Q1. We consider consider the following problem:

$$\frac{\partial u}{\partial x^2} = 4\frac{\partial u}{\partial t}, \quad 0 < x < 2, t > 0$$
$$u(0,t) = 0 \text{ and } u(2,t) = 0 \text{ for } t > 0$$
$$u(x,0) = 2\sin\frac{\pi x}{2} + 4\sin(2\pi x), \quad 0 \le x \le 2$$

Solve for $u(x,t)$.

Answer:

Assume a solution of the form: $u(x,t) = T(t)X(x)$.

Then $u_{xx} = T(t)X''(x)$; $\quad u_t = \dot{T}(t)X(x)$.

Substituting these into the equation:

$$\begin{aligned}
&T(t)X''(x) = 4\dot{T}(t)X(x)\\
\Rightarrow &\frac{4\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda, \text{ for some constant } \lambda\\
\Rightarrow &\dot{T}(t) = \frac{1}{4}\lambda T(t) \quad ; \quad X''(x) = \lambda X(x)
\end{aligned}$$

So for the time-dependent part, its solution is $T(t) = Ce^{\frac{\lambda}{4}t}$, $\quad C$ constant.

Using the boundary conditions, we have:

$$u(x,t) = T(t)X(x) \Rightarrow T(t)X(0) = 0 = T(t)X(2) \Rightarrow X(0) = X(2) = 0$$

So the eigenvalue problem is:

$$\begin{aligned}
&X''(x) = \lambda X(x), \quad \text{with}\\
&X(0) = X(2) = 0, \quad (0 < x < 2).
\end{aligned}$$

CASE 1: $\lambda > 0$:

Let $\lambda = \mu^2$, then:

$$X''(x) - \mu^2 X(x) = 0$$

Since the characteristic equation: $r^2 - \mu^2 = 0 \Rightarrow r = \pm\mu$, the general solution is :

$$X(x) = Ae^{\mu x} + Be^{-\mu x}$$

Applying the boundary conditions:

$$\begin{aligned} &X(0) = A + B = 0 \Rightarrow B = -A \\ \Rightarrow\ &X(x) = A\left(e^{\mu x} - e^{-\mu x}\right) = 2A\sinh(\mu x) \\ &X(2) = 2A\sinh(2\mu) = 0 \end{aligned}$$

Since $\sinh(2\mu) > 0$ for $\mu > 0$, it must be true that $A = 0 = B$, leading to a trivial solution.

CASE 2: $\lambda = 0$:

The equation becomes:

$$X''(x) = 0 \Rightarrow X'(x) = A; \quad X(x) = Ax + B$$

Applying the boundary conditions:

$$\begin{aligned} X(0) &= B = 0 \\ X(2) &= 2A = 0 \Rightarrow A = 0 \end{aligned}$$

$A = B = 0$ gives a trivial solution.

CASE 3: $\lambda < 0$:

Let $\lambda = -\mu^2$, then:

$$X'' + \mu^2 X = 0$$

Since the characteristic equation: $r^2 + \mu^2 = 0 \Rightarrow r = \pm i\mu$, the general solution is:

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

Applying the boundary conditions:

$$\begin{aligned} &X(0) = A = 0 \\ \Rightarrow \quad &X(x) = B\sin(\mu x) \\ &X(2) = B\sin(2\mu) = 0 \end{aligned}$$

For a non-trivial solution ($B \neq 0$), we need:

$$\sin(2\mu) = 0 \quad \Rightarrow \quad 2\mu = n\pi, \quad n = 1, 2, 3, \ldots$$

Thus, nontrivial solution exists for

$$\mu = \frac{n\pi}{2} \text{ and } \lambda_n = -\left(\frac{n\pi}{2}\right)^2$$

The corresponding eigenfunction are:

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{2}\right), \quad n = 1, 2, 3, \ldots$$

Since B_n gets absorbed into the Fourier coefficients later, without the loss of generality, we can choose:

$$X_n(x) = \sin\left(\frac{n\pi x}{2}\right), \quad n = 1, 2, 3, \ldots$$

So the solution will have the form:

$$u_n(x,t) = \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{4}\right)^2 t}, n = 1, 2, 3, \ldots$$

Since the equation is linear, the most general solution is of the form:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{4}\right)^2 t}, \text{ for some coefficients } A_n, n = 1, 2, 3, \ldots$$

Using the initial condition $u(x,0) = 2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)$:

$$2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)$$

Note that $\sin\left(\frac{\pi x}{2}\right)$ corresponds to $n = 1, \sin(\pi x) = \sin\left(\frac{2\pi x}{2}\right)$ corresponds to $n = 2$, and $\sin(2\pi x) = \sin\left(\frac{4\pi x}{2}\right)$ corresponds to $n = 4$.

So the Fourier coefficients are:

$$A_1 = 2, \quad A_2 = -1, A_4 = 4, \text{ and } A_n = 0 \text{ for all the other } n.$$

Thus, the final solution to this equation is:

$$\begin{aligned} u(x,t) &= 2\sin\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2}{16}t} - \sin(\pi x) e^{-\frac{4\pi^2}{16}t} + 4\sin(2\pi x) e^{-\frac{16\pi^2}{16}t} \\ &= 2\sin\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2}{16}t} - \sin(\pi x) e^{-\frac{\pi^2}{4}t} + 4\sin(2\pi x) e^{-\pi^2 t} \end{aligned}$$

Q2. We consider the problem of heat diffusion in a metal rod of length $L = 1$.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \text{ for } 0 < x < 1 \text{ and } t \geq 0$$
$$u(0,t) = 0 \text{ and } \frac{\partial u}{\partial x}(1,t) = 0, \text{ for } t > 0$$
$$u(x,0) = x(1-x) \text{ for } 0 \leq x \leq 1$$

Determine the temperature $u(x,t)$ using the method of separation of variables.

Answer:

Assume a solution of the form: $u(x,t) = X(x)T(t)$

Then $u_t = X(x)\dot{T}(t); \quad u_{xx} = X''(x)T(t)$

Substituting these into the PDE:

$$\begin{aligned} & X(x)\dot{T}(t) - X''(x)T(t) = 0 \\ \Rightarrow \quad & X(x)\dot{T}(t) = X''(x)T(t) \end{aligned}$$

$$\Rightarrow \frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda, \quad \text{for some constant } \lambda$$

So the time-dependent ODE: $\dot{T}(t) = \lambda T(t) \Rightarrow T(t) = e^{\lambda t}$

And the spatial ODE: $X''(x) - \lambda X(x) = 0$

Using the boundary conditions, we have:

$$\left.\begin{array}{l} u(x,t) = X(x)T(t) \Rightarrow u(0,t) = X(0)T(t) = 0 \\ u_x(x,t) = X'(x)T(t) \Rightarrow u_x(1,t) = X'(1)T(t) = 0 \end{array}\right\} \Rightarrow X(0) = 0 = X'(1)$$

So the eigenvalue problem is:

$$\begin{aligned} & X'(x) - \lambda X(x) = 0, \quad \text{with} \\ & X(0) = 0, \quad X'(1) = 0 \end{aligned}$$

CASE 1: $\lambda > 0$:

Let $\lambda = \mu^2$, then: $X'(x) - \mu^2 X(x) = 0$, so the general solution is

$$X(x) = Ae^{\mu x} + Be^{-\mu x}$$

Applying the boundary conditions:

$$\begin{aligned} X(0) &= A + B = 0 \Rightarrow B = -A \\ \Rightarrow X(x) &= A\left(e^{\mu x} - e^{-\mu x}\right) = 2A\sinh(\mu x) \Rightarrow X'(x) = 2\mu A\cosh(\mu x) \\ X'(1) &= 2\mu A\cosh(\mu) \end{aligned}$$

Since $\cosh(\mu) > 0$ for $\mu > 0$, it must be true that $A = 0 = B$, leading to a trivial solution.

CASE 2: $\lambda = 0$:

The equation becomes:

$$X''(x) = 0 \Rightarrow X'(x) = A; X(x) = Ax + B$$

Applying the boundary conditions:

$$\begin{aligned} X(0) &= B = 0 \\ X'(1) &= A = 0 \end{aligned}$$

$A = B = 0$ gives a trivial solution.

CASE 3: $\lambda < 0$:

Let $\lambda = -\mu^2$, then:

$$X'' + \mu^2 X = 0$$

Since the characteristic equation: $r^2 + \mu^2 = 0 \Rightarrow r = \pm i\mu$, the general solution is:

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

Applying the boundary conditions:

$$\begin{aligned} X(0) &= A = 0 \\ \Rightarrow X(x) &= B\sin(\mu x) \Rightarrow X'(x) = \mu B\cos(\mu x) \\ X'(1) &= \mu B\cos(\mu) = 0 \end{aligned}$$

For a nontrivial solution ($B \neq 0$), we require:

$$\cos(\mu) = 0 \Rightarrow \mu = \frac{\pi}{2} + n\pi, \quad n = 0, 1, 2, 3, \ldots$$

Thus, a nontrivial solution exists for

$$\mu = \frac{(2n+1)\pi}{2}, \quad \lambda = -\left[\frac{(2n+1)\pi}{2}\right]^2$$

The corresponding eigenfunctions are:

$$X_n(x) = B_n \sin\left(\frac{(2n+1)\pi}{2}x\right), \quad n = 0, 1, 2, 3, \dots$$

So the solution will have the form:

$$u_n(x,t) = \sin\left(\frac{(2n+1)\pi}{2}x\right) e^{-\frac{(2n+1)^2\pi^2}{4}t}, \quad n = 0, 1, 2, 3, \dots$$

Since the equation is linear, the most general solution is of the form:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{(2n+1)\pi}{2}x\right) e^{-\frac{(22+1)^2\pi^2}{4}t}, \qquad \text{for some constant } A_n, n = 0, 1, 2, 3, \dots$$

Applying the initial condition $u(x,0) = x(1-x)$:

$$\sum_{n=0}^{\infty} A_n \sin\left(\frac{(2n+1)\pi}{2}x\right) = x(1-x) \tag{1}$$

Because $\sin\left(\frac{(2n+1)\pi}{2}x\right)$ is orthogonal over the interval $[0,1]$, we multiply both sides of (1) by $\sin\left(\frac{(2m+1)\pi}{2}x\right)$ and integrate over $[0,1]$:

$$\sum_{n=0}^{\infty} A_n \int_0^1 \sin\left(\frac{(2n+1)\pi}{2}x\right) \sin\left(\frac{(2m+1)\pi}{2}x\right) dx = \int_0^1 \left(x - x^2\right) \sin\left(\frac{(2m+1)\pi}{2}x\right) dx \tag{2}$$

The orthogonal property tells us that:

$$\int_0^1 \sin\left(\frac{(2n+1)\pi}{2}x\right) \sin\left(\frac{(2m+1)\pi}{2}x\right) dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{1}{2}, & \text{if } n = m. \end{cases}$$

So (2) becomes:

$$A_m \frac{1}{2} = \int_0^1 \left(x - x^2\right) \sin\left(\frac{(2m+1)\pi}{2}x\right) dx$$

Thus, the coefficient is

$$
\begin{aligned}
A_m &= 2\int_0^1 \left(x - x^2\right)\sin\left(\frac{(2m+1)\pi}{2}x\right)dx \\
&= 2\left[\int_0^1 x\sin\left(\frac{(2m+1)\pi}{2}x\right)dx - \int_0^1 x^2\sin\left(\frac{(2m+1)\pi}{2}x\right)dx\right]
\end{aligned}
$$

Let $\mu = \frac{(2m+1)\pi}{2}$ for simplicity.

Simplifying A_m using integration by parts:

For $\int_0^1 x\sin(\mu x)dx$ term:

Take $= x, \quad dv = \sin(\mu x)dx$

$\Rightarrow du = dx, \quad v = -\frac{1}{\mu}\cos(\mu x)$

$$
\begin{aligned}
\Rightarrow \int_0^1 x\sin(\mu x)dx &= \left[-\tfrac{x}{\mu}\cos(\mu x)\right]_0^1 + \tfrac{1}{\mu}\int_0^1 \cos(\mu x)dx \\
&= -\tfrac{1}{\mu}\cos(\mu) + \tfrac{1}{\mu}\left[\tfrac{1}{\mu}\sin(\mu x)\right]_0^1 \\
&= -\tfrac{\cos(\mu)}{\mu} + \tfrac{\sin(\mu)}{\mu^2}
\end{aligned}
$$

For $\int_0^1 x^2\sin(\mu x)dx$ term:

Take $u = x^2, \quad dv = \sin(\mu x)dx$

$\Rightarrow du = 2xdx, v = -\frac{1}{\mu}\cos(\mu x)$

$$
\begin{aligned}
\Rightarrow \int_0^1 x^2\sin(\mu x)dx &= \left[-\tfrac{x^2}{\mu}\cos(\mu x)\right]_0^1 + \tfrac{2}{\mu}\int_0^1 x\cos(\mu x)dx \\
&= -\tfrac{\cos(\mu)}{\mu} + \tfrac{2}{\mu}\int_0^1 x\cos(\mu x)dx
\end{aligned}
$$

Take $u = x, \quad dv = \cos(\mu x)dx$

$$\Rightarrow du = dx, \quad v = \tfrac{1}{\mu}\sin(\mu x)$$

$$\Rightarrow -\frac{\cos(\mu)}{\mu} + \frac{2}{\mu}\int_0^1 x\cos(\mu x)dx = -\frac{\cos(\mu)}{\mu} + \frac{2}{\mu}\left\{\left[\frac{x}{\mu}\sin(\mu x)\right]_0^1 - \frac{1}{\mu}\int_0^1 \sin(\mu x)dx\right\}$$

$$= -\frac{\cos(\mu)}{\mu} + \frac{2}{\mu}\left[\frac{\sin(\mu)}{\mu} + \frac{1}{\mu}\left[\frac{1}{\mu}\cos(\mu x)\right]_0^1\right]$$

$$= -\frac{\cos(\mu)}{\mu} + \frac{2}{\mu}\left[\frac{\sin(\mu)}{\mu} + \frac{\cos(\mu)-1}{\mu^2}\right]$$

$$= -\frac{\cos(\mu)}{\mu} + \frac{2\sin(\mu)}{\mu^2} + \frac{2(\cos(\mu)-1)}{\mu^3}$$

So the coefficient is:

$$A_m = 2\left[\int_0^1 x\sin\left(\frac{(2m+1)\pi}{2}x\right)dx - \int_0^1 x^2\sin\left(\frac{(2m+1)\pi}{2}x\right)dx\right]$$

$$= 2\left[-\frac{\cos(\mu)}{\mu} + \frac{\sin(\mu)}{\mu^2} + \frac{\cos(\mu)}{\mu} - \frac{2\sin(\mu)}{\mu^2} - \frac{2(\cos\mu - 1)}{\mu^3}\right]$$

$$= 2\left[-\frac{\sin\mu}{\mu^2} - \frac{2(\cos\mu-1)}{\mu^3}\right]$$

$$= 2\left[-\frac{\sin\left(\frac{(2m+1)\pi}{2}\right)}{\left(\frac{(2m+1)\pi}{2}\right)^2} - \frac{2\left[\cos\left(\frac{(2m+1)\pi}{2}\right)-1\right]}{\left(\frac{(2m+1)\pi}{2}\right)^3}\right]$$

Note that $\sin\left(\frac{(2m+1)\pi}{2}\right) = (-1)^m$ and $\cos\left(\frac{(2m+1)\pi}{2}\right) = 0$, for $m = 0, 1, 2, \ldots$ So

$$A_m = 2\left[-\frac{4(-1)^m}{(2m+1)^2\pi^2} - \frac{16(-1)}{(2m+1)^3\pi^3}\right]$$

$$= \frac{32}{(2m+1)^3\pi^3} - \frac{8(-1)^m}{(2m+1)^2\pi^2}$$

$$= \frac{8}{(2m+1)^2\pi^2}\left[\frac{4}{(2m+1)\pi} - (-1)^m\right], \qquad m = 0, 1, 2, \ldots$$

And the solution is:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8}{(2n+1)^2\pi^2}\left[\frac{4}{(2n+1)\pi} - (-1)^n\right]\sin\left(\frac{(2n+1)\pi}{2}x\right)e^{-\frac{(2n+1)^2\pi^2}{4}t}$$