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Contents

1 Introduction

This is the introduction. It could explain the following:

- Mathematical relevance of CW complexes
- What is Lean in mathlib and why is it relevant
- Any related work (do CW complexes exist in other proof assistants?)

I think Floris was maybe going to write this?

2 Preliminaries

I could also do a subsection here explaining what Lean and mathlib is.

2.1 Basic Topology in Lean

There aren't really a lot of basics that I need. I could write how to get a topological space, continuous maps, open sets... Not sure if I need anything else.

In `Mathlib`, a topological space is a type `X` together with a topology `TopologicalSpace X` on it. This then allows you to describe whether a set `A : Set X` is open or closed by writing `IsOpen A` and `IsClosed A`. A function `f : X → Y` between two topological spaces `X` and `Y` can be described as being continuous and as being continuous on a set `A : Set X` which is expressed by writing `Continuous f` and `ContinuousOn f A`. `Mathlib` also implements various separation axioms: to specify that a topological space `X` is Hausdorff one can write `T2Space X`.

I am not sure if this all isn't a bit too basic and whether I should even include how to write things in Lean

3 Definition of CW complexes

In this section I should first describe mathematically what a CW complex is. Then I show the Lean definition and explain why/how Maybe this can also have proofs of basic lemmas (e.g. a CW complex is closed in its ambient space)

A *CW complex* is a topological space that can be constructed by glueing images of closed disc of different dimensions together along the images of their boundaries. These images of closed discs in the CW complex are called *cells*. To specify that a cell is the image of an n-dimensional disc, one can call it an n-cell. The cells up to dimension n make up what is called the *n-skeleton*. In a relative CW complex these discs can additionally be attached to a specified base set.

The different definitions of CW complexes present in the literature can be broadly categorized into to approaches: firstly there is the "classical" approach that sticks closely in style to Whitehead's original definition in [Whi18]. This definition assumes the cells to all lie in one topological space and then describes how the cells interact with each other and the space. Secondly, there is a popular approach that is more categorical in nature. In this approach the skeletons are regarded as different spaces and the definition describes how to construct the $n+1$-skeleton from the n-skeleton. The CW complex is then defined as the colimit of the skeletons.

At the start of this project neither of the approaches had been formalized in Lean. The authors chose to proceed with the former approach for the following reasons:

References

[Whi18] J. H. C. Whitehead. “Combinatorial homotopy. I”. In: *Bulletin (new series) of the American Mathematical Society* 55.3 (2018), pp. 213–245. ISSN: 0273-0979.