Examples of CW-complexes

Hannah Scholz

Mathematical Institute of the University of Bonn

28.01.2024

Definition of CW-complexes

Let X be a Hausdorff space. A *CW-complex* on X consists of a family of indexing sets $(I_n)_{n\in\mathbb{N}}$ and a family of maps $(Q_i^n\colon D_i^n \to X)_{n\geq 0, i\in I_n}$ s.t.

(i) $Q_i^n|_{\mathrm{int}(D_i^n)} : \mathrm{int}(D_i^n) \to Q_i^n(\mathrm{int}(D_i^n))$ is a homeomorphism. We call $e_i^n := Q_i^n(\mathrm{int}(D_i^n))$ an *(open) n-cell* (or a cell of dimension n) and $\overline{e}_i^n := Q_i^n(D_i^n)$ a *closed n-cell*.

(ii) For all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in I_m$ where $(n, i) \neq (m, j)$ the cells e_i^n and e_j^m are disjoint.

(iii) For each $n \in \mathbb{N}$, $i \in I_n$, $Q_i^n(\partial D_i^n)$ is contained in the union of a finite number of closed cells of dimension less than n.

(iv) $A \subseteq X$ is closed iff $Q_i^n(D_i^n) \cap A$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$.

(v) $\bigcup_{n\geq 0} \bigcup_{i\in I_n} Q_i^n(D_i^n) = X$.

We call Q_i^n a *characteristic map* and $\partial e_i^n := Q_i^n(\partial D_i^n)$ the *frontier of the n-cell* for any i and n.

Examples of CW-complexes

We will look at the CW-complex structures on the following spaces:

- $\varnothing$: The empty set.
- Any finite set.
- $[a, b]$: Any closed interval.
- $\mathbb{R}$: The real line.
- S^n: The n-dimensional sphere.

CW-complexes in Lean

- I have defined and proven statements about CW-complexes as my bachelors thesis and as my work as a student research assistant.
- This version of CW-complexes is not (yet) in mathlib.
- Proven statements include constructions like subcomplexes and products.

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

- Use spherical coordinates

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

- Use spherical coordinates
  - + Easiest map

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

- Use spherical coordinates
  - + Easiest map
  - - Spherical coordinates are not in mathlib

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

- Use spherical coordinates
  - + Easiest map
  - - Spherical coordinates are not in mathlib
- Define the map explicitly

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

- Use spherical coordinates
  - \+ Easiest map
  - \- Spherical coordinates are not in mathlib
- Define the map explicitly
  - \+ Only relies on basic calculation

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

- Use spherical coordinates
  - + Easiest map
  - - Spherical coordinates are not in mathlib
- Define the map explicitly
  - + Only relies on basic calculation
  - - A lot of ugly calculations

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

- Use spherical coordinates
  - + Easiest map
  - - Spherical coordinates are not in mathlib
- Define the map explicitly
  - + Only relies on basic calculation
  - - A lot of ugly calculations
- Compose the stereographic projection with a homeomorphism from $\mathbb{R}^n$ to $\text{int}(D^n)$

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

- Use spherical coordinates
  - + Easiest map
  - - Spherical coordinates are not in mathlib
- Define the map explicitly
  - + Only relies on basic calculation
  - - A lot of ugly calculations
- Compose the stereographic projection with a homeomorphism from $\mathbb{R}^n$ to $\text{int}(D^n)$
  - + All necessary properties on $\text{int}(D^n)$ are already in mathlib

Sphere (Direct version): Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

- Use spherical coordinates
  - + Easiest map
  - - Spherical coordinates are not in mathlib
- Define the map explicitly
  - + Only relies on basic calculation
  - - A lot of ugly calculations
- Compose the stereographic projection with a homeomorphism from $\mathbb{R}^n$ to $\text{int}(D^n)$
  - + All necessary properties on $\text{int}(D^n)$ are already in mathlib
  - - Showing continuity on D^n is hard

Sphere (Direct version): Defining the characteristic map

Let $n \geq 2$,
stereographic' : $S^n \setminus \{p\} \to \mathbb{R}^n$ be the stereographic projection where p is the north pole of the sphere and
unitBall : $\mathbb{R}^n \to \text{int}(D^n)$ be the obvious map.
Then we define:

$$D^n \to S^n, x \mapsto \begin{cases} (\text{unitBall} \circ \text{stereographic'})^{-1} x & x \in \text{int}(D^n) \\ p & x \in S^{n-1} \end{cases}$$

Sphere (Direct version): Difficulties

- I lost a lot of time trying the explicit version of the characteristic map.

Sphere (Direct version): Difficulties

- I lost a lot of time trying the explicit version of the characteristic map.
- The maps stereographic' and unitBall are wrappers for a compositions of a bunch of different functions and are designed for a specific purpose.

Sphere (Direct version): Difficulties

- I lost a lot of time trying the explicit version of the characteristic map.
- The maps stereographic' and unitBall are wrappers for a compositions of a bunch of different functions and are designed for a specific purpose.
- I don't really understand filters.

Sphere (Inductive version): Choosing a characteristic map

I came up with the following two options:

Sphere (Inductive version): Choosing a characteristic map

I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib

Sphere (Inductive version): Choosing a characteristic map

I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib
  - + Possibly existing useful statements about the map

Sphere (Inductive version): Choosing a characteristic map

I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib
  - + Possibly existing useful statements about the map
  - - Very general map, therefore hard to work with explicitly

Sphere (Inductive version): Choosing a characteristic map

I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib
  - + Possibly existing useful statements about the map
  - - Very general map, therefore hard to work with explicitly
- Define the function explicitly

Sphere (Inductive version): Choosing a characteristic map

I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib
  + Possibly existing useful statements about the map
  - Very general map, therefore hard to work with explicitly
- Define the function explicitly
  + Only relies on basic calculations

Sphere (Inductive version): Choosing a characteristic map

I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib
  - \+ Possibly existing useful statements about the map
  - \- Very general map, therefore hard to work with explicitly
- Define the function explicitly
  - \+ Only relies on basic calculations
  - \- Need to describe properties of the map myself

Sphere (Inductive Version): Difficulties

- Construction is relatively technical, it was hard to write code that looked okay and wasn't too slow.

Sphere (Inductive Version): Difficulties

- Construction is relatively technical, it was hard to write code that looked okay and wasn't too slow.
- I didn't expect to have to do actual work for the inclusion.

Sphere (Inductive Version): Difficulties

- Construction is relatively technical, it was hard to write code that looked okay and wasn't too slow.
- I didn't expect to have to do actual work for the inclusion.
- Finding partial maps is hard.

Comparison the two definitions

- Direct version

- Inductive version

Comparison the two definitions

- Direct version
  + Simpler construction

- Inductive version

Comparison the two definitions

- Direct version
  - + Simpler construction
  - - The characteristic map is considerably harder.

- Inductive version

Comparison the two definitions

- Direct version
  - + Simpler construction
  - - The characteristic map is considerably harder.
  - Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Inductive version

Comparison the two definitions

- Direct version
  - + Simpler construction
  - - The characteristic map is considerably harder.
  - Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Inductive version
  - + Significantly easier characteristic maps

Comparison the two definitions

- Direct version
  - + Simpler construction
  - - The characteristic map is considerably harder.
  - Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Inductive version
  - + Significantly easier characteristic maps
  - - More technical construction

Comparison the two definitions

- Direct version
  - + Simpler construction
  - - The characteristic map is considerably harder.
  - Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Inductive version
  - + Significantly easier characteristic maps
  - - More technical construction
  - - Would be a nightmare to unfold.

Comparison the two definitions

- Direct version
  - \+ Simpler construction
  - \- The characteristic map is considerably harder.
  - Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Inductive version
  - \+ Significantly easier characteristic maps
  - \- More technical construction
  - \- Would be a nightmare to unfold.
- I decided to set the direct version as the default

Future work

- Generalize from unit spheres in euclidean space to all spheres under more metrics.
- Do more examples.