal{L}$. Thus it now makes sense to speak of a Lie algebra of the type of $\mathcal{L}$ over arbitrary commutative rings by tensoring said ring with $\mathcal{L}_\mathbb{Z}$.

**Theorem A.2.3 ([SFW67, §5])** *Let* $G_\mathbb{C}$ *be a connected semisimple linear algebraic group over* $\mathbb{C}$. *There is a unique (up to isomorphism) affine group scheme* $\mathbb{G}$ *over* $\mathbb{Z}$ *with the following properties.*

- $\mathbb{G}(\mathbb{C}) \cong G_{\mathbb{C}}$.
- *For every algebraically closed field $k$, the group $\mathbb{G}(k)$ is connected semisimple of the same type as $G_{\mathbb{C}}$, and split over the prime field of $k$ .*

$\mathbb{G}$ *is called the Chevalley-Demazure group scheme associated with $G_{\mathbb{C}}$ (or equivalently with the root datum of $G_{\mathbb{C}}$).*

The Chevalley-Demazure group scheme associated with the connected semisimple linear algebraic group $G_{\mathbb{C}}$ can be constructed in the following way:

Take $\rho$ a finite dimensional faithful representation of $G_{\mathbb{C}}$ on some vector space $V$ and denote by $d\rho$ the corresponding representation of $\mathrm{Lie}(G_{\mathbb{C}})$ on $V$. Then there is a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} = \langle v_1, ..., v_n \rangle_{\mathbb{Z}}$ in $V$ with the following properties:

- $\frac{1}{m!} d\rho(x_\alpha)^m V_{\mathbb{Z}} \subset V_{\mathbb{Z}}$ for all $\alpha \in \Phi, m \in \mathbb{N}$.
- $d\rho(h_\alpha)v_i = \eta_i(\alpha)v_i$ with $\eta_i(\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$ and $1 \leq i \leq n$.

The choice of the basis $v_1, ..., v_n$ gives $\mathbb{C}[G_{\mathbb{C}}] \subset \mathbb{C}[\mathrm{GL}_n]$ an integral structure, i.e. determines a $\mathbb{Z}$-subalgebra $\mathbb{Z}[\mathbb{G}]$ with $\mathbb{Z}[\mathbb{G}] \otimes \mathbb{C} = \mathbb{C}[G_{\mathbb{C}}]$ and this subalgebra is closed under the Hopf algebra operations of $\mathbb{C}[G_{\mathbb{C}}]$. Now the Chevalley-Demazure group scheme is the functor represented by $\mathbb{Z}[\mathbb{G}]$.

# B. Semisimple Groups over Local Fields

In this appendix we review some parts of the theory of linear algebraic groups over a local ground field. This situation is well-studied. The first major article on the topic is due to Iwahori and Matsumoto [IM65] who started by looking at the semisimple split case. Later Hijikata [Hij75] and probably most notably Bruhat and Tits [BT72, BT84] extended the theory to work for reductive groups over essentially arbitrary local ground fields.

Here we do not want to present the theory in the most general way possible as this would probably add more than a hundred pages. Instead we focus on the case that is most important for this thesis. This means in most parts of this chapter we only concern ourselves with split semisimple groups over locally compact local fields and to that end we will follow the exposition in [Iwa66].

## B.1. Algebraic Groups over Local Fields

Let $F$ be a local field of characteristic zero with ring of integers $\mathcal{O}_F$, prime ideal $\mathfrak{p} = \pi\mathcal{O}_F$, and finite residue class field $k = \mathcal{O}/\mathfrak{p}$. Furthermore let $\mathbb{G}$ be a reductive linear algebraic group defined over $F$ which we view as a subgroup of a general linear group $\mathrm{GL}_n$ for some $n$. The $F$-rational points $G := \mathbb{G}(F)$ form a subgroup of $\mathrm{GL}_n(F)$ which can be seen as a subset of the $F$-vector space $F^{n\times n}$ and is thus endowed with a topology (coming from the topology on $F$). We agree upon the following convention: The terms open, bounded, and compact will always refer to this topology, while the terms connected and closed always refer to the Zariski topology.

**Theorem B.1.1 ([Bru66, §1])** *Let $K$ be a subgroup of $G$. The following are equivalent:*

1. *There is a faithful linear representation $\rho$ of $\mathbb{G}$ defined over $F$ such that the matrix entries of $\rho(K)$ are bounded.*

2. *For any faithful linear representation $\rho$ of $\mathbb{G}$ defined over $F$ the matrix entries of $\rho(K)$ are bounded.*

3. *For any faithful linear representation $\rho$ of $\mathbb{G}$ defined over $F$ there is an $\mathcal{O}_F$-lattice $L$ in the representation space which is fixed by $\rho(K)$.*

A subgroup fulfilling the equivalent conditions of the last theorem will be called bounded. Since stabilizers of $\mathcal{O}_F$-lattices are open we see that every bounded subgroup is contained in an open, bounded subgroup. These groups actually arise from algebraic affine group schemes over $\mathcal{O}_F$ in the following way (cf. Definition 2.2.1): Under the choice of a basis of $L \leq V \cong F^n$ its stabilizer in $\mathrm{GL}(V)$ arises as the $\mathcal{O}_F$-group scheme represented by $\mathcal{O}_F[\mathrm{GL}_n] \subset F[\mathrm{GL}_n]$. The group scheme yielding the stabilizer of $L$ in $\mathbb{G}$ is then simply represented by the image of $\mathcal{O}_F[\mathrm{GL}_n]$ in $F[\mathbb{G}]$ (under the homomorphism $F[\mathrm{GL}_n] \to F[\mathbb{G}]$ corresponding to $\mathbb{G} \hookrightarrow \mathrm{GL}_n$).

In our situation where $k$ is finite (and hence $F$ locally compact) and $\mathbb{G}$ is reductive we obtain the following result.

**Theorem B.1.2 ([Bru66, §1])** *Every compact subgroup of $G$ is contained in a maximal compact subgroup.*

## B.2. The Affine Building

We now want to associate to our group $\mathbb{G}$ over a local field a combinatorial object with a $G$-action - called the building of $\mathbb{G}$ - capturing in essence the whole structure of $G$. In principle we could do this for arbitrary reductive groups over arbitrary local fields. However we will focus on the situation that is most important for this thesis and thus only look at split semisimple groups. The primary source for the following construction is [Tit79].

In keeping with the previous section let $F$ be a local field of characteristic zero with finite residue class field $k$ and discrete valuation $\omega$. We normalize $\omega$ in such a way that its value group $\omega(F^\times)$ is equal to $\mathbb{Z}$.

Let $\mathbb{G}$ be a split, connected, semisimple, linear algebraic group over $F$. We choose an $F$-split maximal torus $\mathbb{T}$ in $\mathbb{G}$ and denote by $\mathbb{N}$ its normalizer in $\mathbb{G}$. The finite Weyl group $W_0 := \mathbb{N}(F)/\mathbb{T}(F)$ is isomorphic to the Weyl group $W(\Phi)$ of the root system $\Phi = \Phi(\mathbb{G}, \mathbb{T}) \subset X^*(\mathbb{T})$, and for $a \in \Phi$ we denote by $\mathbb{U}_a$ the corresponding root subgroup of $\mathbb{G}$. Since $\mathbb{G}$ is split, $\mathbb{U}_a$ is of dimension one and isomorphic to $\mathbb{G}_a$. We set $V := X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ and denote the underlying affine $\mathbb{R}$-space by $A$.

### B.2.1. The Apartment

We start by constructing the fundamental segments of which our building will consist.

The choice of a Chevalley basis for the Lie algebra of $\mathbb{G}$ determines isomorphisms $x_a : \mathbb{G}_a \to \mathbb{U}_a$. Since $F$ (or rather its additive group) is filtered (via $\omega$) the same holds for the groups $\mathbb{U}_a(F) = x_a(F)$. The terms of these filtrations are in bijection with affine forms on A of the form $a + \gamma : x \mapsto a(x) + \gamma$ with $a \in \Phi$ and $\gamma \in \mathbb{Z}$ via the definition

$$X_{a+\gamma} := x_a(\omega^{-1}([\gamma, \infty])). \tag{B.1}$$

If we look instead at the new Chevalley basis we obtain by transforming the chosen one with an element of $t \in \mathbb{T}(F)$ (or rather its image under the adjoint representation), we have to replace the set $\{x_a\}$ by $\{x'_a\}$ where $x'_a = x_a \circ a(t)$. If we set $X'_{a+\gamma} := x'_a(\omega^{-1}([\gamma, \infty]))$ we thus get

$$X'_{a+\gamma} = X_{a+\gamma+\omega(a(t))}. \tag{B.2}$$

In particular the terms of the filtrations have not changed, however their indexation has.

**Definition B.2.1** *1. The space $A = A(\mathbb{G}, \mathbb{T}, F)$ is called the apartment (of $\mathbb{G}$ with respect to $\mathbb{T}$ over the field $F$).*

*2. The affine root system (of $\mathbb{G}$ with respect to $\mathbb{T}$) is the set*

$$\Phi_{af} := \{a + \gamma \mid a \in \Phi, \gamma \in \mathbb{Z}\}. \tag{B.3}$$

The set $\Phi_{af}$ consists of affine forms on the $\mathbb{R}$-space $A$ and to each element $\alpha \in \Phi_{af}$ we have associated a subgroup $X_\alpha$ of $\mathbb{G}(F)$. Whenever we choose a Chevalley basis we have an associated point $0 \in A$ such that $\Phi_{af}$ consists exactly of the forms

$$x \mapsto a(x - 0) + \gamma, \;\; a \in \Phi, \gamma \in \mathbb{Z}. \tag{B.4}$$

The normalizer $\mathbb{N}(F)$ acts on $A$ as a group of affine transformations (we call the corresponding homomorphism from $\mathbb{N}(F)$ to the group of affine transformations $\nu$) such that

$$n^{-1} X_\alpha n = X_{\alpha \circ \nu(n)} \;\; \alpha \in \Phi_{af}, n \in \mathbb{N}(F) \tag{B.5}$$

and $\mathbb{T}(F) \subset \mathbb{N}(F)$ acts as a subgroup of translations subject to the rule

$$a(\nu(t)) = -\omega(a(t)) \text{ for all } a \in \Phi, t \in \mathbb{T}(F). \tag{B.6}$$

Let us denote by $Z$ the kernel of $\nu : \mathbb{T}(F) \to V$. Then $\mathbb{T}(F)/Z =: \Lambda$ is a free Abelian group of rank $\dim(V)$. Furthermore $\widetilde{W} := \mathbb{N}(F)/Z$ is an extension of $W_0$ by $\Lambda$ and $\mathbb{N}(F)$ acts on $A$ through $\widetilde{W}$.

We now decompose the space $A$ with respect to the structure suggested by $\Phi_{af}$. To that end let $\alpha$ be an arbitrary affine form on $A$ with linear part $a \in \Phi$. We denote by $A_\alpha$ the subset $\alpha^{-1}([0,\infty])$ of $A$, by $\partial A_\alpha = \alpha^{-1}(0)$ its boundary and by $r_\alpha$ the reflection on $A$ with fixed point set $\partial A_\alpha$. The sets $A_\alpha$ with $\alpha \in \Phi_{af}$ are called the half-apartments of $A$ and the corresponding $\partial A_\alpha$ the walls of $A$. The connected components of the complement of the union of all walls of $A$ are called chambers and their facets will also be called facets of $A$ (making the chambers into the facets of maximal dimension in $A$). We note that for almost simple $\mathbb{G}$ all facets are simplices while in general (for semisimple $\mathbb{G}$) we get polysimplices (i.e. products of simplices, one for each almost simple constituent of $\mathbb{G}$).

**Definition B.2.2** *The group $W_{af} := \langle r_\alpha \mid \alpha \in \Phi_{af}\rangle$ is called the Weyl group of the affine root system $\Phi_{af}$ (or the affine Weyl group of $\mathbb{G}$).*

Since $\Phi_{af}$ is stable under the action of $\widetilde{W}$ the same holds for the sets of walls, half-apartments, and chambers and $W_{af}$ is a normal subgroup of $\widetilde{W}$. Moreover the group $W_{af}$ acts simply transitively on the set of chambers of $A$. Let $C$ be an arbitrary chamber of $A$ and denote the walls bounding $C$ by $L_0, ..., L_l$. Then there are unique roots $\alpha_i$, $0 \leq i \leq l$, such that $L_i = \partial A_{\alpha_i}$ and we call the set $\{\alpha_i \mid 1 \leq i \leq l\}$ the basis of $\Phi_{af}$ associated with $C$. As in the case of usual root systems we can now define a (local) Dynkin diagram capturing the information contained in $\Phi_{af}$.

**Definition B.2.3** *The local Dynkin diagram of $\Phi_{af}$ is given as follows: Denote (in the above notation) the linear part of $\alpha_i$ by $a_i$ and choose some $W$-invariant scalar product on the dual of $V$. The vertices $v_i$ of the Dynkin diagram are in bijection with the basis $\{\alpha_i\}$ and we join two vertices with no edge, a single edge, a double edge, a triple edge or a thick edge according to the possible values $0, 1, 2, 3, 4$ of the integer*

$$\lambda_{i,j} := \frac{4(a_i, a_j)}{(a_i, a_i)(a_j, a_j)}. \tag{B.7}$$

*In addition we direct the edge towards the shorter of the two roots in the cases $\lambda_{i,j} = 2, 3$ or $\lambda_{i,j} = 4$ and $a_i \neq -a_j$.*

Since $W_{af}$ is simply transitive on the set of chambers this definition is independent of our choice of $C$. If we disregard all directions and multiplicities in the local Dynkin diagram we obtain the ordinary Dynkin diagram of the reflection group $W_{af}$.

For a point $x \in A$ we will denote by $W_{af,x}$ the subgroup of $W_{af}$ generated by the $r_\alpha$ with $\alpha \in \Phi_{af}$ and $\alpha(x) = 0$. Furthermore set $\Phi_x$ the set of linear parts of affine roots vanishing on $x$ and associate to $x$ the subset $I_x$ of the local Dynkin diagram in the following way: Choose some $w \in W_{af}$ transforming $x$ into an element of the fundamental chamber $C$ and then define $I_x := \{v_i \mid xw \notin L_i\}$. Since all of these sets do only depend on the facet $f$ containing $x$ we will also denote them by $W_{af,f}, \Phi_f$ and $I_f$. Furthermore the set $\Phi_x$ is a subroot system of $\Phi$ whose Weyl group is simply the linear part of $W_{af,x}$ and whose Dynkin diagram we can obtain from the local Dynkin diagram of $\Phi_{af}$ by deleting all vertices in $I_x$ (and their attached edges).

**Definition B.2.4** *A point $x \in A$ is called hyperspecial if every element of $\Phi$ is proportional to an element of $\Phi_x$.*

Note that $x$ is hyperspecial if and only if $\Phi_x$ and $\Phi$ have the same Weyl groups; in this situation $W_{af}$ decomposes as the semidirect product of $W_0$ and $\Lambda = \mathbb{T}(F)/Z$. Geometrically speaking the point $x$ is hyperspecial if for any reflection hyperplane $\partial A_\alpha$ there is a parallel reflection hyperplane $\partial A_{\alpha'}$ containing $x$.

The hyperspecial points of $A$ are also characterized by their corresponding sets $I_x$ in the following way: We call a vertex in the Coxeter diagram of an irreducible affine reflection group hyperspecial if by deleting it we obtain the Coxeter diagram of the corresponding finite (spherical) reflection group. Then $x \in A$ is hyperspecial if $I_x$ consists of one hyperspecial vertex in each irreducible component of the local Dynkin diagram.

**Example B.2.5** *We want to make some of the definitions in this section more approachable by doing an explicit example in coordinates. To that end let $\mathrm{Sp}_{2m}$ be the symplectic group of degree $2m$ over $F$ which we embed into $\mathrm{GL}_{2m}$ as the stabilizer of the bilinear form*

$$J := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}. \tag{B.8}$$

*The set $\mathbb{T}$ of diagonal matrices in $\mathrm{Sp}_{2m}$ is a split maximal torus with set of characters $X^*(\mathbb{T}) = \langle \chi_1, ..., \chi_m \rangle_{\mathbb{Z}}$ where $\chi_i(t) = t_{i,i}$ for all $1 \leq i \leq m$. The set of cocharacters is then generated by $\eta_1, ..., \eta_m$ where $\langle \eta_i, \chi_j \rangle = \delta_{i,j}$.*

*Consider the root $a := \chi_1 - \chi_2$ with root space generated by $\rho := E_{2,1} - E_{3,4}$ and corresponding isomorphism $x_a : F \to \mathbb{U}_a(F),\ u \mapsto I_{2m} + u\rho$. Then for $\gamma \in \mathbb{Z}$ the group $X_{a+\gamma}$ is given by*

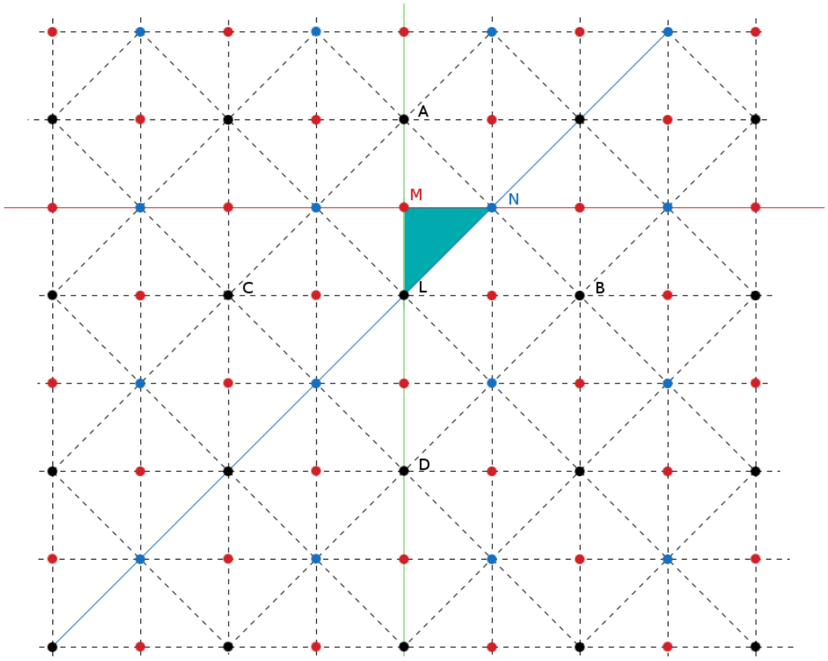
$$X_{a+\gamma} = I_{2m} + \pi^\gamma \mathcal{O}_F \rho. \tag{B.9}$$

*Now let $t = \mathrm{diag}(x_1, ..., x_m, x_1^{-1}, ..., x_m^{-1})$ be an element of $\mathbb{T}(F)$. Then $t$ acts on $A$ through the translation $\nu(t) \in V = \langle \eta_1, ..., \eta_m \rangle_{\mathbb{R}}$ where we have (in this basis):*

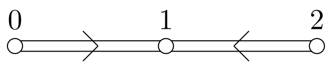
$$\nu(t) = (c_1, ..., c_m) \text{ with } c_i = \chi_i(\nu(t)) = -\omega(\chi(t)) = -\omega(x_i). \tag{B.10}$$

*In particular we see that the kernel $Z$ of $\nu$ is the set of all elements of $\mathbb{T}$ whose diagonal elements all lie in $\mathcal{O}_F$.*

*For $m = 2$ we can visualize the decomposition of $A$ with respect to the affine root system $\Phi_{af}$ (since it is a 2-dimensional space) as follows:*




*The turquois triangle is a chamber and the Dynkin diagram is of the form*

$\widetilde{C}_2$ : 


*The hyperspecial points of the apartment are the blue and the black points. Note that $W_{af}$ has two orbits on the hyperspecial points as indicated by the fact that the above Dynkin diagram has two hyperspecial vertices (vertices 0 and 2).*

### B.2.2. The Building

We keep the notation from the last subsection and now want to glue the apartments of all the $F$-split maximal tori together in a way that the resulting space has nice properties.

**Theorem B.2.6 ([Tit79, §2])** *There is (up to unique isomorphism) a unique (right-)$\mathbb{G}(F)$-set $\mathfrak{B} = \mathfrak{B}(\mathbb{G}, F)$ containing $A$ - called the Euclidean or local building of $\mathbb{G}$ over $F$ - with the following properties:*

1. $\mathfrak{B} = \cup_{g \in \mathbb{G}(F)} Ag$.

2. $\mathbb{N}(F)$ *fixes* $A$ *and acts on it via* $\nu$.

3. *For any* $\alpha \in \Phi_{af}$ *with linear part* $a$*, the group* $X_\alpha$ *fixes the half-apartment* $A_\alpha$.

One can construct $\mathfrak{B}$ by taking the quotient of $A \times \mathbb{G}(F)$ modulo the equivalence relation suggested by the properties above. The subsets $Ag$ of $\mathfrak{B}$ will be called the apartments of $\mathfrak{B}$ and we can think of $Ag$ as the apartment of the (again $F$-split maximal) torus $\mathbb{T}^g$ in a canonical way. In fact, $Ag$ is the only apartment that is stable under $\mathbb{T}^g$ (thus yielding a bijection between apartments and $F$-split maximal tori) and its stabilizer in $\mathbb{G}(F)$ is given by $\mathbb{N}(F)^g$.

By construction each $Ag$ is endowed with the structure of an affine $\mathbb{R}$-space and decomposes into facets; however it is a priori not clear that these structures are actually compatible with one another, i.e. agree on the intersection of apartments.

**Theorem B.2.7 ([Tit79, 2.2.1])** *Let* $A$ *and* $A'$ *be two apartments in the Euclidean building* $\mathfrak{B}$ *of* $\mathbb{G}$*. Then there is an element* $g \in \mathbb{G}(F)$ *with* $A' = Ag$ *that fixes* $A \cap A'$ *pointwise. Moreover* $A \cap A'$ *is a closed convex union of facets in* $A$.

Following this result it makes sense to speak of the facets of the building $\mathfrak{B}$ and $\mathfrak{B}$ is again a simplicial (polysimplicial) complex in the case where $\mathbb{G}$ is almost simple (semisimple). Furthermore for two facets $f, f'$ of $\mathfrak{B}$ there is always an apartment of $\mathfrak{B}$ containing them both.

Let us now take a scalar product on $V$ which is invariant under the action of the finite Weyl group $W_0$. The action of $\mathbb{G}(F)$ on $\mathfrak{B}$ then transports that scalar product to all apartments of $\mathfrak{B}$. This makes $\mathfrak{B}$ into a metric space where we define the distance between to points via the scalar product in an arbitrary apartment containing them both. This gives a well-defined metric by the above theorem. The topology induced by this metric is $\mathbb{G}(F)$-invariant and makes $\mathfrak{B}$ into a locally compact and contractible space.

**Theorem B.2.8 ([Tit79, 2.3.1])** *Every compact subgroup of* $\mathbb{G}(F)$ *has a fixed point in* $\mathfrak{B}$.

If $C$ is a chamber of $\mathfrak{B}$, we can choose an apartment containing $C$ and thus carry over the definition of the local Dynkin diagram to the whole building. However this definition is clearly independent of the chosen apartment and thus it makes sense to speak of the local Dynkin diagram $\Delta(\mathbb{G})$ of $\mathbb{G}$ (over $F$). Analogously the definition of hyperspecial points carries over to the whole building and the same characterizations as in the apartment case are still valid.

The group $(\mathrm{Aut}(\mathbb{G}))(F)$ acts on $\mathfrak{B}$ and thus on $\Delta(\mathbb{G})$. By the standard morphism $\mathbb{G} \to \mathrm{Int}(\mathbb{G}) \hookrightarrow \mathrm{Aut}(\mathbb{G})$ we hence get alternative actions of $\mathbb{G}(F)$ on $\mathfrak{B}$

and $\Delta(\mathbb{G})$, both of which coincide with the standard action. Let us denote by $\Omega$ the image of $\mathbb{G}(F)$ in (the finite group) $\mathrm{Aut}(\Delta(\mathbb{G}))$. We summarize some aspects of this action in the following theorem.

**Theorem B.2.9 ([Tit79, 2.5])**

1. *If $\mathbb{G}$ is semisimple and simply connected we have $\Omega = \{1\}$.*
2. *The group $\Omega$ coincides with the image of $\mathbb{T}(F)$ in $\mathrm{Aut}(\Delta(\mathbb{G}))$.*
3. *Two facets $f, f'$ of $\mathfrak{B}$ are in the same $\mathbb{G}(F)$-orbit if and only if the two sets $I_f, I_{f'}$ are in the same $\Omega$-orbit.*

### B.2.3. Stabilizers

As seen earlier, every compact subgroup of $\mathbb{G}(F)$ has a fixed point in $\mathfrak{B}$. Denote now for bounded $M \subset \mathfrak{B}$ the pointwise stabilizer of $M$ in $\mathbb{G}(F)$ by $\mathbb{G}(F)^M$. Then we have as a converse to the first statement that $\mathbb{G}(F)^M$ is always compact. This means that maximal compact subgroups are always stabilizers of points and if $f$ is a facet of $\mathfrak{B}$ of minimal dimension and $x \in f$ then $\mathbb{G}(F)^x$ is a maximal compact subgroup. In particular the stabilizers of hyperspecial points are always maximal compact.

**Definition B.2.10** *The pointwise stabilizers in $\mathbb{G}(F)$ of hyperspecial points of $\mathfrak{B}$ are called hyperspecial subgroups of $\mathbb{G}(F)$.*

Now let $C$ be a chamber in $\mathfrak{B}$ and $B := \mathbb{G}(F)^C$. Then we have the Bruhat decomposition

$$\mathbb{G}(F) = B\mathbb{N}(F)B \tag{B.11}$$

and $BnB \mapsto \nu(n)$ ($n \in \mathbb{N}(F)$) is a bijection $B\backslash\mathbb{G}(F)/B \to \widetilde{W}$.

**Theorem B.2.11 ([Tit79, 3.4.1])** *Let $M \subset \mathfrak{B}(\mathbb{G}, F)$ be bounded. Then there is a smooth affine group scheme $\underline{G}_M$ over $\mathcal{O}_F$ with:*

1. *The generic fiber $\underline{G}_{M,F}$ is $\mathbb{G}$.*
2. *For any extension $F_1/F$ we have $\underline{G}_M(\mathcal{O}_{F_1}) = \mathbb{G}(F_1)^M$.*

The group schemes $\mathbb{G}_x$ with hyperspecial $x \in \mathfrak{B}$ are precisely the Chevalley group schemes with generic fiber $\mathbb{G}$.

**Theorem B.2.12 ([Tit79, 3.4.4])** *Let $M$ be as above. The reduction mod $\pi$ homomorphism $\underline{G}_M(\mathcal{O}_F) = \mathbb{G}(F)^M \to \underline{G}_{M,k}(k)$ is surjective.*

Now let $C$ be a chamber in $\mathfrak{B}$. Then we call $I(C) := \underline{G}_C(\mathcal{O}_F) \leq \mathbb{G}(F)$ the Iwahori subgroup of $\mathbb{G}(F)$ with respect to $C$. Equivalently we could take a point $x$ in the closure of $C$ and define $I(C)$ to be the inverse image of a Borel subgroup of $\underline{G}_{x,k}(k)$ (under the reduction homomorphism). Since $\mathbb{G}(F)$ is transitive on the chambers in $\mathfrak{B}$ all Iwahori subgroups are conjugate. We call a compact subgroup of $\mathbb{G}(F)$ containing an Iwahori subgroup a parahoric subgroup.[1]

The group $\mathbb{G}(F)$ is locally compact and unimodular thus there is a unique Haar measure giving the Iwahori subgroups measure 1. We get the following characterization of hyperspecial subgroups:

**Theorem B.2.13 ([Tit79, 3.8])** *Among the compact subgroups of $\mathbb{G}(F)$ the hyperspecial subgroups are characterized as those with maximal volume (with respect to the Haar measure above).*

## B.3. The Generalized Tits System

We now want to look at the structure of the group itself. Again let $\mathbb{G}$ be a connected, semisimple, linear algebraic group over $F$ of Chevalley type. We fix an $F$-split maximal torus $\mathbb{T}$ in $\mathbb{G}$ and denote by $\Phi$ and $\Phi^\vee$ the root system of $\mathbb{G}$ with respect to $\mathbb{T}$ and the set of coroots, respectively. For simplicity we set $G := \mathbb{G}(F), T := \mathbb{T}(F)$. This yields the usual chains

$$\langle\Phi\rangle \leq X^*(T) \leq \left(\langle\Phi^\vee\rangle\right)^\# \quad \text{and } \langle\Phi^\vee\rangle \leq X_*(T) \leq \left(\langle\Phi\rangle\right)^\# \tag{B.12}$$

where we take the dual with respect to the usual pairing. Since $\mathbb{G}$ is semisimple, the Abelian group $\left(\langle\Phi^\vee\rangle\right)^\# / \langle\Phi\rangle \cong \left(\langle\Phi\rangle\right)^\# / \langle\Phi^\vee\rangle$ is finite. Thus also the group $\Omega' := X_*(T)/\langle\Phi^\vee\rangle$ is finite. Remember that $\mathbb{G}$ is adjoint if and only if $\langle\Phi\rangle = X^*(T)$ and simply connected if and only if $X_*(T) = \langle\Phi^\vee\rangle$ (so if $\Omega'$ is of order 1).

Denote by $\mathfrak{g}_F$ the Lie algebra of $\mathbb{G}$ over $F$ and by $\mathfrak{g}_\mathcal{O}$ the Chevalley lattice in $\mathfrak{g}_F$. We define $G_\mathcal{O}$ to be the stabilizer of $\mathfrak{g}_\mathcal{O}$ in $G$, i.e.

$$G_\mathcal{O} = \{g \in G \mid \mathfrak{g}_\mathcal{O}\mathrm{Ad}(g) \subset \mathfrak{g}_\mathcal{O}\}. \tag{B.13}$$

**Theorem B.3.1 ([Iwa66, §2])** *The group $G_\mathcal{O}$ is generated by the elements*

$$x_\alpha(r),\ \alpha \in \Phi, r \in \mathcal{O} \text{ and } \eta(s),\ \eta \in X_*(T), s \in \mathcal{O}^\times \tag{B.14}$$

*where for $\alpha \in \Phi$ we denote by $x_\alpha$ the associated ($F$-rational) isomorphism $\mathbb{G}_a \to \mathbb{U}_\alpha$.*

This description of $G_\mathcal{O}$ implies the following result:

[1] The explicit definition of parahoric subgroups varies in the literature. However for our purposes it is fine to work with this one.

**Corollary B.3.2 ([Iwa66, §2])** *The morphism $\phi$, given by reduction modulo $\mathfrak{p}$, maps $G_{\mathcal{O}}$ onto the Chevalley group $G_k$ over $k$ associated with the root datum of $\mathbb{G}$. In particular $G_{\mathcal{O}}$ has good reduction and is a hyperspecial subgroup of $G$.*

We now fix a Borel subgroup $B_k$ (associated with a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$) containing the reduction of $T \cap G_{\mathcal{O}}$ and set $I := \phi^{-1}(B_k) \subset G_{\mathcal{O}}$. Since $G_{\mathcal{O}}$ is hyperspecial the group $I$ is an Iwahori subgroup of $G$.

**Theorem B.3.3 ([Iwa66, §2])** *The group $I$ is generated by the elements*

$$x_\alpha(r),\ \alpha \in \Phi^+, r \in \mathcal{O},\ x_\beta(r),\ \beta \in \Phi^-, r \in \mathfrak{p},\ \text{and } \eta(s),\ \eta \in X_*(T), s \in \mathcal{O}^\times \tag{B.15}$$

In addition to our notation so far we define $N$ to be the normalizer of $T$ in $G$. The main result regarding the structure of $G$ is as follows.

**Theorem B.3.4 ([Iwa66, §2])** *The triple $(G, I, N)$ is a generalized Tits system. That means it fulfills the following axioms:*

1. *$H := N \cap I$ is normal in $N$.*
2. *The quotient $N/H$ is a semidirect product of an Abelian subgroup $\Omega$ and a normal subgroup $W_{af}$ generated by involutions $w_i \in S$ such that*
   a) *For any $\sigma \in \Omega W_{af}$ and $w_i \in S$ we have $\sigma I w_i \subset I\sigma w_i I \cup I\sigma I$.*
   b) *$w_i^{-1} I w_i \neq I$ for all $w_i \in S$.*
3. *The elements of $\Omega$ normalize $S$.*
4. *The elements of $\Omega$ normalize $I$ but $I\rho \neq I$ for all $\rho \in \Omega$.*

*As usual all cosets are written in suggestive notation (e.g. $Iw_iI$ means the double coset $I\widetilde{w_i}I$ for some representative of $w_i$).*

The group $\Omega W_{af}$ coincides with the generalized or extended affine Weyl group of $G$ which we denoted by $\widetilde{W}$. Before we come to describing the situation more precisely we first note some immediate consequences of the axioms in the above theorem.

**Corollary B.3.5 ([Iwa66, §1])** *In the notation of the above theorem the following holds:*

1. *$G = \bigsqcup_{w \in \widetilde{W}} IwI$.*
2. *The normalizer of $I$ is $N_G(I) = I\Omega$ and $N_G(I)/I \cong \Omega$.*

3. *If $H$ is a subgroup of $G$ containing $I$, there is a subset $S_H \subset S$ and a subgroup $\Omega_H \leq \Omega$ fixing $S_H$ such that $H = I\Omega_H W(S_H)I$ where $W(S_H)$ is the subgroup of $W$ generated by $S_H$.*

4. *Conversely for every pair $(S', \Omega')$ with the above properties the set $H := I\Omega' S' I$ is a subgroup of $G$ containing $I$ with $S_H = S'$ and $\Omega_H = \Omega'$. In particular there is a bijection between such pairs and subgroups of $G$ containing $I$.*

5. *Two subgroups $H_1$ and $H_2$ of $G$ containing $I$ are conjugate if and only if they are conjugate under $N_G(I)$ which is the case if and only if $\Omega_{H_1} = \Omega_{H_2}$ and $S_{H_1}$ and $S_{H_2}$ are conjugate under an element of $\Omega$.*

6. *The group $G_0 := IW_{af}I$ is a normal subgroup of $G$ (with quotient isomorphic to $\Omega$) and $(G_0, I, N_0)$ is a Tits system in the usual sense.*

In our situation we can describe the structure of $\widetilde{W}$ a little more explicitly (cf.[Iwa66, §2]). The group $\widetilde{W}$ acts on the lattice $X_*(T)$ and thus on the affine $\mathbb{R}$-space $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ as a group of affine transformations isomorphic to $X_*(T) \rtimes W_0$ where we embed $X_*(T)$ as a group of translations and $W_0 = W(\Phi)$ denotes the (finite) Weyl group of $\Phi$. The subgroup $W_{af}$ is isomorphic to the affine Weyl group associated with $\Phi$, i.e. the group generated by all the reflections

$$w_{\alpha,k} : X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \to X_*(T) \otimes_{\mathbb{Z}} \mathbb{R},\ v \mapsto v - \langle v, \alpha \rangle \alpha^\vee + k\alpha^\vee,\ \alpha \in \Phi, k \in \mathbb{Z}. \tag{B.16}$$

The set $S$ of involutions generating $W_{af}$ (and making it into a Coxeter group) is given as follows: Let $\Phi = \Phi_1 \sqcup ... \sqcup \Phi_r$ be the decomposition of $\Phi$ into its irreducible components and fix the notation $\Delta_i = \{\alpha_j^{(i)} \mid 1 \leq j \leq r_i\}$ where $\Delta_i$ is the simple system in $\Phi_i$ associated with the choice of our Borel subgroup. Furthermore denote by $\alpha_0^{(i)}$ the highest root of the root system $\Phi_i$ (with respect to $\Delta_i$). Then $S$ is the disjoint union of the sets $S_i,\ 1 \leq i \leq r$, where

$$S_i = \left\{ w_{\alpha_j^{(i)},0} \mid 1 \leq j \leq r_i \right\} \sqcup \left\{ w_{\alpha_0^{(i)},1} \right\}. \tag{B.17}$$

In the decomposition $\widetilde{W} \cong X_*(T) \rtimes W_0$ the elements $w_{\alpha_j^{(i)},0}$ correspond to the standard generators of $W_0$, while $w_{\alpha_0^{(i)},1}$ acts as $t_{-\alpha_0^\vee} w_{\alpha_0}$ on $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ where by $t_{-\alpha_0^\vee}$ we mean the translation by $-\alpha_0^\vee$.

In particular we can read off the number of conjugacy classes of maximal compact subgroups containing a conjugate of $I$ in this situation from the Dynkin diagram (and the action of $\Omega$ on $\Phi$). If $\mathbb{G}$ is simply connected (so $\Omega = \{1\}$) the number of such conjugacy classes is simply $\prod_{i=1}^r (r_i + 1)$ and the $r_i$ are the ranks of the almost simple components of $\mathbb{G}$. If $\mathbb{G}$ is adjoint and almost simple we give the corresponding numbers in the following table:

| Dynkin type | Number of classes |
|---|---|
| $A_n,\ n \geq 1$ | # of divisors of $n+1$ |
| $B_n, C_n,\ n \geq 2$ | $n+1$ |
| $D_n, n \geq 4$ even | $n+2$ |
| $D_n, n \geq 4$ odd | $n+2$ |
| $E_6$ | 5 |
| $E_7$ | 8 |
| $E_8$ | 9 |
| $F_4$ | 5 |
| $G_2$ | 3 |

**Example B.3.6** *We return briefly to Example B.2.5. In this case the group* $\mathrm{Sp}_{2m,\mathcal{O}}$ *consists of all symplectic matrices having all entries in* $\mathcal{O}$*. Thus it is the stabilizer of the hyperspecial point given by the lattice*

$$L := \langle e_1, ..., e_m, f_1, ..., f_m \rangle \tag{B.18}$$

*in the building we constructed in Section 6.2.2.*

*The Iwahori subgroup* $I$ *consists of all symplectic matrices that have entries only in* $\mathcal{O}$ *and are upper triangular after reduction modulo* $\pi$*. In particular* $I$ *is the stabilizer of the chamber given by the lattice chain*

$$\langle e_1, ..., e_m, f_1, ..., f_m \rangle \supset \langle \pi e_1, e_2 ..., e_m, f_1, ..., f_m \rangle \supset ... \supset \langle \pi e_1, ..., \pi e_m, f_1, ..., f_m \rangle. \tag{B.19}$$

# C. Implementation

## C.1. A Guide to the Programs

Here we want to provide the potential user with a short handbook on how to use our programs to compute algebraic modular forms for compact forms of symplectic groups. Note that in general these functions make only a limited effort to ensure that the input is as desired.

### C.1.1. Working with Hermitian Lattices

The programs for working with Hermitian lattices (i.e. $\mathcal{O}_D$-lattices in $D^m$ where $D$ is a definite quaternion algebra) draw heavily on the internal arithmetic for quaternion algebras as well as Markus Kirschmer's “forms”-package ([Kir15]) for working with lattices of number fields.

We will first describe the most important commands and then give an example at the end of this subsection.

The user may specify a Hermitian lattice in one of two ways: Option one is to use the command `HermitianLattice` which takes as an argument a matrix in $D^{n\times m}$ where $D$ is a definite quaternion algebra over a totally real number field. The result is then the $\mathcal{O}_D$ lattice (where $\mathcal{O}_D$ is either computed along the way or stored as `D‘MaxOrd`) generated by the rows of the matrix.

The second option is to use the command `GenusRepresentative` which takes a single argument `GS` of type `rec` and has three assigned fields `GS‘Dimension`, `GS‘Discriminant` and `GS‘Labels`. The result will be a Hermitian lattice in the specified dimension over a maximal order of the quaternion algebra with the specified discriminant. The list `GS‘Labels` consists of tuples of the form `<P,i>` where $P$ is a prime ideal not dividing the discriminant an $i$ lies between 1 and `1+GS‘Dimension`. The resulting lattice will then have label $i$ at $P$ and label 1 at all other primes.

There are a few commands available for dealing with such a lattice once it has been initialized. The command `RegularLattice` computes the corresponding $\mathcal{O}_k$-lattice in $k^{4m}$ under the regular representation of $D$. The command

`RegularToHermLattice` does the opposite, taking a lattice in $k^{4m}$ and a definite quaternion algebra as its input and computing the Hermitian lattice that is generated by the inverse image under the regular representation.

The stabilizer of an Hermitian lattice in the unitary group can be computed via the command `UStab` which takes as an argument either a Hermitian lattice or a tuple of lattices (in which case the pointwise stabilizer of this tuple will be computed). The result is a (finite) matrix group of degree $4m$ over the ground field of the quaternion algebra. Similarly one can test two Hermitian lattices (or tuples thereof) for being isomorphic (under the unitary group) via the command `IsUIsomorphic(L::HermLat,M::HermLat)`. The result is a boolean and a unitary matrix `g` (in case the boolean value is `true`) which fulfills `L=Mg`.

For transforming matrices between the quaternion infrastructure and the regular representation we provide the two commands `QuatToRegular(mat::Mtrx)`, which computes the regular representation (with respect to the quaternion algebra underlying `mat`), and `RegularToQuat(mat::Mtrx,H::AlgQuat)` performing the inverse.

**Example:**

```
> K:=QNF();
> R:=Integers(K);
> H:=QuaternionAlgebra(K,-1,-1);
> L:=HermitianLattice(MatrixRing(H,2)!1);
> UL:=UStab(L);
> #UL;
1152
> GS:=recformat<Discriminant,Dimension,Labels>;
> gs:=rec<Dimension:=2, Discriminant:=2*R,Labels:=[]>;
> M:=GenusRepresentative(gs);
> IsUIsomorphic(L,M);
true
[1 0]
[0 1]
> gs:=rec<GS |Dimension:=2, Discriminant:=2*R,Labels:=[<3*R,2>]>;
> M:=GenusRepresentative(gs);
> Label(M,3*R);
2
> IsUIsomorphic(L,M);
false
> ULM:=UStab(<L,M>);
> #ULM;
144
> ULM eq UStab(L) meet UStab(M);
true
```

## C.1.2. Genera of Hermitian Lattices

We provide programs for computing a system of representatives in the genus of a Hermitian lattice.

The command `Label(L::HermLat,P::RngOrdIdl)` computes the label of the Hermitian lattice `L` at the prime ideal `P` (see Section 6.2.2) provided the prime ideal does not divide the discriminant of the underlying quaternion algebra and `L` defines a maximal integral form.

The command `MassSP(L::HermLat)` computes the mass of the genus of `L` under the same assumption on `L` as above.

The command `ComputeGenus(L::HermLat)` computes a system of representatives for the isomorphism classes in the genus of `L`. The result is a list of tuples; each tuple consisting of a Hermitian lattice, a (symplectic) matrix over a completion of the ground field and the prime ideal at which we completed. The matrix, say `g`, is chosen such that the representative coincides with `L` away from the given prime ideal and corresponds to `Lg` there. As optional arguments the user may specify the prime ideal at which he wishes the representatives to be computed or the mass of the genus (in case `L` does not define a maximal integral form, but the user precomputed the mass himself).

**Example:**

```
> K:=QNF();
> R:=Integers(K);
> H:=QuaternionAlgebra(K,-2,-5);
> L:=HermitianLattice(MatrixRing(H,2)!1);
> GenL:=ComputeGenus(L);
New lattice found, mass left:  0
> MassSP(L);
13/720
> &+[1/#UStab(r[1]): r in GenL];
13/720
> L eq GenL[1][1];
true
> IsUIsomorphic(L,GenL[2][1]);
false
> GenL2:=ComputeGenus(L: Prime:=3*R,Mass:=13/720);
New lattice found, mass left:  0
> IsUIsomorphic(GenL[2][1],GenL2[2][1]);
true
[-1/6 - 11/24*i - 1/12*j + 1/24*k 1/6 - 3/8*i + 1/12*j + 1/8*k]
[5/12 - 7/24*i - 1/6*j - 1/24*k -5/12 + 1/8*i + 1/6*j - 1/8*k]
```

### C.1.3. Hecke Operators and Algebraic Modular Forms

The main task that can be tackled with our program is the computation of spaces of algebraic modular forms and Hecke operators acting on these.

Again $k$ will be a totally real number field, $m$ a natural number and $D$ a definite quaternion algebra over $k$.

The most basic command is `BasisOfFormSpace(L::HermLat,Rho::Map)` which computes a basis for the space of algebraic modular forms of level Stab(`L`) and weight given by `Rho`. The map `Rho` has to be the representation the user wants to work with and has to be applicable to elements of $\mathrm{GL}_{4m}(k)$ (or at least the images of $\mathrm{U}_m(D)$ under the regular representation). The output consists of two sequences. The second one contains representatives for the genus of `L` and the first one bases for the subspaces of invariants under the various stabilizers of the representatives (in the representation space). This will be interpreted as a basis of the form space as follows:

Denote the $i$-th representative of the genus of `L` by $\mathtt{L}_i$, then $\mathtt{L}_i = \mathtt{L}\mu_i$ for certain $\mu_i$ in the adelic group (e.g. one can take for $\mu_i$ the second entry of the tuple containing $\mathtt{L}_i$ in the sequence of genus representatives). We label the basis for the invariants (under `Rho`) of the stabilizer of $\mathtt{L}_i$ as $b_1^{(i)}, ..., b_{d_i}^{(i)}$, then a basis of the space of algebraic modular forms is given the functions

$$f_{i,j}, 1 \leq i \leq h, 1 \leq j \leq d_i, \text{ where } f_{i,j}(\mu_i') = \delta_{i,i'} b_j^{(i)}. \tag{C.1}$$

Given the sequence of representatives all commands will interpret the space of modular forms as endowed with this basis (ordered first by $i$, then by $j$).

For the computation of Hecke operators we provide two methods: The first one is the standard method via computing a decomposition of double cosets into right cosets by employing the formulas in [Lan01]. The command to do this is

```
HeckeOperator(L::HermLat,P::RngOrdIdl, w::MonStgElt, Rho::Map)
```

which computes the Hecke operator corresponding to `w` at the prime `P` acting on the space of algebraic modular forms of level described by `L` and weight given by `Rho`. The inputs `L` and `Rho` are to be interpreted as above, and the prime ideal `P` states that we are computing a Hecke operator of a double coset which is only supported at this prime. The string `w` consists of integers between 0 and $m$ and corresponds to an element of the affine Weyl group in the following sense. Say `w:=‘‘010’’`, then we are computing the Hecke operator (supported only at `P`) which corresponds to the double coset of $s_0 s_1 s_0$ in the affine Weyl group. It is the users responsibility to ensure that `L` is maximal at `P` and that `w` is the element of minimal length in its double coset with respect to the subgroup of $W_{af}$ corresponding to $\mathrm{Stab}(L)$.

The output consists of a matrix (describing the action of the Hecke operator), a sequence (containing representatives for the genus of `L`) and another sequence (containing - as above - bases for the invariant subspaces under the stabilizers).

In addition the user may already hand over the genus of `L` or a $U_m(D)$-invariant form on the representation space. In the former case the basis of the form space will be ordered in the order suggested by the given representatives; in the latter case there will be a fourth output which is the Gram matrix of the Peterson inner product on the space of algebraic modular forms. There is no need to input an invariant form in the case of the trivial representation as in this case the Peterson product will be computed anyway.

**Example:**

```
> K:=QNF();
> R:=Integers(K);
> H:=QuaternionAlgebra(K,-2,-5);
> B:=MatrixRing(H,2)!1;
> L:=HermitianLattice(B);
> RhoTriv:=hom<GL(4*2,K)->GL(1,K)|x:->GL(1,K)!1>;
> h,GenL,FB,Pet:=HeckeOperator(L,2*R,"0",RhoTriv);
New isometry class found. Mass left:  0
> h;
[21 30]
[ 9  0]
> Pet;
[ 1/72        0]
[       0 1/240]
> h*Pet eq Pet*Transpose(h);
true
> hh:=HeckeOperator(L,2*R,"010",RhoTriv : Genus:=GenL);
> hh;
[96 80]
[24 40]
> hh*Pet eq Pet*Transpose(hh);
true
> h*hh eq hh*h;
true
```

The second method for computing Hecke operators is via the Eichler method. To that end we describe how to compute intertwining operators. The command to do this is

```
IntertwiningOperator(L::HermLat, P::RngOrdIdl,
                gen2::RngIntElt,Rho::Map)
```

where `L,P,Rho` all are as above and `gen2` specifies the label of the second genus we want to compute with at `P`. As before the user may hand over representatives for both genera and an invariant form (the latter being imperative if he wants to employ the Eichler method).

Due to the considerate number of computed objects the output is compiled into a record with the following fields: `T21`, `Genus1`, `Genus2`, `FormSpaceBasis1`, `FormSpaceBasis2`, `T12`, `Peterson1`, and `Peterson2`. The last three will only be computed if the user provides an invariant form on the representation space. The matrix `T21` is the intertwining operator between the spaces of algebraic modular forms whose bases are given by the sequences `FormSpaceBasis1` and `FormSpaceBasis2` (in the order suggested by `Genus1` and `Genus2`). The opposite operator `T12` is computed as its adjoint if an invariant form is given (in which case the gram matrices for the Peterson products will be computed also).

As before it is not necessary to provide the inner product in the case of the trivial representation.

Additionally we provide a function
`FullActionEichlerMethod(L::HermLat,P::RngOrdIdl,Rho::Map)` which computes the action of the standard generators of the Hecke algebra at `P` via the Eichler method as long as `L` is of label 1 at `P`. For this command it is imperative that the user provides an invariant form (if the representation is not the trivial one).

**Example:**

```
> K:=QNF();
> R:=Integers(K);
> H:=QuaternionAlgebra(K,-2,-5);
> B:=MatrixRing(H,2)!1;
> L:=HermitianLattice(B);
> RhoTriv:=hom<GL(4*2,K)->GL(1,K)|x:->GL(1,K)!1>;
> T:=IntertwiningOperator(L,2*R,2,RhoTriv);
New lattice found, mass left:  0
New isometry class in second genus found. Mass left:  0
> T`T21;
[ 6  0]
[ 9 15]
> T`T12;
[3 2]
[0 1]
```

```
> T‘T12*T‘T21;
[36 30]
[ 9 15]
> h+15; //h as defined in the last example.
[36 30]
[ 9 15]
> T2:=IntertwiningOperator(L,2*R,3,RhoTriv: Genus1:=T‘Genus1);
New isometry class in second genus found. Mass left:  0
> T2‘T12*T2‘T21;
[174 170]
[ 51  55]
> hh+3*h+15;
[174 170]
[ 51  55]
> [h,hh];
[
    [21 30]
    [ 9  0],

    [96 80]
    [24 40]
]
> FullActionEichlerMethod(L,2*R,RhoTriv);
New lattice found, mass left:  0
New isometry class in second genus found. Mass left:  0
New isometry class in second genus found. Mass left:  0
[
    [21 30]
    [ 9  0],

    [96 80]
    [24 40]
]
```

## C.2. Source Code

Due to space constraints and the very limited benefit of printing source code, the programs are only available on the author's webpage:

http://www.math.rwth-aachen.de/~Sebastian.Schoennenbeck/

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