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Examples of CW-complexes

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Definition of CW-complexes

Let X be a Hausdorff space. A *CW-complex* on X consists of a family of indexing sets $(I_n)_{n \in \mathbb{N}}$ and a family of maps $(Q_i^n \colon D_i^n \to X)_{n \geq 0, i \in I_n}$ s.t.

(i) $Q_i^n|_{\text{int}(D_i^n)} \colon \text{int}(D_i^n) \to Q_i^n(\text{int}(D_i^n))$ is a homeomorphism. We call $e_i^n \coloneqq Q_i^n(\text{int}(D_i^n))$ an *(open) n-cell* (or a cell of dimension n) and $\overline{e}_i^n \coloneqq Q_i^n(D_i^n)$ a *closed n-cell*.

(ii) For all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in I_m$ where $(n, i) \neq (m, j)$ the cells e_i^n and e_j^m are disjoint.

(iii) For each $n \in \mathbb{N}$, $i \in I_n$, $Q_i^n(\partial D_i^n)$ is contained in the union of a finite number of closed cells of dimension less than n.

(iv) $A \subseteq X$ is closed iff $Q_i^n(D_i^n) \cap A$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$.

(v) $\bigcup_{n \geq 0} \bigcup_{i \in I_n} Q_i^n(D_i^n) = X$.

We call Q_i^n a *characteristic map* and $\partial e_i^n \coloneqq Q_i^n(\partial D_i^n)$ the *frontier of the n-cell* for any i and n.

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Examples of CW-complexes

We will look at the CW-complex structures on the following spaces:

- $\varnothing$: The empty set.
- Any finite set.
- $[a, b]$: Any closed interval.
- $\mathbb{R}$: The real line.
- S^n: The n-dimensional sphere.

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CW-complexes in Lean

- I have defined and proven statements about CW-complexes as my bachelors thesis and as my work as a student research assistant.
- This version of CW-complexes is not (yet) in mathlib.
- Proven statements include constructions like subcomplexes and products.

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Sphere Version 1: Choosing a characteristic map

The main issue is how to define the characteristic map. The three options I came up with are:

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Sphere Version 1: Choosing a characteristic map

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- Use spherical coordinates

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Sphere Version 1: Choosing a characteristic map

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- Use spherical coordinates
  - $+$ Easiest map

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  - \- Spherical coordinates are not in mathlib
- Define the map explicitly

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- Use spherical coordinates
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    - $+$ Only relies on basic calculation

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  - $-$ A lot of ugly calculations

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- Compose the stereographic projection with a homeomorphism from $\mathbb{R}^n$ to $\text{int}(D^n)$

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  from $\mathbb{R}^n$ to $\text{int}(D^n)$
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- Use spherical coordinates
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- Define the map explicitly
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    - $-$ A lot of ugly calculations
- Compose the stereographic projection with a homeomorphism from $\mathbb{R}^n$ to $\text{int}(D^n)$
    - $+$ All necessary properties on $\text{int}(D^n)$ are already in mathlib
    - $-$ Showing continuity on D^n is hard

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Sphere Version 1: Defining the characteristic map

Let $n \geq 2$,
stereographic' $: S^n \setminus \{p\} \to \mathbb{R}^n$ be the stereographic projection
where p is the north pole of the sphere and
unitBall $: \mathbb{R}^n \to \text{int}(D^n)$ be the obvious map.
Then we define:

$$D^n \to S^n, x \mapsto \begin{cases} (\text{stereographic'} \circ \text{unitBall})^{-1} x & x \in \text{int}(D^n) \\ p & x \in S^{n-1} \end{cases}$$

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Sphere Version 1: Difficulties

- I lost a lot of time trying the explicit version of the characteristic map.

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Sphere Version 1: Difficulties

- I lost a lot of time trying the explicit version of the characteristic map.
- The maps stereographic' and unitBall are wrappers for a compositions of a bunch of different functions and are designed for a specific purpose.

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Sphere Version 1: Difficulties

- I lost a lot of time trying the explicit version of the characteristic map.
- The maps stereographic' and unitBall are wrappers for a compositions of a bunch of different functions and are designed for a specific purpose.
- I don't really understand filters.

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Sphere Version 2: Choosing a characteristic map

I came up with the following two options:

Sphere Version 2: Choosing a characteristic map

I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib

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Sphere Version 2: Choosing a characteristic map

I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib
    - + Possibly existing useful statements about the map

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Sphere Version 2: Choosing a characteristic map

I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib
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    - $-$ Very general map, therefore hard to work with explicitly

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I came up with the following two options:

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- Define the function explicitly
    - $+$ Only relies on basic calculations

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I came up with the following two options:

- Use the function **orthogonalProjection** from mathlib
    - $+$ Possibly existing useful statements about the map
    - $-$ Very general map, therefore hard to work with explicitly
- Define the function explicitly
    - $+$ Only relies on basic calculations
    - $-$ Need to describe properties of the map myself

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Sphere Version 2: Difficulties

- Construction is relatively technical, it was hard to write code that looked okay and wasn't too slow.

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Sphere Version 2: Difficulties

- Construction is relatively technical, it was hard to write code that looked okay and wasn't too slow.
- I didn't expect to have to do actual work for the inclusion.

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Sphere Version 2: Difficulties

- Construction is relatively technical, it was hard to write code that looked okay and wasn't too slow.
- I didn't expect to have to do actual work for the inclusion.
- Finding partial maps is hard.

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Comparison the two definitions

- Version 1 (Two cells overall)

- Version 2 (Two cells in each dimension)

Comparison the two definitions

- Version 1 (Two cells overall)
  - $+$ Simpler construction

- Version 2 (Two cells in each dimension)

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Comparison the two definitions

- Version 1 (Two cells overall)
    - \+ Simpler construction
    - \- The characteristic map is considerably harder.

- Version 2 (Two cells in each dimension)

Comparison the two definitions

- Version 1 (Two cells overall)
    - + Simpler construction
    - - The characteristic map is considerably harder.
    - Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Version 2 (Two cells in each dimension)

Comparison the two definitions

- Version 1 (Two cells overall)
    - \+ Simpler construction
    - \- The characteristic map is considerably harder.
    - Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Version 2 (Two cells in each dimension)
    - \+ Significantly easier characteristic maps

Comparison the two definitions

- Version 1 (Two cells overall)
    - \+ Simpler construction
    - \- The characteristic map is considerably harder.
    - Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Version 2 (Two cells in each dimension)
    - \+ Significantly easier characteristic maps
    - \- More technical construction

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Comparison the two definitions

- Version 1 (Two cells overall)
  - + Simpler construction
  - - The characteristic map is considerably harder.
  - ■ Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Version 2 (Two cells in each dimension)
  - + Significantly easier characteristic maps
  - - More technical construction
  - - Would be a nightmare to unfold.

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Comparison the two definitions

- Version 1 (Two cells overall)
    + Simpler construction
    - The characteristic map is considerably harder.
    - Spherical coordinated would probably eliminate that issue. The construction should be redone once they exist.
- Version 2 (Two cells in each dimension)
    + Significantly easier characteristic maps
    - More technical construction
    - Would be a nightmare to unfold.
- I decided to set version 1 as the default

Future work

- Generalize from unit spheres in euclidean space to all spheres under more metrics.
- Do more examples.