

Examples of CW-complexes

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Definition of CW-complexes

Let X be a Hausdorff space. A *CW-complex* on X consists of a family of indexing sets $(I_n)_{n \in \mathbb{N}}$ and a family of maps $(Q_i^n: D_i^n \rightarrow X)_{n \geq 0, i \in I_n}$ s.t.

- (i) $Q_i^n|_{\text{int}(D_i^n)}: \text{int}(D_i^n) \rightarrow Q_i^n(\text{int}(D_i^n))$ is a homeomorphism. We call $e_i^n := Q_i^n(\text{int}(D_i^n))$ an *(open) n -cell* (or a cell of dimension n) and $\bar{e}_i^n := Q_i^n(D_i^n)$ a *closed n -cell*.
- (ii) For all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in I_m$ where $(n, i) \neq (m, j)$ the cells e_i^n and e_j^m are disjoint.
- (iii) For each $n \in \mathbb{N}$, $i \in I_n$, $Q_i^n(\partial D_i^n)$ is contained in the union of a finite number of closed cells of dimension less than n .
- (iv) $A \subseteq X$ is closed iff $Q_i^n(D_i^n) \cap A$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$.
- (v) $\bigcup_{n \geq 0} \bigcup_{i \in I_n} Q_i^n(D_i^n) = X$.

We call Q_i^n a *characteristic map* and $\partial e_i^n := Q_i^n(\partial D_i^n)$ the *frontier of the n -cell* for any i and n . Additionally we define $X_n := \bigcup_{m < n+1} \bigcup_{i \in I_m} \bar{e}_i^m$ and call it the *n -skeleton* of X for $-1 \leq n \leq \infty$.

Examples of CW-complexes

We will look at the CW-complex structures on the following spaces:

- \emptyset : The empty set.
- Any finite set.
- $[a, b]$: Any closed interval.
- \mathbb{R} : The real line.
- S^n : The n -dimensional sphere.

- I have defined and proven statements about CW-complexes as my bachelors thesis and as my work as a student research assistant.
- CW-complexes are not (yet) in mathlib.
- Proven statements include constructions like subcomplexes and products.

Equivalence of the definitions

Lemma

Let C be a CW-complex in a Hausdorff space X as in the definition in the formalisation. Then C is a CW-complex as in the paper definition.

Lemma

Let X be a Hausdorff space and C a CW-complex in X as in the formalised definition. Then C is closed.

Closedness of CW-complexes

Proof

- 1 By the weak topology it is enough to show that the intersection with every closed cell is closed.
- 2 Take any closed cell of C .
- 3 Since the closed cell is a subset of C , the intersection is just the closed cell.
- 4 Every closed cell is closed.

Proof (Lean)

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1 rw [closed _ (by rfl)]
2 intros
3 rw [inter_eq_right.2 (closedCell_subset_complex _ _)]
4 exact isClosed_closedCell
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The topology of the product

Definition k -space

Let X be a topological space. We call X a k -space if

$A \subseteq X$ is closed \iff for all compact sets $C \subseteq X$ the intersection $A \cap C$ is closed in C .

Lemma

Let X be a CW-complex and $C \subseteq X$ a compact set. Then C is disjoint with all but finitely many cells of X .

Lemma

If $X \times Y$ is a k -space then it has weak topology with respect to the characteristic maps $(Q_i^n \times P_j^m: D_i^n \times D_j^m \rightarrow (X \times Y)_c)_{n,m \in \mathbb{N}, i \in I_n, j \in J_m}$, i.e. $A \subseteq X \times Y$ is closed iff $A \cap (\bar{e}_i^n \times \bar{f}_j^m)$ is closed for every $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in J_m$.