Dominance and Rationalizability

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Notation

a strategic form (or "normal form") game Γ consists of

- set of players I
- an action set for each player $\{S_i\}_{i \in I}$ (usually written $\{S_i\}$)
- Bernoulli utility functions $\{u_i\}_{i \in I}$ (usually written $\{u_i\}$) where $u_i : S_1 \times \cdots \times S_I \to \Re$)

we write $\Gamma = [I, \{S_i\}, \{u_i\}]$

Example: Prisoner's dilemma

	C	D
C	-2,-2	0,-3
D	-3,-0	-1,-1

- I is $\{P_1, P_2\}$
- S_i is $\{C, D\}$
- u_1 is defined by: $u_1(D, D) = -1$, $u_1(D, C) = -3$, $u_1(C, D) = 0$, $u_1(C, C) = -2$

Interpretation of a strategic form game

1. one shot game
   - one time event
   - each player knows game
   - rationality is common knowledge
   - actions are chosen simultaneously and independently
   - a player can base his expectation of other player's play only on primitives of the game
2. repeated play without strategic link
   - game is played repeatedly but with different opponents
   - no intertemporal strategic link between games
   - players can have expectations how rational players play based on past plays

Assumptions maintained throughout the course

- game itself is common knowledge among the players
- players are rational
  - clearly defined objective/preferences
  - unlimited analytical capabilities
- players are strategic
  - take into account that other players are similarly rational (rationality is common knowledge)
- players maximize expected utility (i.e. their preferences satisfy assumptions of von Neumann and Morgenstern's expected utility theorem)

Language and notation

- *action*: an element of S_i (a pure strategy)
- *action profile*: a vector $(s_1, \ldots, s_I)$ with one action for every player
- *mixed strategy*: σ_i is a probability distribution over S_i, i.e. $\sigma_i \in \Delta(S_i)$
- *(mixed) strategy profile*: a vector $(\sigma_1, \ldots, \sigma_I)$ with one strategy for every player
- *support of a mixed strategy σ_i*: set of all actions on which σ_i puts strictly positive probability
- s_{-i}: like an action profile but not including an action for player i, e.g. $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$
- S_{-i}: set of all possible s_{-i}
- σ_{-i}: as s_{-i} but with mixed strategies instead of actions

Dominant action

Definition (Strictly dominant action)

An action $s_i \in S_i$ is *strictly dominant* for player i in game $[I, \{S_i\}, \{u_i\}]$ if for all $s_i' \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i}$.

Example: Prisoner's dilemma

	D	C
D	-2,-2	0,-3
C	-3,-0	-1,-1

Strictly dominated action

Definition (Strictly dominated action)

An action $s_i \in S_i$ is *strictly dominated* for player i in game $[I, \{S_i\}, \{u_i\}]$ if there exists a $\sigma_i' \in \Delta(S_i)$, such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

- note: due to expected utility assumption, there is no difference between the definition above and using $u_i(\sigma_i', \sigma_{-i}) > u_i(s_i, \sigma_{-i})$ for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$ (check!)

Example

	L	R
T	3,0	0,1
M	1,2	1,0
B	0,0	3,1

Relevance

what will a rational player do if

- he has a strictly dominant action?
- he has a strictly dominated action?

Iterative elimination of strictly dominated strategies (IESDS) I

Example

Which action will a rational (and strategic) player 1 play?

	L	R
T	3,0	0,1
M	1,2	1,0
B	0,0	3,1

Iterative elimination of strictly dominated strategies (IESDS) II

Procedure:

- eliminate all strictly dominated actions
- eliminate all actions that are strictly dominated in the remaining game
- continue until no strictly dominated action left
- (eliminate all mixed strategies that are dominated)

note:

- order of elimination does not matter for outcome of the procedure (check!)
- if rationality is common knowledge, players will not use actions eliminated in the process above

(Never) best response

Definition ((Never) Best response)

In game $[I, \{S_i\}, \{u_i\}]$, strategy σ_i is a *best response* for player i to the other players' strategies σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$$

for all $\sigma'_i \in \Delta(S_i)$. Strategy σ_i is a *never best response* if there is no σ_{-i} to which σ_i is a best response.

- a mixed strategy is only a best response if every pure strategy in its support is a best response (by the expected utility assumption) (check!)
- a strictly dominated action is a never best response

Rationalizability

a rational player has

- a belief about other players' strategies
- plays best response given his belief

if rationality is common knowledge, a rational player's belief can only put positive probability on actions that are themselve best response to a belief (of the other players) that puts only positive probability on best responses to a belief. . .

Definition (Rationalizable actions)

In game $[I, \{S_i\}, \{u_i\}]$, the strategies surviving iterative elimination of never best responses are called *rationalizable strategies*.

Common knowledge of rationality implies that players play rationalizable strategies!

- remark: "rationalizable actions" of player i are those that are in the support of i's rationalizable strategies
  (these actions are also the pure strategies that are rationalizable)

Rationalizability: examples

Example: finite game

	L	R
T	3,0	0,1
M	1,2	1,0
B	0,0	3,1

Example: homogenous good Bertrand competition

- 2 firms with marginal costs $c \geq 0$ (zero fixed costs) compete in prices
- one consumer with unit demand and willingness to pay $v > c$

Example: Cournot competition

- 2 firms with marginal costs $c \in (0, 1)$ (zero fixed costs) compete in quantities $q_i \geq 0$
- inverse demand $P(q_1 + q_2) = 1 - q_1 - q_2$
- firm i profit: $u_i(q_1, q_2) = (1 - q_1 - q_2 - c)q_i$

Rationalizability and IESDS

- strictly dominated actions are never best responses
- only actions surviving IESDS can be rationalizable
- in game with more than 2 players, some actions surviving IESDS might not be rationalizable (in the way we defined it)
- for 2 player games: set of rationalizable actions and set of actions surviving IESDS are identical