Formalisation of CW complexes

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January 4, 2026

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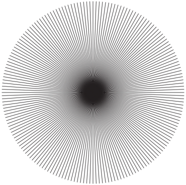
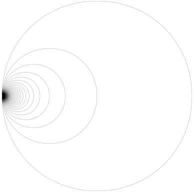
Why CW complexes?

- Very general class of spaces
  - Examples of CW complexes: $\mathbb{R}^n$, S^n, $\mathbb{CP}^n$, $\mathbb{RP}^\infty$
  - Homotopy type of CW complexes: differentiable manifolds
  - Not a CW complex: hedgehog space
  - Not homotopy equivalent to a CW complex: Hawaiian earring



hedgehog space



Hawaiian earring

Why CW complexes?

- A lot of strong results about CW complexes

Theorem (Whitehead theorem, 1949)

A continuous map between two CW complexes that induces isomorphisms on all homotopy groups is a homotopy equivalence.

Theorem (Cellular homology)

Let X be a CW complex. Then the cellular and singular homology of X agree.

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- Glue n-cells (i.e. continuous images of n-discs) together along their boundaries

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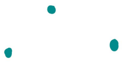

0-cells

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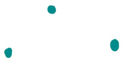

0-cells

1-cells

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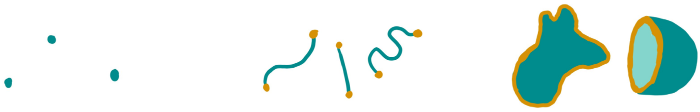

0-cells 1-cells 2-cells

Examples: What is a CW complex?

Interval

Examples: What is a CW complex?

Interval

Real line

Examples: What is a CW complex?

Interval

Real line

2-sphere

Definition: What is a CW complex?

Let X be a Hausdorff space. A *CW complex* on X consists of a family of indexing sets $(I_n)_{n\in\mathbb{N}}$ and a family of continuous maps $(Q_i^n\colon D^n \to X)_{n\in\mathbb{N}, i\in I_n}$ called *characteristic maps* with the following properties:

(i) $Q_i^n|_{\mathrm{int}(D^n)} : \mathrm{int}(D^n) \to Q_i^n(\mathrm{int}(D^n))$ is a homeomorphism for every $n \in \mathbb{N}$ and $i \in I_n$. We call $e_i^n := Q_i^n(\mathrm{int}(D^n))$ an *(open) n-cell* and $\overline{e}_i^n := Q_i^n(D^n)$ a *closed n-cell*.

(ii) Two different open cells are disjoint.

(iii) For each $n \in \mathbb{N}$ and $i \in I_n$ the *cell frontier* $\partial e_i^n := Q_i^n(\partial D^n)$ is contained in the union of a finite number of closed cells of a lower dimension.

(iv) A set $A \subseteq X$ is closed if the intersections $A \cap \overline{e}_i^n$ are closed for all $n \in \mathbb{N}$ and $i \in I_n$.

(v) The union of all closed cells is X.

Lean: What is a CW complex?

```
class CWComplex.{u} {X : Type u} [TopologicalSpace X] (C : Set X) where
  cell (n : ℕ) : Type u
  map (n : ℕ) (i : cell n) : PartialEquiv (Fin n → ℝ) X
  source_eq (n : ℕ) (i : cell n) : (map n i).source = ball 0 1
  continuousOn (n : ℕ) (i : cell n) : ContinuousOn (map n i) (closedBall 0 1)
  continuousOn_symm (n : ℕ) (i : cell n) : ContinuousOn (map n i).symm (map n i).target
  pairwiseDisjoint' :
    (univ : Set (Σ n, cell n)).PairwiseDisjoint (fun ni ↦ map ni.1 ni.2 '' ball 0 1)
  mapsTo' (n : ℕ) (i : cell n) : ∃ I : Π m, Finset (cell m),
    MapsTo (map n i) (sphere 0 1) (⋃ (m < n) (j ∈ I m), map m j '' closedBall 0 1)
  closed' (A : Set X) (hAC : A ⊆ C) :
    (∀ n j, IsClosed (A ∩ map n j '' closedBall 0 1)) → IsClosed A
  union' : ⋃ (n : ℕ) (j : cell n), map n j '' closedBall 0 1 = C
```

What has been done in Lean?

By other people (that I am aware of):

- Categorical definition `TopCat.RelativeCWComplex` by Jiazhen Xia and Elliot Dean Young and refactored by Joël Riou: in Mathlib
- Equivalence of the definitions by Robert Maxton: PRs
- Definition and basic properties: In mathlib
- Finiteness notions on CW Complexes: PR
- Subcomplexes: Done
- Basic constructions: Done
- KSpaces: PR
- Additional Lemmas: Done
- Product: Done
- Examples: Needs clean-up
- Maps: Work in progress