# Algorithms for algebraic modular forms

UPenn Algebra seminar

Sebastian Schönnenbeck

March 21, 2016

# A classical problem

$V = \mathbb{Q}^n$, $q : V \to \mathbb{Q}$ a positive definite quadratic form.
$L, L' \subset V$ lattices (i.e. $\mathbb{Z}$-submodules in $V$ of rank $n$)

- $L, L'$ *isometric*, $L \cong L'$, iff $gL = L'$ for some $g \in O(q)$; $\mathrm{class}(L) = \{M \mid M \cong L\}$.
- $L, L'$ in the *same genus*, iff for all $p$ prime there is $g_p \in O(\mathbb{Q}_p \otimes V, \mathbb{Q}_p \otimes q)$ with $g_p(\mathbb{Z}_p \otimes L) = \mathbb{Z}_p \otimes L'$.
- $L, L'$ isometric implies $L, L'$ in the same genus **but** the converse is false.
- The genus of $L$ decomposes into finitely many isometry classes; $\mathrm{genus}(L) = \mathrm{class}(L_1) \sqcup \ldots \sqcup \mathrm{class}(L_r)$, $r$ the *class number* of $L$.

**Question:** How do we find representatives for the isometry classes in a given genus?

### Theorem (Eichler, Kneser)

Assume $L$ even (i.e. $q(L) \subset 2\mathbb{Z}$) of rank greater or equal $3$, $\det(L)$ squarefree, and $p \nmid \det(L)$ prime. Then every class in the genus of $L$ is represented by a lattice $M$ such that

$$\mathbb{Z}_\ell \otimes L = \mathbb{Z}_\ell \otimes M \;\forall\, \ell \neq p.$$

# Strong approximation and the Kneser method

## Neighbours

$L$ as before. $M, N \in \mathrm{genus}(L)$ are called $p$-neighbours, $M \overset{p}{-} N$ if

$$[M : M \cap N] = [N : M \cap N] = p.$$

## Theorem (Kneser)

For $M \in \mathrm{genus}(L)$ there is an $M' \in \mathrm{class}(M)$ and a chain of lattices $L = L_0, L_1, ..., L_k = M'$ such that

$$L_0 \overset{p}{-} L_1 \overset{p}{-} L_2 \overset{p}{-} ... \overset{p}{-} M'.$$

## Neighbouring Graph

The $p$-neighbouring graph of $\mathrm{genus}(L)$ is the directed weighted graph with vertices the isometry classes in $\mathrm{genus}(L)$ and edges $\mathrm{class}(L_i) \to \mathrm{class}(L_j)$ weighted by the number of $p$-neighbours of $L_i$ isometric to $L_j$.

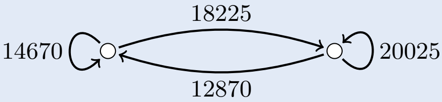
# Example

## Dimension $16$

There are two isometry classes in the set of even unimodular lattices of rank $16$:

$$\mathrm{class}(E_8 \perp E_8) \sqcup \mathrm{class}(D_{16}^+).$$

The $2$-neighbouring graph of this genus is:




## Remark

- Note that $18225 + 14670 = 12870 + 20025$, number of neighbours does not depend on the class.
- Note that $\frac{18225}{12870} = \frac{|\mathrm{Aut}(E_8 \perp E_8)|}{|\mathrm{Aut}(D_{16}^+)|}$.
- The adjacency matrix of the graph acts as a Hecke operator on the space of modular forms generated by the theta series of the two lattices (Eichler, Andrianov, Yoshida).

# Algebraic modular forms

**Notation:** $\mathbb{G}$ almost simple linear algebraic group defined over $\mathbb{Q}$, $\mathbb{G}(\mathbb{R})$ compact, $V$ a f.d. $\mathbb{Q}$-rational representation of $\mathbb{G}$, $\mathbb{A}_f$ the finite adeles of $\mathbb{Q}$.

## Definition [Gross '99]

$K \leq \mathbb{G}(\mathbb{A}_f)$ open and compact.

$$M(V,K) := \{f : \mathbb{G}(\mathbb{A}_f) \to V \mid f(gxk) = gf(x) \text{ for all } g \in \mathbb{G}(\mathbb{Q}), x \in \mathbb{G}(\mathbb{A}_f), k \in K\},$$

the space of *algebraic modular forms* of *level* $K$ and *weight* $V$.

## Remark

- $|gKg^{-1} \cap \mathbb{G}(\mathbb{Q})| < \infty$ for all $g \in \mathbb{G}(\mathbb{A}_f)$.
- $|\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_f)/K| < \infty$.
- Let $\mathbb{G}(\mathbb{A}_f) = \bigsqcup_{i=1}^{r} \mathbb{G}(\mathbb{Q})\gamma_i K$ and $\Gamma_i := \gamma_i K \gamma_i^{-1} \cap \mathbb{G}(\mathbb{Q})$ then

$$M(V,K) \cong_{\mathbb{Q}} \bigoplus_{i=1}^{r} V^{\Gamma_i}.$$

# Example (cont.)

$\mathbb{G} = \mathrm{SO}_{16}$, $K = \prod_p \mathrm{Stab}_{\mathrm{SO}_{16}(\mathbb{Q}_p)}(\mathbb{Z}_p \otimes (E_8 \perp E_8))$. Then:

$$\mathbb{G}(\mathbb{A}_f) = \mathbb{G}(\mathbb{Q})K \sqcup \mathbb{G}(\mathbb{Q})\gamma K,$$

where $\gamma \in \mathbb{G}(\mathbb{A}_f)$ with $\gamma(E_8 \perp E_8) = D_{16}^+$.
Set

$$\Gamma_1 := \mathrm{Stab}_{\mathrm{SO}_{16}(\mathbb{Q})}(E_8 \perp E_8) = K \cap \mathbb{G}(\mathbb{Q}),\ |\Gamma_1| = 2^{28} \cdot 3^{10} \cdot 5^4 \cdot 7^2,$$
$$\Gamma_2 := \mathrm{Stab}_{\mathrm{SO}_{16}(\mathbb{Q})}(D_{16}^+) = \gamma K \gamma^{-1} \cap \mathbb{G}(\mathbb{Q}),\ |\Gamma_2| = 2^{29} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13.$$

- $M(\text{triv.}, K) \cong \mathbb{Q}^{\Gamma_1} \oplus \mathbb{Q}^{\Gamma_2} \cong \mathbb{Q} \oplus \mathbb{Q}$.
- $M(\mathbb{Q}^{16}, K) \cong (\mathbb{Q}^{16})^{\Gamma_1} \oplus (\mathbb{Q}^{16})^{\Gamma_2} = \{0\}$.
- $M(\mathrm{Sym}^2(\mathbb{Q}^{16}), K) \cong \mathbb{Q} \oplus \mathbb{Q}$

# The Hecke algebra

## Definition

$K \leq \mathbb{G}(\mathbb{A}_f)$ open and compact.

$$H_K := \{f : \mathbb{G}(\mathbb{A}_f) \to \mathbb{Q} \mid f \text{ compactly supported and } K\text{-bi-invariant}\}$$

with multiplication by convolution. The *Hecke algebra* of $\mathbb{G}$ with respect to $K$.

## Remark

- $H_K$ has the natural basis $\mathbb{1}_{K\gamma K}$, $K\gamma K \in \mathbb{G}(\mathbb{A}_f)/\!\!/K$.
- Let $\gamma_1, \gamma_2 \in \mathbb{G}(\mathbb{A}_f)$ with $K\gamma_i K = \bigsqcup_j \gamma_{i,j} K$. The multiplication in $H_K$ is

$$\mathbb{1}_{K\gamma_1 K}\mathbb{1}_{K\gamma_2 K} = \sum_{j,j'} \mathbb{1}_{\gamma_{1,j}\gamma_{2,j'}K}.$$

- If $K = \prod_p K_p$ is a product of local factors, the Hecke algebra is the restricted tensor product

$$H_K = \otimes'_p H_{K_p}.$$

# The action of the Hecke algebra

## Definition

For $\gamma \in \mathbb{G}(\mathbb{A}_f)$ we define the linear operator $T(\gamma) \in \mathrm{End}_{\mathbb{Q}}(M(V, K))$ via

$$(T(\gamma)f)(x) = \sum_i f(x\gamma_i)$$

where $f \in M(V, K)$ and $K\gamma K = \bigsqcup_i \gamma_i K$.

## Remark

- The additive extension of $\mathbb{1}_{K\gamma K} \mapsto T(\gamma)$ defines an algebra homomorphism $H_K \to \mathrm{End}_{\mathbb{Q}}(M(V, K))$.
- The is a scalar product on $M(V, K)$ with respect to which $T(\gamma)^{ad} = T(\gamma^{-1})$.
- $M(V, K)$ is a semi-simple $H_K$-module.

# Integral forms

What would be “interesting” / computationally well-suited open compact subgroups to consider?

## Definition

Let $\mathbb{G} \hookrightarrow \mathrm{GL}_n$ be a faithful representation. An *integral form* $\mathbb{G}_L$ of $\mathbb{G}$ is given by a lattice $L \leq_{\mathbb{Z}} \mathbb{Q}^n$ via

$$\mathbb{G}_L(\mathcal{O}_k) = \mathrm{Stab}_{\mathbb{G}(k)}(\mathcal{O}_k \otimes L),\ \mathbb{G}_L(\mathbb{Z}_p) = \mathrm{Stab}_{\mathbb{G}(\mathbb{Q}_p)}(\mathbb{Z}_p \otimes L)$$

for every finite extension $k$ of $\mathbb{Q}$ and every prime $p$.

## Remark

- $\mathbb{G}(\mathbb{A}_f)$ acts on the integral forms via $(g_p)_p L = L'$ where $\mathbb{Z}_p \otimes L' = g_p(\mathbb{Z}_p \otimes L)$ for all $p$.
- $\mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L) = \prod_p \mathbb{G}_L(\mathbb{Z}_p)$ is an open compact subgroup of $\mathbb{G}(\mathbb{A}_f)$.
- $\mathbb{G}_L(\mathbb{Z}_p)$ is a hyperspecial maximal compact subgroup of $\mathbb{G}(\mathbb{Q}_p)$ for all but finitely many $p$.
- Call $L, L'$ ($\mathbb{G}$-)*isomorphic* if $gL = L'$ for some $g \in \mathbb{G}(\mathbb{Q})$. Say $L, L'$ in the same *genus* if $\gamma L = L'$ for some $\gamma \in \mathbb{G}(\mathbb{A}_f)$.

# Algorithmic questions

**Aim:** Compute the action of $T(\gamma)$ on $M(V, K)$ (where $K$ comes from an integral form $\mathbb{G}_L$).

## Approach

- Decompose $\mathbb{G}(\mathbb{A}_f) = \bigsqcup_i \mathbb{G}(\mathbb{Q})\mu_i K$, compute $\Gamma_i = \mu_i K \mu_i^{-1} \cap \mathbb{G}(\mathbb{Q})$ and $V^{\Gamma_i}$.
- Decompose $K\gamma K = \bigsqcup_j \gamma_j K$.
- For $i, j$ write $\gamma_j \mu_i$ as $g' \mu_{i'} k$ for some $i'$.

## Aspects to consider

What do we have to know in order to make this work?

- Decide for two integral forms if they are $\mathbb{G}$-isomorphic / compute the stabilizer of a lattice in $\mathbb{G}(\mathbb{Q})$.
- Be able to compute a system of representatives for $\mathrm{genus}(L)$.
- Decompose double cosets into left cosets.

# Stabilizers and isometries

$\mathbb{G} \hookrightarrow \mathrm{GL}_n$, $L, L' < \mathbb{Q}^n$ lattices.

- $\mathbb{G}(\mathbb{Q}) \subset \mathrm{GL}_n(\mathbb{Q})$ compact $\rightsquigarrow$ $\mathbb{G}(\mathbb{Q})$ fixes a definite inner product on $\mathbb{Q}^n$.
- $\mathbb{G}(\mathbb{Q}) \subset O_n(\mathbb{Q}) \rightsquigarrow \mathrm{Stab}_{\mathbb{G}(\mathbb{Q})}(L) \subset \mathrm{Stab}_{O_n(\mathbb{Q})}(L)$.
- $\mathrm{Stab}_{O_n(\mathbb{Q})}(L)$ computable (Plesken-Souvignier-algorithm) and finite $\rightsquigarrow$ Finding $\mathrm{Stab}_{\mathbb{G}(\mathbb{Q})}(L)$ reduced to a finite problem.
- Same idea for isometry testing: Find $O_n$-isometry $g : L \to L'$ $\rightsquigarrow$ All isometries are given by $g\,\mathrm{Stab}_{O_n(\mathbb{Q})}(L)$ $\rightsquigarrow$ Finite problem.

## Example: $G_2$

The group $G_2$ can be realized as the automorphism group of the ($8$-dim.) octonion algebra $\mathbb{O}$ (Dickson-double of the Hamilton quaternions). $G_2$ fixes the inner product $(x, y) \mapsto \mathrm{Tr}(x\bar{y})$. $L < \mathbb{O}$ lattice then $\mathrm{Stab}_{G_2}(L)$ is the stabilizer of the multiplication (which can be thought of as an element of $V^* \otimes V^* \otimes V$) in $\mathrm{Stab}_{O_8}(L)$. E.g. $L$ a maximal order then

$$|\mathrm{Stab}_{O_8}(L)| = 696729600, \ |\mathrm{Stab}_{G_2}(L)| = 12096.$$

# Genus enumeration

**Question:** How do you compute representatives of $\mathrm{genus}(L)$ starting at $L$?

## Almost Strong Approximation [Chan, Hsia]

If $\mathbb{G}$ is simply connected and of certain type then there is a finite set $\Omega$ of primes such that for all $p \notin \Omega$ the lattice $L_p$ is hyperspecial and we can find representatives of $\mathrm{genus}(L)$ as "$p$-neighbours" of $L$.

**Question:** How do you know when to stop?

## Mass Formula

Let $\mathrm{genus}(L) = \mathrm{class}(L_1) \sqcup \ldots \sqcup \mathrm{class}(L_r)$ and set

$$\mathrm{mass}(\mathrm{genus}(L)) := \sum_{i=1}^{r} \frac{1}{|\,\mathrm{Stab}_{\mathbb{G}}(\mathbb{Q})(L_i)|}.$$

Then we can compute $\mathrm{mass}(\mathrm{genus}(L))$ from information only on the local structure of $L$.

# Examples

## Some Genera for $G_2$

| Genus | Class Number | Mass Decomposition |
|---|---|---|
| max. order | $1$ | $\frac{1}{12096}$ |
| type $2$ at $3$, max. else | $2$ | $\frac{1}{192} + \frac{1}{432}$ |
| type $2$ at $5$, max. else | $3$ | $\frac{1}{192} + \frac{1}{48} + \frac{1}{36}$ |
| type $3$ at $7$, max. else | $2$ | $\frac{1}{216} + \frac{1}{42}$ |

## Some Genera for $\mathrm{Sp}_4$

Compact forms of $\mathrm{Sp}$ can be found as unitary groups over (definite) quaternion algebras $D$, integral forms via $\mathcal{O}_D$-lattices (where $\mathcal{O}_D$ is a maximal order).

| Genus Representative | Class Number | Mass Decomposition |
|---|---|---|
| $\mathcal{O}_D^2$, $D = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ | $1$ | $\frac{1}{1152}$ |
| $\mathcal{O}_D^2$, $D = \left(\frac{-2,-5}{\mathbb{Q}}\right)$ | $2$ | $\frac{1}{240} + \frac{1}{72}$ |
| $\mathcal{O}_D^2$, $D = \left(\frac{-2,-13}{\mathbb{Q}}\right)$ | $4$ | $\frac{1}{48} + \frac{1}{12} + \frac{2}{8}$ |
| $\mathcal{O}_D^2$, $D = \left(\frac{-1,-23}{\mathbb{Q}}\right)$ | $16$ | $\frac{5}{4} + \frac{3}{8} + \frac{3}{12} + \frac{2}{24} + \frac{1}{32} + \frac{1}{48} + \frac{1}{72}$ |

# Coset decomposition

**Aim:** Decompose a double coset $K\gamma K$ into left cosets.
First observation: Since $H_K = \otimes'_p H_{K_p}$ we only need to do this locally.
For simplicity assume: $\mathbb{G}$ split at $p$, $K_p$ "nice".

## Structure of $p$-adic Groups (Bruhat-Tits)

$I \leq K_p \leq \mathbb{G}(\mathbb{Q}_p)$ Iwahori subgroup, $\tilde{W}(= X^\vee \rtimes W_0)$ the extended affine Weyl group.

- $\mathbb{G}(\mathbb{Q}_p) = \bigsqcup_{w\in\tilde{W}} IwI$, $K_p = \bigsqcup_{w\in W_{K_p}} IwI$ for some $W_{K_p} \leq \tilde{W}$.
- $H_I$ is an algebra with basis $T_w,\ w \in \tilde{W}$ and multiplication $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$, $T_s^2 = (p-1)T_s + p$ for the simple reflections $s$.
- $e := [K_p : I]^{-1} \sum_{w\in W_{K_p}} T_w \in H_I$ is an idempotent and $H_{K_p} \cong eH_Ie$.

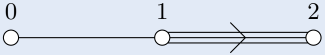
## Coset Decomposition

- Bruhat-Tits ('65): Explicit formula to decompose $IwI,\ w \in \tilde{W}$ into $I$-left cosets.
- Lansky-Pollack (2001): Explicit formula to decompose $K_p w K_p,\ w \in \tilde{W}$ into $K_p$-left cosets.

# Example

## $G_2$

$\mathbb{G}$ of type $G_2$ (simply connected and adjoint) with extended Dynkin diagram (at split prime $p$):

$\tilde{G}_2$ :



If $K$ open compact, with $K_p$ hyperspecial maximal compact ($W_{K_p} = \langle s_1, s_2 \rangle$), then the local Hecke algebra $H_{K_p}$ is a polynomial ring in two variables generated by the characteristic functions on the double cosets

$$T_1 := K_p s_0 K_p \text{ and } T_2 := K_p s_0 s_1 s_2 s_1 s_0 K_p.$$

$T_1$ decomposes into $p(p^5 + p^4 + p^3 + p^2 + p + 1)$ left cosets.
$T_2$ decomposes into $p^5(p^5 + p^4 + p^3 + p^2 + p + 1)$ left cosets.

## $\mathrm{Sp}_4$

In the analogous situation for $\mathrm{Sp}_4$ (simply connected but not adjoint) there are also two generators which decompose into $p(p^3 + p^2 + p + 1)$ and $p^3(p^3 + p^2 + p + 1)$ left cosets, respectively.

# The Eichler method

**Aim:** Find a way to compute two Hecke operators at once using the incidence relation on the affine building of $\mathbb{G}$.
**Idea:** Use a generalization of the following idea attributed to Eichler: $L, L'$ two lattices, then

$$\mathrm{mass}(\mathrm{genus}(L))\frac{[\mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L) : \mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L, L')]}{[\mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L') : \mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L, L')]} = \mathrm{mass}(\mathrm{genus}(L')).$$

Let $K_1, K_2$ be two open, compact subgroups of $\mathbb{G}(\mathbb{A}_f)$ and $K_2 = \bigsqcup_i m_i(K_1 \cap K_2)$.

## Observation

If $\mathbb{G}(\mathbb{A}_f) = \bigsqcup_j K_1\gamma_j\mathbb{G}(\mathbb{Q})$, then one can find a system of representatives for $K_2\backslash\mathbb{G}(\mathbb{A}_f)/\mathbb{G}(\mathbb{Q})$ in the collection $m_i\gamma_j$.

## Interpretation (Ex.)

Is $L \leq E_8 \perp E_8$ a lattice, one can find representatives of the isometry classes in $\mathrm{genus}(L)$ as sublattices of $E_8 \perp E_8$ and $D_{16}^+$ (and all of these representatives have index $[(E_8 \perp E_8) : L]$ in either $E_8 \perp E_8$ or $D_{16}^+$).

# The intertwining operator

## Definition

We define the intertwining operator $T_2^1 := T(K_1, K_2)$ (w.r.t. $K_1$ and $K_2$) via

$$T_2^1 : M(V, K_1) \to M(V, K_2),\ f \mapsto f' \text{ where } f'(x) = \sum_i f(xm_i).$$

## Lemma

The operators $T(K_1, K_2)$ und $T(K_2, K_1)$ are adjoint to each other with respect to the scalar products on $M(V, K_1)$ and $M(V, K_2)$. In particular $T(K_2, K_1)$ is uniquely determined by $T(K_1, K_2)$.

## Definition

Let $K_1 = \bigsqcup_{i'} l_{i'}(K_1 \cap K_2)$, then we call $\nu_{1,2} := \nu(K_1, K_2) := \sum_{i,i'} \mathbb{1}_{l_{i'} m_i K_1}$ the *Eichler element* w.r.t. $K_1$ and $K_2$.

## Proposition

- $\nu_{1,2}$ is an element of $H_{K_1}$.
- $T_1^2 T_2^1 = T(\nu_{1,2})$, in particular we see that $T_1^2 T_2^1$ acts as a (self adjoint) Hecke operator on $M(V, K_1)$.

**Question:** Which operators are obtainable in this fashion?

## Theorem [S.]

Let $\mathbb{G}$ be simply connected, $K_i = \prod_p K_{i,p}$ products of local factors with $K_{1,p} = K_{2,p}$ for all $p \neq q$ and $K_{1,q}, K_{2,q}$ parahoric subgroups of $\mathbb{G}(\mathbb{Q}_p)$, which contain a common Iwahori subgroup $I$. Let $\widetilde{W}$ be the extended affine Weyl group and $W_i \leq \widetilde{W}$ with $K_{i,q} = IW_iI$, $W_{1,2} = W_1 \cap W_2$ and $[W_{1,2} \backslash W_2 / W_{1,2}]$ a system of representatives of elements of shortest lengths. Then the following holds:

$$\nu_{1,2} = \sum_{\kappa \in [W_{1,2} \backslash W_2 / W_{1,2}]} [I(W_1 \cap {}^{\kappa}W_1)I : I(W_1 \cap {}^{\kappa}W_1 \cap W_2)I] \mathbb{1}_{K_1 \kappa K_1}.$$

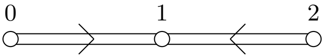
## Theorem [S.]

Let $\mathbb{G}$ be of type $C_n$ simply connected, $K_1$ as above with $K_{1,q}$ hyperspecial. If $K_{i,q}, 2 \leq i \leq n+1$, runs through the $n$ further conjucacy classes of maximal parahoric subgroups, then the corresponding elements $\nu(K_1, K_i)$ form a minimal generating system for the local Hecke algebra $H_{K_{1,q}}$.

# Example

Generators for the local (hyperspecial) Hecke algebra for $\mathrm{Sp}_4$ (Type $C_2$, s.c.):
Extended Dynkin diagram:

$\widetilde{C}_2$ :




$W_1 := \langle s_1, s_2 \rangle, W_2 := \langle s_0, s_2 \rangle, W_3 := \langle s_0, s_1 \rangle$. $I \leq \mathrm{Sp}_4(\mathbb{Q}_q)$ Iwahori subgroup,
$K_{i,q} = IW_iI, i = 1, 2, 3$.
$H_{K_{1,q}}$ is generated by $\mathbb{1}_{K_1 s_0 K_1}, \mathbb{1}_{K_1 s_0 s_1 s_0 K_1}$.

Coset decomposition and Eichler elements:

- $[W_{1,2} \backslash W_2 / W_{1,2}] = \{1, s_0\}$, ${}^{s_0}W_1 \cap W_1 = \langle s_2 \rangle = {}^{s_0}W_1 \cap W_1 \cap W_2$.
- $\nu_{1,2} = (q^3 + q^2 + q + 1)\mathbb{1}_{K_1} + \mathbb{1}_{K_1 s_0 K_1}$.
- $[W_{1,3} \backslash W_3 / W_{1,3}] = \{1, s_0, s_0 s_1 s_0\}$, ${}^{s_0 s_1 s_0}W_1 \cap W_1 = \langle s_1 \rangle = {}^{s_0 s_1 s_0}W_1 \cap W_1 \cap W_3$.
- $\nu_{1,3} = (q^3 + q^2 + q + 1)\mathbb{1}_{K_1} + (q + 1)\mathbb{1}_{K_1 s_0 K_1} + \mathbb{1}_{K_1 s_0 s_1 s_0 K_1}$.

# Example (cont.)

$L_1 = \mathcal{O}_D^2$, $D = \left(\frac{-2,-5}{\mathbb{Q}}\right)$, hyperspecial at $p \neq 5$. There are lattices $L_3 \leq L_2 \leq L_1$, such that $L_i$ differ only at $2$ and $\mathrm{Stab}_{\mathbb{G}(\mathbb{Q}_2)}(L_i) = K_{i,2}$ (as above).
Compute $T(K_1 s_0 K_1)$ and $T(K_1 s_0 s_1 s_0 K_1)$ acting on $M(\text{triv.}, K_1)$:

- $\mathrm{genus}(L_i) = \mathrm{class}(L_i) \sqcup \mathrm{class}(L_i')$, for $1 \leq i \leq 3$.
- Classical method: $T(K_1 s_0 K_1) = \begin{pmatrix} 21 & 9 \\ 30 & 0 \end{pmatrix}$, $T(K_1 s_0 s_1 s_0 K_1) = \begin{pmatrix} 96 & 24 \\ 80 & 40 \end{pmatrix}$.
- For each class in $\mathrm{genus}(L_1)$ we had to construct $30$ (resp. $120$) lattices and test them for isometry.
- $T_1^2 = \begin{pmatrix} 9 & 6 \\ 15 & 0 \end{pmatrix}$, $T_1^3 = \begin{pmatrix} 12 & 3 \\ 10 & 5 \end{pmatrix}$ $\left( \rightsquigarrow T_2^1 = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}, T_3^1 = \begin{pmatrix} 12 & 3 \\ 10 & 5 \end{pmatrix} \right)$.
- For each class in in $\mathrm{genus}(L_1)$ we had to construct $15$ (resp. $15$) lattices and test them for isometry.
- $15I_2 + T(K_1 s_0 K_1) = \begin{pmatrix} 36 & 9 \\ 30 & 15 \end{pmatrix} = T_1^2 T_2^1$.
- $15I_2 + 3T(K_1 s_0 K_1) + T(K_1 s_0 s_1 s_0 K_1) = \begin{pmatrix} 174 & 51 \\ 170 & 55 \end{pmatrix} = T_1^3 T_3^1$.
- We also obtain the Hecke operators $T_2^1 T_1^2 = 3I_2 + T(K_2 s_1 K_2) + T(K_2 s_1 s_2 s_1 K_2)$ and $T_3^1 T_1^3 = 15 I_2 + T(K_3 s_2 K_3)$ acting on $M(\text{triv.}, K_2)$ and $M(\text{triv.}, K_3)$ respectively.