

Formalisation of CW complexes

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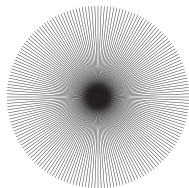
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22.01.2026

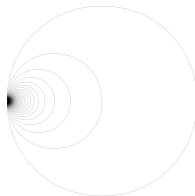
Joint work with and supervised by Prof. Floris van Doorn

Why CW complexes?

- Very general class of spaces
 - Examples of CW complexes: \mathbb{R}^n , S^n , $\mathbb{C}P^n$, $\mathbb{R}P^\infty$
 - Homotopy type of CW complexes: differentiable manifolds
 - Not a CW complex: hedgehog space
 - Not homotopy equivalent to a CW complex: Hawaiian earring



hedgehog space



Hawaiian earring

Why CW complexes?

- A lot of strong results about CW complexes

Theorem (Whitehead theorem, 1949)

A continuous map between two CW complexes that induces isomorphisms on all homotopy groups is a homotopy equivalence.

Theorem (Cellular homology)

Let X be a CW complex. Then the cellular and singular homology of X agree.

Intuition: What is a CW complex?

- Glue n -cells (i.e. continuous images of n -discs) together along their boundaries

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0-cells

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1-cells

Intuition: What is a CW complex?

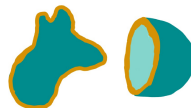
- Glue n -cells (i.e. continuous images of n -discs) together along their boundaries



0-cells



1-cells



2-cells

Examples: What is a CW complex?



Interval

Examples: What is a CW complex?



Interval



Real line

Examples: What is a CW complex?



Interval



Real line



2-sphere

Definition: What is a CW complex?

Let X be a Hausdorff space. An (*absolute*) *CW complex* on X consists of a family of indexing sets $(I_n)_{n \in \mathbb{N}}$ and a family of continuous maps $(Q_i^n : D^n \rightarrow X)_{n \in \mathbb{N}, i \in I_n}$ called *characteristic maps* with the following properties:

- (i) $Q_i^n|_{\text{int}(D^n)} : \text{int}(D^n) \rightarrow Q_i^n(\text{int}(D^n))$ is a homeomorphism for every $n \in \mathbb{N}$ and $i \in I_n$. We call $e_i^n := Q_i^n(\text{int}(D^n))$ an (*open*) n -cell and $\bar{e}_i^n := Q_i^n(D^n)$ a *closed* n -cell.
- (ii) Two different open cells are disjoint.
- (iii) For each $n \in \mathbb{N}$ and $i \in I_n$ the *cell frontier* $\partial e_i^n := Q_i^n(\partial D^n)$ is contained in the union of a finite number of closed cells of a lower dimension.
- (iv) A set $A \subseteq X$ is closed if the intersections $A \cap \bar{e}_i^n$ are closed for all $n \in \mathbb{N}$ and $i \in I_n$.
- (v) The union of all closed cells is X .

Lean: What is a CW complex?

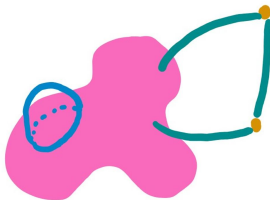
```

class CWComplex.{u} {X : Type u} [TopologicalSpace X] (C : Set X) where
  cell (n : ℕ) : Type u
  map (n : ℕ) (i : cell n) : PartialEquiv (Fin n → ℝ) X
  source_eq (n : ℕ) (i : cell n) : (map n i).source = ball 0 1
  continuousOn (n : ℕ) (i : cell n) : ContinuousOn (map n i) (closedBall 0 1)
  continuousOn_symm (n : ℕ) (i : cell n) : ContinuousOn (map n i).symm (map n i).target
  pairwiseDisjoint' :
    (univ : Set (Σ n, cell n)).PairwiseDisjoint (fun ni ↦ map ni.1 ni.2 '' ball 0 1)
  mapsTo' (n : ℕ) (i : cell n) : ∃ I : Π m, Finset (cell m),
    MapsTo (map n i) (sphere 0 1) (U (m < n) (j ∈ I m), map m j '' closedBall 0 1)
  closed' (A : Set X) (hAC : A ⊆ C) :
    (∀ n j, IsClosed (A ∩ map n j '' closedBall 0 1)) → IsClosed A
  union' : U (n : ℕ) (j : cell n), map n j '' closedBall 0 1 = C

```

Intuition: What is a relative CW complex?

A relative CW complex additionally has a base set that the boundaries can attach to.



An example of a relative CW complex

Lean: What is a relative CW complex?

```

class RelCWComplex.{u} {X : Type u} [TopologicalSpace X] (C : Set X) (D : outParam (Set X)) where
  cell (n : ℕ) : Type u
  map (n : ℕ) (i : cell n) : PartialEquiv (Fin n → ℝ) X
  source_eq (n : ℕ) (i : cell n) : (map n i).source = ball 0 1
  continuousOn (n : ℕ) (i : cell n) : ContinuousOn (map n i) (closedBall 0 1)
  continuousOn_symm (n : ℕ) (i : cell n) : ContinuousOn (map n i).symm (map n i).target
  pairwiseDisjoint' :
  | (univ : Set (Σ n, cell n)).PairwiseDisjoint (fun ni ↦ map ni.1 ni.2 '' ball 0 1)
  disjointBase' (n : ℕ) (i : cell n) : Disjoint (map n i '' ball 0 1) D
  mapsTo (n : ℕ) (i : cell n) : ∃ I : Π m, Finset (cell m),
  | MapsTo (map n i) (sphere 0 1) (D ∪ ∪ (m < n) (j ∈ I m), map m j '' closedBall 0 1)
  closed' (A : Set X) (hAC : A ⊆ C) :
  | ((∀ n j, IsClosed (A ∩ map n j '' closedBall 0 1)) ∧ IsClosed (A ∩ D)) → IsClosed A
  isClosedBase : IsClosed D
  union' : D ∪ ∪ (n : ℕ) (j : cell n), map n j '' closedBall 0 1 = C

```

Implementation: general situation

Situation: We have a general and a specific definition where

- the specific definition is a lot more commonly used
- the specific case provides significant simplifications
- the differentiating parameter is an `outParam`

Implementation: Issues with naive definition

- Naive approach: define an absolute CW complex as a relative one with empty base
- Issues with naive approach:
 - repeated simplifications
 - instances where the base is provably but not definitionally equal to empty set

Implementation: Issues with naive definition

- Product of two relative CW complexes (C, \emptyset) and (E, \emptyset) has type:

$$\text{RelCWComplex } (C \times^s E) (\emptyset \times^s E \cup C \times^s \emptyset)$$

- Product of two absolute CW complexes C and E has type:

$$\text{CWComplex } (C \times^s E)$$

- With the naive approach this would be definitionally the same as:

$$\text{RelCWComplex } (C \times^s E) \emptyset$$

What has been done in Lean?

By other people (that I am aware of):

- Categorical definition `TopCat.RelativeCWComplex` by Jiazhen Xia and Elliot Dean Young and refactored by Joël Riou: in Mathlib
- Whitehead theorem in model categories by Joël Riou: in Mathlib
- Equivalence of the definitions by Robert Maxton: PRs

What has been done in Lean?

By us:

- Definition and basic properties (~ 600 LOC): in Mathlib
- Finiteness notions (~ 300 LOC): in Mathlib
- Subcomplexes (~ 800 LOC): in Mathlib/PRs
- Compactly coherent spaces (~ 200 LOC): in Mathlib/PRs
- Product (~ 600 LOC): done
- Examples (~ 1000 LOC): needs refactor
- Rest of the Project (~ 3000 LOC)

Products of CW complexes

Let X and Y be CW complexes. The respective families of characteristic maps are $(Q_i^n: D^n \rightarrow X)_{n \in \mathbb{N}, i \in I_n}$ and $(P_j^m: D^m \rightarrow Y)_{m \in \mathbb{N}, j \in J_m}$.

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Theorem

Assume that $X \times Y$ is compactly coherent. Then $X \times Y$ is a CW complex with characteristic maps

$(Q_i^n \times P_j^m: D^n \times D^m \rightarrow C \times E)_{n,m \in \mathbb{N}, i \in I_n, j \in J_m}$ and indexing sets $K_l = \bigcup_{n+m=l} I_n \times J_m$.

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Theorem

In general, the compact coherentification of $X \times Y$ is a CW complex.

Compactly coherent spaces

Compactly coherent spaces

Abbreviation	Meaning summary
CG-1	Topology coherent with family of its compact subspaces
CG-2	Topology same as final topology with respect to continuous maps from arbitrary compact Hausdorff spaces
CG-3	Topology coherent with family of its compact Hausdorff subspaces

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Definition

Let X be a topological space. We call X *compactly coherent* if a set $A \subseteq X$ is open iff for all compact sets $C \subseteq X$, the intersection $A \cap C$ is open in C .

Compactly coherent spaces

```
class CompactlyCoherentSpace (X : Type*) [TopologicalSpace X] : Prop where
| isCoherentWith : IsCoherentWith (X := X) {K | IsCompact K}
```

```
structure IsCoherentWith (S : Set (Set X)) : Prop where
| isOpen_of_forall_induced (u : Set X) :
| (∀ s ∈ S, IsOpen ((↑)⁻¹' u : Set s)) → IsOpen u
```

Summary

- CW complexes are an important class of topological spaces
- a CW complex is made up of a lot of discs glued together
- the product of two CW complexes is in general **not** a CW complex