Simultaneous Computation of Hecke Operators

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Experimental and constructive algebra

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Algebraic Modular Forms

Notation: $\mathbb{G}$ almost simple, linear algebraic group defined over $\mathbb{Q}$, $\mathbb{G}(\mathbb{R})$ compact, V a f.d. $\mathbb{Q}$-rational representation of $\mathbb{G}$, $\mathbb{A}_f$ the finite adeles of $\mathbb{Q}$.

Definition [Gross '99]

$K \leq \mathbb{G}(\mathbb{A}_f)$ open and compact.

$$M(V,K) := \{f : \mathbb{G}(\mathbb{A}_f) \to V \mid f(gxk) = gf(x) \text{ for all } g \in \mathbb{G}(\mathbb{Q}), x \in \mathbb{G}(\mathbb{A}_f), k \in K\},$$

the space of *algebraic modular forms* of *level* K and *weight* V.

Remark

- $|gKg^{-1} \cap \mathbb{G}(\mathbb{Q})| < \infty$ for all $g \in \mathbb{G}(\mathbb{A}_f)$.
- $|\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_f)/K| < \infty$.
- Let $\mathbb{G}(\mathbb{A}_f) = \bigsqcup_{i=1}^{r} \mathbb{G}(\mathbb{Q})\gamma_i K$ and $\Gamma_i := \gamma_i K \gamma_i^{-1} \cap \mathbb{G}(\mathbb{Q})$ then we have

$$M(V,K) \cong_{\mathbb{Q}} \bigoplus_{i=1}^{r} V^{\Gamma_i}.$$

The Hecke Algebra

Definition

$K \leq \mathbb{G}(\mathbb{A}_f)$ open and compact.

$$H_K := \{f : \mathbb{G}(\mathbb{A}_f) \to \mathbb{Q} \mid f\ K\text{-biinvariant with compact support}\}$$

with multiplication given by convolution is called the *Hecke algebra* of $\mathbb{G}$ w.r.t. K.

Remark

- H_K has the natural basis $\mathbb{1}_{K\gamma K},\ K\gamma K \in \mathbb{G}(\mathbb{A}_f)/\!\!/K$.
- Let $\gamma_1, \gamma_2 \in \mathbb{G}(\mathbb{A}_f)$ and $K\gamma_i K = \bigsqcup_j \gamma_{i,j} K$. Then the multiplication in H_K is given by
$$\mathbb{1}_{K\gamma_1 K}\mathbb{1}_{K\gamma_2 K} = \sum_{j,j'} \mathbb{1}_{\gamma_{1,j}\gamma_{2,j'}K}.$$
- If $K = \prod_p K_p$ is a product of local factors, the Hecke algebra is the restricted tensor product
$$H_K = \otimes'_p H_{K_p}.$$

The Action of the Hecke Algebra

Definition

For $\gamma \in \mathbb{G}(\mathbb{A}_f)$ we define the linear map $T(\gamma) \in \mathrm{End}_{\mathbb{Q}}(M(V,K))$ via

$$(T(\gamma)f)(x) = \sum_i f(x\gamma_i)$$

where $f \in M(V,K)$ and $K\gamma K = \bigsqcup_i \gamma_i K$.

Remark

- The additive extension of $\mathbb{1}_{K\gamma K} \mapsto T(\gamma)$ yields an algebra morphism $H_K \to \mathrm{End}_{\mathbb{Q}}(M(V,K))$.
- $M(V,K)$ carries a scalar product, with respect to which $T(\gamma)' = T(\gamma^{-1})$.
- $M(V,K)$ is a semisimple H_K-module.

The Venkov Method

Aim: Find a way to compute two Hecke operators at once using the incidence relation on the affine building of $\mathbb{G}$.
Idea: Use a generalization of the following idea attributed to Venkov: L, L' two lattices, then

$$\mathrm{mass}(\mathrm{genus}(L))\frac{[\mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L) : \mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L,L')]}{[\mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L') : \mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L,L')]} = \mathrm{mass}(\mathrm{genus}(L')).$$

Remark

Let U be a f.d. faithful representation of $\mathbb{G}$ and $L \subset U$ a lattice. Then $K_L := \mathrm{Stab}_{\mathbb{G}(\mathbb{A}_f)}(L) = \prod_p \mathrm{Stab}_{\mathbb{G}(\mathbb{Q}_p)}(L \otimes \mathbb{Z}_p)$ is an open compact subgroup of $\mathbb{G}(\mathbb{A}_f)$.

Let K_1, K_2 be two open, compact subgroups of $\mathbb{G}(\mathbb{A}_f)$ and $K_2 = \bigsqcup_i m_i(K_1 \cap K_2)$.

Observation

If $\mathbb{G}(\mathbb{A}_f) = \bigsqcup_j K_1\gamma_j\mathbb{G}(\mathbb{Q})$, then one can find a system of representatives for $K_2\backslash\mathbb{G}(\mathbb{A}_f)/\mathbb{G}(\mathbb{Q})$ in the collection $m_i\gamma_j$.

The Transfer Operator

Definition

We define the transfer operator $T_2^1 := T(K_1, K_2)$ (w.r.t. K_1 and K_2) via

$$T_2^1 : M(V, K_1) \to M(V, K_2),\ f \mapsto f' \text{ where } f'(x) = \sum_i f(xm_i).$$

Lemma [S.]

The operators $T(K_1, K_2)$ und $T(K_2, K_1)$ are adjoint to each other with respect to the scalar products on $M(V, K_1)$ and $M(V, K_2)$. In particular $T(K_2, K_1)$ is uniquely determined by $T(K_1, K_2)$.

Definition

Let $K_1 = \bigsqcup_{i'} l_{i'}(K_1 \cap K_2)$, then we call $\nu_{1,2} := \nu(K_1, K_2) := \sum_{i,i'} \mathbb{1}_{l_{i'} m_i K_1}$ the *Venkov element* w.r.t. K_1 und K_2.

Proposition

- $\nu_{1,2}$ is an element of H_{K_1}.
- $T_1^2 T_2^1 = T(\nu_{1,2})$, in particular we see that $T_1^2 T_2^1$ acts as a (self adjoint) Hecke operator on $M(V, K_1)$.

Question: Which operators are obtainable in this fashion?

Theorem [S.]

Let $\mathbb{G}$ be simply connected, $K_i = \prod_p K_{i,p}$ products of local factors with $K_{1,p} = K_{2,p}$ for all $p \neq q$ and $K_{1,q}, K_{2,q}$ parahoric subgroups of $\mathbb{G}(\mathbb{Q}_p)$, which contain a common Iwahori subgroup I. Let $\widetilde{W}$ be the extended affine Weyl group and $W_i \leq \widetilde{W}$ with $K_{i,q} = IW_iI$, $W_{1,2} = W_1 \cap W_2$ and $[W_{1,2} \backslash W_2 / W_{1,2}]$ a system of representatives of elements of shortest lengths. Then the following holds:

$$\nu_{1,2} = \sum_{\kappa \in [W_{1,2} \backslash W_2 / W_{1,2}]} [I(W_1 \cap {}^{\kappa}W_1)I : I(W_1 \cap {}^{\kappa}W_1 \cap W_2)I] \mathbb{1}_{K_1 \kappa K_1}.$$

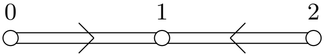
Theorem [S.]

Let $\mathbb{G}$ be of type C_n simply connected, K_1 as above with $K_{1,q}$ hyperspecial. If $K_{i,q}, 2 \leq i \leq n+1$, runs through the n further conjucacy classes of maximal parahoric subgroups, then the corresponding elements $\nu(K_1, K_i)$ form a minimal generating system for the local Hecke algebra $H_{K_{1,q}}$.

Example

Generators for the local (hyperspecial) Hecke algebra for Sp_4 (Type C_2, s.c.):
Extended Dynkin diagram:

$\widetilde{C}_2$:



$W_1 := \langle s_1, s_2 \rangle, W_2 := \langle s_0, s_2 \rangle, W_3 := \langle s_0, s_1 \rangle$. $I \leq \mathrm{Sp}_4(\mathbb{Q}_q)$ Iwahori subgroup,
$K_{i,q} = I W_i I, i = 1, 2, 3$.
$H_{K_{1,q}}$ is generated by $\mathbb{1}_{K_1 s_0 K_1}, \mathbb{1}_{K_1 s_0 s_1 s_0 K_1}$.

Coset decomposition and Venkov elements:

- $[W_{1,2} \backslash W_2 / W_{1,2}] = \{1, s_0\}$, ${}^{s_0}W_1 \cap W_1 = \langle s_2 \rangle = {}^{s_0}W_1 \cap W_1 \cap W_2$.
- $\nu_{1,2} = (q^3 + q^2 + q + 1)\mathbb{1}_{K_1} + \mathbb{1}_{K_1 s_0 K_1}$.
- $[W_{1,3} \backslash W_3 / W_{1,3}] = \{1, s_0, s_0 s_1 s_0\}$, ${}^{s_0 s_1 s_0}W_1 \cap W_1 = \langle s_1 \rangle = {}^{s_0 s_1 s_0}W_1 \cap W_1 \cap W_3$.
- $\nu_{1,3} = (q^3 + q^2 + q + 1)\mathbb{1}_{K_1} + (q + 1)\mathbb{1}_{K_1 s_0 K_1} + \mathbb{1}_{K_1 s_0 s_1 s_0 K_1}$.

Example (cont.)

$L_1 = \mathcal{O}_D^2$, $D = \left(\frac{-2,-5}{\mathbb{Q}}\right)$, hyperspecial at $p \neq 5$. There are lattices $L_3 \leq L_2 \leq L_1$, such that L_i differ only at 2 and $\mathrm{Stab}_{\mathbb{G}(\mathbb{Q}_2)}(L_i) = K_{i,2}$ (as above).
Compute $T(K_1 s_0 K_1)$ and $T(K_1 s_0 s_1 s_0 K_1)$ acting on $M(\text{triv.}, K_1)$:

- $\mathrm{genus}(L_i) = \mathrm{class}(L_i) \sqcup \mathrm{class}(L_i')$, for $1 \leq i \leq 3$.
- Classical method: $T(K_1 s_0 K_1) = \begin{pmatrix} 21 & 9 \\ 30 & 0 \end{pmatrix}$, $T(K_1 s_0 s_1 s_0 K_1) = \begin{pmatrix} 96 & 24 \\ 80 & 40 \end{pmatrix}$.
- For each class in $\mathrm{genus}(L_1)$ we had to construct 30 (resp. 120) lattices and test them for isometry.
- $T_1^2 = \begin{pmatrix} 9 & 6 \\ 15 & 0 \end{pmatrix}$, $T_1^3 = \begin{pmatrix} 12 & 3 \\ 10 & 5 \end{pmatrix}$ $\left(\rightsquigarrow T_2^1 = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}, T_3^1 = \begin{pmatrix} 12 & 3 \\ 10 & 5 \end{pmatrix} \right)$.
- For each class in in $\mathrm{genus}(L_1)$ we had to construct 15 (resp. 15) lattices and test them for isometry.
- $15 I_2 + T(K_1 s_0 K_1) = \begin{pmatrix} 36 & 9 \\ 30 & 15 \end{pmatrix} = T_1^2 T_2^1$.
- $15 I_2 + 3T(K_1 s_0 K_1) + T(K_1 s_0 s_1 s_0 K_1) = \begin{pmatrix} 174 & 51 \\ 170 & 55 \end{pmatrix} = T_1^3 T_3^1$.
- We also obtain the Hecke operators $T_2^1 T_1^2 = 3I_2 + T(K_2 s_1 K_2) + T(K_2 s_1 s_2 s_1 K_2)$ and $T_3^1 T_1^3 = 15 I_2 + T(K_3 s_2 K_3)$ acting on $M(\text{triv.}, K_2)$ and $M(\text{triv.}, K_3)$ respectively.