INTENTIONALLY SCUFFED NOTES

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Contents

REMEMBER THAT AT ANY POINT, THESE EQUATIONS, FORMULAE AND NOTES IN GENERAL MAY CONTAIN INTENTIONAL ERRORS. WE ARE NOT LIABLE FOR ANYTHING

1 Introduction

Do you want to learn mathematics quickly and easily? Do you find textbooks to be too long and uninteresting to read? Are you in need of some quick revision before a test? Well this is the perfect resource for you! This document is contains simplified explanations accompanied with rigorous, well explained proofs to aid your understanding. At the end of each section are challenging practice questions for you to test your skills and abilities.

2 Basics

As with any subject, mathematics should begin with a solid foundation of the most basic concepts, beginning with the basic operations; addition, subtraction, multiplication and division.

2.1 Basic Arithmetic Operations

2.1.1 Addition

$$a + b = b + a \ \forall a, b \in \mathbb{C}$$

If we want to be extra fancy, this is called a commutative property, as it does not matter whether a is written first or if b is written first.

Exercise 2.1. Find a and b that satisfies $a + b = b + a$.

You may have discovered in the previous exercise that for $a \in \mathbb{C}$, $a + 0 = a$. This is the additive identity, whereby any number added by zero is equal to the original number itself.

2.1.2 Subtraction

Notice that subtraction is merely addition but with negative numbers involved. To prove this, we can refer to the additive inverse property:

$$\forall a \in \mathbb{C} \ \exists! b \in \mathbb{C} \text{ s.t. } a + b = 0$$

If you did not understand that, that's okay, because here we have it in simpler terms:

For every a in the set of complex numbers, there exists a unique b in the set of complex numbers such that the addition of a and b equals 0 (they cancel each other out!).

2.1.3 Multiplication

Multiplication is easy, simply refer to the provided multiplication table.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

Question 2.1. Why does the multiplication table only go up to 12×12?

From the provided multiplication table, you may have realised this:

$$ab = ba \ \forall a, b \in \mathbb{C}$$

Again, if we wanted to be fancy, this is also referred to as a commutative property.

Exercise 2.2. Find a and b that satisfies $ab = ba$.

From the exercise, you may have realised that $a \times 1 = a$ for all $a \in \mathbb{C}$. Why? Because anything multiplied by 1 is that number itself. Amazing!

2.1.4 Division

Notice that division is merely division but with the multiplicand being flipped upside down so that a 1 is on top (is the numerator). In proper terms, we say that the multiplicand is the reciprocal of the multiplier. To prove this, we can refer to the multiplicative inverse property:

$$\forall a \in \mathbb{C} \ a \neq 0, \ \exists! b \in \mathbb{C} \text{ s.t. } ab = 1$$

2.2 Order of Operations

We have strict rules for the order in which we do operations, since the mathematicians like rules and they get mad if we don't follow them. Although it goes by many names such as BODMAS, BEDMAS and PEDMAS, we will focus on BEDMAS for today. BEDMAS stands for:

- Brackets
- Exponents
- Division
- Multiplication
- Addition
- Subtraction

Knowing this, we can explore associativity:

$$(a + b) + c = a + (b + c) \land (ab)c = a(bc) \ \forall a, b, c \in \mathbb{C}$$

You see, having brackets is trivial when all you're doing is adding and multiplying.

Knowing order of operations also allows us to explore distributive properties:

$$a(b + c) = ab + ac \ \forall a, b, c \in \mathbb{C}$$

Challenge 2.1. What is $6 \div 2(1 + 2)$?

3 Proof by Induction

3.1 The Principle of Mathematical Induction

When inducting, we must follow strict protocols. We first have the initial proposition (usually $P(1)$) where we must prove that LHS = RHS. Then we prove the kth case, then we assume the k+1th case to be true.

We can then conclude that by principle of mathematical induction, what we said is true because we proved and then assumed it to be. Simple as that.

3.2 Sequences and Series

4 Functions

Let us focus on functions.

4.1 Linear equation

A linear equation is a polynomial degree 1. It can be shown in several ways.

- General form: $ax + by = c$
- Slope-intercept form: $y = mx + c$

4.2 Dealing With Infinity

For functions such as the inverse function $(1/x)$, then x cannot be zero otherwise we will have $1/0 = \infty$.

Question 4.1. Why is $1/0 = \infty$?

5 Trigonometry

5.1 Pythagorean Theorem

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

5.2 Angle Sum and Difference

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$$

5.3 Double Angle Identities

Double angle identities are derived from the angle and sum difference equations.

We know that

$$\sin(2x) = 2\sin(x)\cos(x)$$

therefore, we can say that

$$\sin(4x) = 4\sin^2(x)\cos^2(x)$$

5.4 n Angle Identities

Using De Moivre's theorem it is known that

$$(|z|\cos\theta)^n = |z|^n \cos n\theta$$

The identity $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ can be applied to deduce identites for sin.

$$\left(|z|\sin\left(\frac{\pi}{2} - \theta\right)\right)^n = |z|^n \sin\left(\frac{\pi}{2} - \theta\right)$$

Since the unit circle has radius 1, $z = 1$. Hence, the following identities are derived.

$$\cos n\theta = (|n|\cos\theta)^n \qquad \sin n\theta = \left(|n|\sin\left(\frac{\pi}{2} - \theta\right)\right)^n$$

How useful!

6 Limits

6.1 Trigonometric Limits

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

$$\lim_{\theta \to 0} \sin^{\theta}(\theta) = 1$$

$$\lim_{\theta \to \infty} \sin^{\theta}(\theta) = 0$$

Question 6.1. What would $\lim_{\theta \to 0} \cos^{\theta}(\theta)$ be equal to?

7 Calculus of the Differential Kind

7.1 Deriving from First Principles

Deriving from first principles is the fundamentals of calculus. In fact, we refer to this as the fundamental theorem of calculus.

7.1.1 First Principles with $\sin(x)$

Recall the formula for first principles:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} \frac{\mathrm{d}\sin(x)}{\mathrm{d}x} &= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\ &= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} \\ &= \cos x \end{aligned}$$

8 Integration

8.1 Integration Rules

$$\int k \ \mathrm{d}x = kx + C$$

8.1.1 The Sum Rule

$$\int i + j + k \ \mathrm{d}x = \int i \ \mathrm{d}x + \int j \ \mathrm{d}x + \int k \ \mathrm{d}x = ix + jx + kx + 3C$$

8.1.2 The Difference Rule

Similar to the sum rule, the difference rule is essentially the sum of additive inverses:

$$\int i - j - k \ \mathrm{d}x = \int i \ \mathrm{d}x - \int j \ \mathrm{d}x - \int k \ \mathrm{d}x = ix - jx - kx - 3C$$

8.2 Substitution

8.3 By Parts (Product Rule)

Integration by parts is when you split it to solve by parts. Suppose we have a function $f(x) = u(x)v(x)$ which we want to find the integral of:

$$\int f(x)\ \mathrm{d}x = \int u(x)v(x)\ \mathrm{d}x$$

We can use integration by parts, also known as the product rule of integrals:

$$\int u(x)v(x)\ \mathrm{d}x = \int u(x)\ \mathrm{d}x \times \int v(x)\ \mathrm{d}x$$

Exercise 8.1. Solve $\int x^2\ \mathrm{d}x$ using integration by parts.

9 Further Calculus

9.1 Oily-Macaroni Constant

The Oily-Macaroni constant is a mathematical constant denoted by gamma (γ).

$$\begin{aligned}
\gamma &= \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n}\frac{1}{k}\right) \\
&= \int_1^\infty \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right)\ \mathrm{d}x
\end{aligned}$$

Theorem 9.1. *γ is irrational.*

Proof.

$$\begin{aligned}
\gamma &= \int_1^\infty \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right)\ \mathrm{d}x \\
&= \int_1^\infty \frac{1}{\lfloor x \rfloor}\ \mathrm{d}x - \int_1^\infty \frac{1}{x}\ \mathrm{d}x \\
&= \int_1^\infty \frac{1}{\lfloor x \rfloor}\ \mathrm{d}x - \ln(2) \\
&= \left(\sum_{i=1}^{\infty}\frac{1}{i}\right) - \ln(2)
\end{aligned}$$

Let $P(n) : \sum_{i=1}^{n}\frac{1}{i} = \frac{p}{q} \forall n \in \mathbb{Z}^+$ where $p, q \in \mathbb{Z}^+ \wedge p, q$ are coprime, i.e. $\sum_{i=1}^{n}\frac{1}{i}$ is rational.

Base case:

$$P(1) : \sum_{i=1}^{1}\frac{1}{i} = \frac{1}{1}$$

$$\therefore P(1) \text{ is true.}$$

Assume $P(k) : \sum_{i=1}^{k}\frac{1}{i} = \frac{p}{q} \forall k \in \mathbb{Z}^+$ where $p, q \in \mathbb{Z}^+ \wedge p, q$ are coprime.

Induction step:

$$
\begin{aligned}
P(k+1) &: \sum_{i=1}^{k+1} \frac{1}{i} \\
&= \frac{1}{k+1} + \sum_{i=1}^{k} \frac{1}{i} \\
&= \frac{1}{k+1} + \frac{p}{q} \quad \text{[From P(k)]} \\
&= \frac{pk+p+q}{qk+q}
\end{aligned}
$$

$$\therefore P(k) \implies P(k+1)$$

$$\because P(1) \text{ is true and } P(k) \implies P(k+1) \forall k \in \mathbb{Z}^+$$

$$\therefore \text{By induction, } P(n) \text{ is true.}$$

Therefore, it trivially follows that $\sum\limits_{i=1}^{\infty} \frac{1}{i}$ is rational.

However, since $\ln(2)$ is irrational, and because the difference of a rational number and an irrational number is always irrational, this means that $\left(\sum\limits_{i=1}^{\infty} \frac{1}{i}\right) - \ln(2) = \gamma$ is irrational. □

10 Complex Numbers

10.1 Euler's Form

We know that

$$\sin^2\theta + \cos^2\theta = 1$$

We also know the famous equation known as Euler's formula. The imaginary number one. With the pi. No, not the one with polyhedrons. To be more precise, we mean Euler's identity, the "beautiful" equation.

$$e^{i\pi} + 1 = 0$$

Therefore, we can then conclude that:

$$
\begin{gathered}
e^{i\theta} + \sin^2\theta + \cos^2\theta = 0 \\
\frac{e^{i\theta} + \sin^2\theta}{-\cos^2\theta} = 1 \\
-\frac{e^{i\theta}}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} = 1 \\
-e^{i\theta}\sec^2\theta - \tan^2\theta = 1 \\
-e^{i\theta}\sec^2\theta = 1 + \tan^2\theta \\
-e^{i\theta}\sec^2\theta = \sec^2\theta \\
-e^{i\theta} = 1 \implies e^{i\theta} = -1
\end{gathered}
$$

What a beautiful conclusion!

10.2 The Cauchy-Riemann Equations (Essential Maths Knowledge)

This is essential knowledge when dealing with the complex world.

The Cauchy-Riemann Equations state that $\forall f : \mathbb{C} \to \mathbb{C}$ where $f : x + iy \mapsto u(x, y) + v(x, y)$:

$$f \text{ is holomorphic} \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{and} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Cauchy-Riemann Equations can also be written using Wirtinger derivatives, which are defined as follows:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Therefore, it is trivial to see that for any holomorphic function f:

$$\frac{\partial}{\partial \bar{z}} = 0$$

Exercise 10.1. Prove the Cauchy-Riemann Equations using first principles.

11 Linear Algebruh

11.1 Equation of a line

Linear algebra is named after its most important mathematical object, the linear algebruh function. As you already know, linear function can be written using the slope-intercept form $y = ax + b$. The a determines the slope of the line and b determined the y-intercept.

Although we have covered this back in Chapter 4, this is a deeper dive into linear algebra.