Bayesian Nash equilibrium

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A first example I

- an incumbent decides whether to build a new plant (I for invest) at cost c_I
- entrant simultaneously decides whether to enter (E)
- entrant does not know whether c is "low" (l) or "high" (h)

Table: Payoffs with $c_I = h$

	E	NE
I	0,-1	2,0
NI	2,1	3,0

Table: Payoffs with $c_I = l$

	E	NE
I	1.5,-1	3.5,0
NI	2,1	3,0

- p: prior probability entrant assigns to h

A first example II

- what will the incumbent do if $c_I = h$?
- what should the incumbent do if $c_I = l$?
- what should entrant do?

Harsanyi's trick I

- suppose "nature" chooses the incumbent's type in a first step: h with prob p and l with prob $1 - p$
- then players play simultaneous move game but only incumbent observes nature's move
- strategy: complete plan of action, i.e. action choice for every type
  - incumbent: prob of investment when h and prob of investment when l
- solve for Nash equilibrium in new game in which players' strategy sets are complete plans of actions for each and every type (they could have had)

- Harsanyi's trick transforms incomplete information (not knowing c) into imperfect information (not knowing nature's move)

Harsanyi's trick II

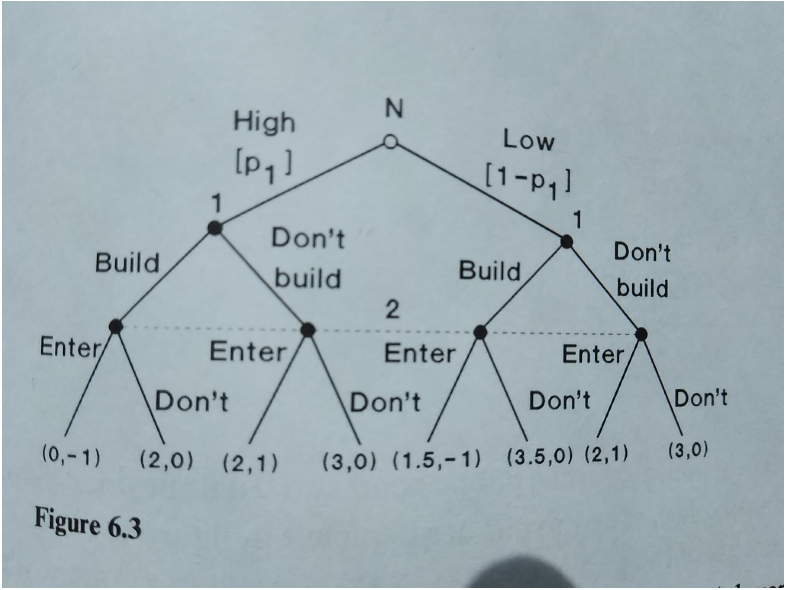

Figure 6.3 from Fudenberg and Tirole, "Game Theory", 1991, MIT Press

A first example III

- optimal not to invest if $h \rightarrow$ only prob of investment when l relevant, called x
- entrant optimally enters if $p + (1-p)(-1x + 1(1-x)) > 0$ $\Leftrightarrow x < 1/(2(1-p))$
- low cost incumbent optimally invests if $y < 1/2$
- equilibria:

Bayesian game/ Static game of incomplete information

- Bayesian game is denoted by $[I, \{S_i\}, \{u_i\}, \Theta, F]$ where
  - $\Theta = \Theta_1 \times \cdots \times \Theta_I$ and Θ_i is player i's set of possible types
  - F is a probability distribution over Θ
  - $u_i : S \times \Theta_i \to \Re$
- let $\mathcal{S}_i = \{s : \Theta_i \to S_i\}$ be the set of functions assigning to each type of player i a strategy in S_i
- let $\tilde{u}_i(s_1(\cdot), \ldots, s_I(\cdot)) = \mathbb{E}_\theta[u_i(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i)]$ be the expected utility of i if players use strategies $\{s_i\}$

Definition (Bayesian Nash equilibrium (BNE))

A (pure) Bayesian Nash equilibrium in $[I, \{S_i\}, \{u_i\}, \Theta, F]$ is a profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ that constitutes a (pure) Nash equilibrium in the game $[I, \{\mathcal{S}_i\}, \{\tilde{u}_i\}]$, i.e.

$$\tilde{u}_i(s_i(\cdot), \ldots, s_{-i}(\cdot)) \geq \tilde{u}_i(s_i'(\cdot), \ldots, s_{-i}(\cdot))$$

for all $s_i' \in \mathcal{S}_i$.

Another characterization of BNE

- $(s_1(\cdot), \ldots, s_I(\cdot))$ is BNE if no type (of any player) can increase his expected payoff by deviating

Lemma (BNE in finite games)

A decision profile $(s_1(\cdot), \ldots, s_I(\cdot))$ is a BNE if and only if for all i and all $\bar{\theta}_i \in \Theta_i$ occurring with positive probability

$$\mathbb{E}_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \theta_i)|\bar{\theta}_i] \geq \mathbb{E}_{\theta_{-i}}[u_i(s_i', s_{-i}(\theta_{-i}), \theta_i)|\bar{\theta}_i]$$

for all $s_i' \in S_i$.

Public good game I

- two neighboring cities have to decide whether to build an airport
- airport can be used by citizens of both cities and this gives each city a payoff normalized to 1
- cost of city i of building airport is c_i and is i's private information
- prior: $c_i \sim u[0, 1]$ (drawn independently)
- payoffs

	B	NB
B	$1\text{-}c_1$,$1\text{-}c_2$	$1\text{-}c_1$,1
NB	1,$1\text{-}c_2$	0,0

- strategy: $s_i : [0, 1] \to \{B, NB\}$

Public good game II

- B is optimal for P_i if $1 - c_i > x \Leftrightarrow c_i < 1 - x$ where x is the belief that $-i$ builds
  - optimal strategy is a cutoff rule
- at cutoff: indifference
- prob of B is prob of c_i below cutoff
- what is equilibrium cutoff?

Generalized public good game

- I players decide to contribute (C) or not (N)
- payoff 1 if at least one player contributes
- costs c_i if contributing distributed on $[0, 1]$ with continuous cdf Φ
  - full support assumption: density $\phi > 0$ on $[0, 1]$
- C is optimal if $1 - c_i > x \quad \Leftrightarrow \quad c_i < 1 - x$
  - cutoff strategy is optimal
- what is probability that no other player contributes in a symmetric equilibrium with cutoff c^*?
- which condition has to be satisfied in symmetric equilibrium?
- if the number of players increases, will the equilibrium cutoff be higher or lower?

2 player all pay auction with exponential type distribution

- 2 players choose effort s_i
- payoff: $\theta_i - s_i$ if $s_i > s_j$ and $-s_i$ else
- θ_i are private info and distributed independently on $[0, \infty)$ with cdf $\Phi(\theta_i) = 1 - e^{-\theta_i}$ (pdf $e^{-\theta_i}$)
- we derive symmetric equilibrium $s(\theta_i)$
- it can be shown that equilibrium strategy s has to be strictly increasing in θ_i
  - strictly increasing inverse of s exists and is denoted by t
  - what is probability that other player bids less than b (in equilibrium)?
- what is expected payoff of i when bidding b (given other player uses equilibrium strategy)?
- which condition characterizes i's best response to equilibrium strategy s?