Formalisation of CW-complexes

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Introduction: Lean, mathlib and CW-complexes

- Lean: Functional programming language and popular theorem prover
- Mathlib: Leans mathematical library with over 300 contributors and 28 maintainers
- CW-complexes: important concept in topology invented by Whitehead to state and prove the Whitehead theorem
- CW-complexes are not yet in mathlib
- Aim: Provide a basic theory of CW-complexes and potentially contribute it to mathlib

Definition of CW-complexes

Let X be a Hausdorff space. A *CW-complex* on X consists of a family of indexing sets $(I_n)_{n\in\mathbb{N}}$ and a family of maps $(Q_i^n\colon D_i^n \to X)_{n\geq 0, i\in I_n}$ s.t.

(i) $Q_i^n|_{\text{int}(D_i^n)} : \text{int}(D_i^n) \to Q_i^n(\text{int}(D_i^n))$ is a homeomorphism. We call $e_i^n := Q_i^n(\text{int}(D_i^n))$ an *(open) n-cell* (or a cell of dimension n) and $\overline{e}_i^n := Q_i^n(D_i^n)$ a *closed n-cell*.

(ii) For all $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in I_m$ where $(n,i) \neq (m,j)$ the cells e_i^n and e_j^m are disjoint.

(iii) For each $n \in \mathbb{N}$, $i \in I_n$, $Q_i^n(\partial D_i^n)$ is contained in the union of a finite number of closed cells of dimension less than n.

(iv) $A \subseteq X$ is closed iff $Q_i^n(D_i^n) \cap A$ is closed for all $n \in \mathbb{N}$ and $i \in I_n$.

(v) $\bigcup_{n\geq 0} \bigcup_{i \in I_n} Q_i^n(D_i^n) = X$.

We call Q_i^n a *characteristic map* and $\partial e_i^n := Q_i^n(\partial D_i^n)$ the *frontier of the n-cell* for any i and n. Additionally we define $X_n := \bigcup_{m<n+1} \bigcup_{i\in I_m} \overline{e}_i^m$ and call it the *n-skeleton* of X for $-1 \leq n \leq \infty$.

Lean: Definition of CW-complexes

```
class CWComplex.{u} {X : Type u} [TopologicalSpace X] (C : Set X) where
  cell (n : ℕ) : Type u
  map (n : ℕ) (i : cell n) : PartialEquiv (Fin n → ℝ) X
  source_eq (n : ℕ) (i : cell n) : (map n i).source = closedBall 0 1
  cont (n : ℕ) (i : cell n) : ContinuousOn (map n i) (closedBall 0 1)
  cont_symm (n : ℕ) (i : cell n) : ContinuousOn (map n i).symm (map n
    i).target
  pairwiseDisjoint' :
    (univ : Set (Σ n, cell n)).PairwiseDisjoint (fun ni ↦ map ni.1
    ni.2 '' ball 0 1)
  mapsto (n : ℕ) (i : cell n) : ∃ I : Π m, Finset (cell m),
   MapsTo (map n i) (sphere 0 1) (⋃ (m < n) (j ∈ I m), map m j ''
    closedBall 0 1)
  closed' (A : Set X) (asubc : A ⊆ C) : IsClosed A ↔ ∀ n j, IsClosed
    (A ∩ map n j '' closedBall 0 1)
  union' : ⋃ (n : ℕ) (j : cell n), map n j '' closedBall 0 1 = C
```

Equivalence of the definitions

Lemma

Let C be a CW-complex in a Hausdorff space X as in the definition in the formalisation. Then C is a CW-complex as in the paper definition.

Lemma

Let X be a Hausdorff space and and C a CW-complex in X as in the formalised definition. Then C is closed.

Closedness of CW-complexes

Proof

1. By the weak topology it is enough to show that the intersection with every closed cell is closed.
2. Take any closed cell of C.
3. Since the closed cell is a subset of C, the intersection is just the closed cell.
4. Every closed cell is closed.

Proof (Lean)

```
rw [closed _ (by rfl)]
intros
rw [inter_eq_right.2 (closedCell_subset_complex _ _)]
exact isClosed_closedCell
```

The topology of the product

Definition k-space

Let X be a topological space. We call X a *k-space* if

$$A \subseteq X \text{ is closed } \iff \text{for all compact sets } C \subseteq X \text{ the intersection } A \cap C \text{ is closed in } C.$$

Lemma

Let X be a CW-complex and $C \subseteq X$ a compact set. Then C is disjoint with all but finitely many cells of X.

Lemma

If $X \times Y$ is a k-space then it has weak topology with respect to the characteristic maps $(Q_i^n \times P_j^m \colon D_i^n \times D_j^m \to (X \times Y)_c)_{n,m \in \mathbb{N}, i \in I_n, j \in J_m}$, i.e. $A \subseteq X \times Y$ is closed iff $A \cap (\overline{e}_i^n \times \overline{f}_j^m)$ is closed for every $n, m \in \mathbb{N}$, $i \in I_n$ and $j \in J_m$.