

# Examples of CW-complexes

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# Definition of CW-complexes

Let  $X$  be a Hausdorff space. A *CW-complex* on  $X$  consists of a family of indexing sets  $(I_n)_{n \in \mathbb{N}}$  and a family of maps  $(Q_i^n: D_i^n \rightarrow X)_{n \geq 0, i \in I_n}$  s.t.

- (i)  $Q_i^n|_{\text{int}(D_i^n)}: \text{int}(D_i^n) \rightarrow Q_i^n(\text{int}(D_i^n))$  is a homeomorphism. We call  $e_i^n := Q_i^n(\text{int}(D_i^n))$  an *(open)  $n$ -cell* (or a cell of dimension  $n$ ) and  $\bar{e}_i^n := Q_i^n(D_i^n)$  a *closed  $n$ -cell*.
- (ii) For all  $n, m \in \mathbb{N}$ ,  $i \in I_n$  and  $j \in I_m$  where  $(n, i) \neq (m, j)$  the cells  $e_i^n$  and  $e_j^m$  are disjoint.
- (iii) For each  $n \in \mathbb{N}$ ,  $i \in I_n$ ,  $Q_i^n(\partial D_i^n)$  is contained in the union of a finite number of closed cells of dimension less than  $n$ .
- (iv)  $A \subseteq X$  is closed iff  $Q_i^n(D_i^n) \cap A$  is closed for all  $n \in \mathbb{N}$  and  $i \in I_n$ .
- (v)  $\bigcup_{n \geq 0} \bigcup_{i \in I_n} Q_i^n(D_i^n) = X$ .

We call  $Q_i^n$  a *characteristic map* and  $\partial e_i^n := Q_i^n(\partial D_i^n)$  the *frontier of the  $n$ -cell* for any  $i$  and  $n$ .

# Examples of CW-complexes

We will look at the CW-complex structures on the following spaces:

- $\emptyset$ : The empty set.
- Any finite set.
- $[a, b]$ : Any closed interval.
- $\mathbb{R}$ : The real line.
- $S^n$ : The  $n$ -dimensional sphere.

# CW-complexes in Lean

- I have defined and proven statements about CW-complexes as my bachelors thesis and as my work as a student research assistant.
- This version of CW-complexes is not (yet) in mathlib.
- Proven statements include constructions like subcomplexes and products.

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    - Showing continuity on  $D^n$  is hard

# Sphere Version 1: Defining the characteristic map

Let  $n \geq 2$ ,

**stereographic'** :  $S^n \setminus \{p\} \rightarrow \mathbb{R}^n$  be the stereographic projection where  $p$  is the north pole of the sphere and

**unitBall** :  $\mathbb{R}^n \rightarrow \text{int}(D^n)$  be the obvious map.

Then we define:

$$D^n \rightarrow S^n, x \mapsto \begin{cases} (\text{unitBall} \circ \text{stereographic}')^{-1}x & x \in \text{int}(D^n) \\ p & x \in S^{n-1} \end{cases}$$

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- The maps `stereographic` and `unitBall` are wrappers for a compositions of a bunch of different functions and are designed for a specific purpose.
- I don't really understand filters.

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    - Need to describe properties of the map myself

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- I didn't expect to have to do actual work for the inclusion.
- Finding partial maps is hard.

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    - More technical construction
    - Would be a nightmare to unfold.
- I decided to set version 1 as the default

# Future work

- Generalize from unit spheres in euclidean space to all spheres under more metrics.
- Do more examples.