Auctions

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Setup: Independent private value (IPV) auction

- one indivisible object is allocated through an auction
- n bidders
- bidder i draws valuation v_i from distribution Φ with continuous density ϕ and support $[0, 1]$
- v_i is i's private information
- valuations of bidders are drawn independently
- payoff: $v_i x_i - t_i$
  - x_i: probability of getting the good
  - t_i: (expected) transfer that i pays to auctioneer

Second price sealed bid auction (IPV)

- allocation: highest bidder gets object
- transfer: highest bidder pays *second highest bid*, others pay zero
- how much should a bidder with valuation v_i bid?

First price sealed bid (IPV)

- allocation: highest bidder gets object
- transfer: highest bidder pays *his own bid*, others pay zero

- solve for symmetric BNE in strictly increasing and continuous strategies
  - bidder with valuation v_i bids $s(v_i)$ which is strictly increasing and continuous
  - let t be the inverse of s, $t(b)$ is type bidding b
  - given $s()$, what is the probability that bidder i wins the auction when bidding b?
  - what is bider i's expected payoff when bidding b?

- what is bidder i's first order condition for optimal bidding?

First price sealed bid (IPV) II

Theorem (Equilibrium)

There exists a symmetric BNE in which all bidders use the bidding strategy

$$s(v_i) = \int_0^{v_i} (n-1)x\Phi(x)^{n-2}\phi(x)\,dx/\Phi(v_i)^{n-1} \quad \textit{for} \quad v_i > 0$$

and $s(0) = 0$.

s is strictly increasing:

$$\begin{aligned}
s'(v_i) &= (n-1)\frac{v_i\Phi^{n-2}(v_i)\phi(v_i)\Phi^{n-1}(v_i) - (n-1)\Phi(v_i)^{n-2}\phi(v_i)\int_0^{v_i} x\Phi(x)^{n-2}\phi(x)\,dx}{(\Phi^{n-1}(v_i))^2} \\
&= (n-1)\frac{\phi(v_i)}{\Phi^n(v_i)}\left(v_i\Phi^{n-1}(v_i) - \int_0^{v_i}(n-1)x\Phi(x)^{n-2}\phi(x)\,dx\right) \\
&= (n-1)\frac{\phi(v_i)}{\Phi^n(v_i)}\left(v_i\Phi^{n-1}(v_i) - \left[x\Phi^{n-1}(x)\right]_0^{v_i} + \int_0^{v_i}\Phi^{n-1}(x)\,dx\right) \\
&= (n-1)\frac{\phi(v_i)}{\Phi^n(v_i)}\left(\int_0^{v_i}\Phi^{n-1}(x)\,dx\right) > 0 \qquad \text{if } v_i > 0
\end{aligned}$$

Common value auction: setup

- one indivisible object is allocated through an auction
- 2 bidders
- value v of the object is either 1 *for all bidders* or 0 *for all bidders* (both probability 1/2)
- bidder i draws signal v_i from distribution $\Phi(v_i|v)$ with density $\phi(v_i|v)$
  - $\phi(v_i|0) = 2 - 2v_i$ for $v_i \in [0, 1]$
  - $\phi(v_i|1) = 2v_i$ for $v_i \in [0, 1]$
- v_i is i's private information
- signals of bidders are drawn independently conditional on v
- payoff: $vx_i - t_i$
  - x_i: probability of getting the good
  - t_i: (expected) transfer that i pays to auctioneer

Common value auction: second price sealed bid

$$\mathbb{E}[v|v_i] = \frac{\phi(v_i|0)}{\phi(v_i|0)+\phi(v_i|1)}0 + \frac{\phi(v_i|1)}{\phi(v_i|0)+\phi(v_i|1)}1 = (1-v_i)0 + v_i 1 = v_i$$

$$\phi(v_j|v_i) = \frac{\phi(v_i|0)}{\phi(v_i|0)+\phi(v_i|1)}\phi(v_j|0) + \frac{\phi(v_i|1)}{\phi(v_i|0)+\phi(v_i|1)}\phi(v_j|1) = (1-v_i)(2-2v_j) + v_i(2v_j) = 2 - 2v_i - 2v_j + 4v_i v_j$$

$$\mathbb{E}[v|v_i, v_j] = \frac{\phi(v_i|0)\phi(v_j|0)}{\phi(v_i|0)\phi(v_j|0)+\phi(v_i|1)\phi(v_j|1)}0 + \frac{\phi(v_i|1)\phi(v_j|1)}{\phi(v_i|0)\phi(v_j|0)+\phi(v_i|1)\phi(v_j|1)}1 = \frac{v_i v_j}{1-v_i-v_j+2v_i v_j}$$

- aside: $\mathbb{E}[v|v_i, v_j] \leq \mathbb{E}[v|v_i]$ if and only if $v_j \leq 1/2$ (check!)

Equilibrium

Both bidders bidding $s(v_i) = \mathbb{E}[v|v_i, v_j = v_i]$ is a symmetric equilibrium in increasing strategies.

Expected payoff bid b: $\int_0^{s^{-1}(b)} (\mathbb{E}[v|v_i, v_j] - \mathbb{E}[v|v_j, v_j])\phi(v_j|v_i)\, dv_j$

Common value auction: English auction

- auctioneer continuously increases price starting from zero
- bidders have hands raised at start and drop their hand if they are no longer interested in the item (i.e. if the price is too high)
- the last bidder with his hand raised gets the object at a price equal to the auctioneer price at the time he lowers his hand

- what should you do when you are the only bidder with your hand raised?

- we look for a symmetric equilibrium in strictly increasing strategies

- at what price should i drop his hand given that j still has his hand up in a symmetric equilibrium?

Common value auction: first price sealed bid

- we search for a symmetric equilibrium in strictly increasing and continuous strategies $s(v_i)$
- let t be the inverse of the equilibrium strategy s
- $prob(v_j \leq x | v_i) = \int_0^x \phi(v_j|v_i)\, dv_j = 2x - 2v_i x - x^2 + 2v_i x^2$
- $\mathbb{E}[v|v_i, v_j \leq x] = \int_0^x \mathbb{E}[v|v_i, v_j] \frac{\phi(v_j|v_i)}{\int_0^x \phi(\tilde{v}_j|v_i)\, d\tilde{v}_j}\, dv_j =$ $\int_0^x \frac{2v_i v_j}{\int_0^x 2 - 2v_i - 2\tilde{v}_j + 4v_i\tilde{v}_j\, d\tilde{v}_j}\, dv_j = \frac{v_i x}{2 - 2v_i - x + 2v_i x}$
- i's expected payoff when bidding b (given that j uses equilibrium strategy s):

$$prob(v_j \leq t(b)) * (\mathbb{E}[v|v_i, v_j \leq t(b)] - b) =$$
$$(2t(b) - 2v_i t(b) - t(b)^2 + 2v_i t(b)^2) * \left(\frac{v_i t(b)}{2 - 2v_i - t(b) + 2v_i t(b)} - b\right) =$$
$$v_i t(b)^2 - (2t(b) - 2v_i t(b) - t(b)^2 + 2v_i t(b)^2) b$$

- i's first order condition for an optimal bid:

$$2v_i t(b) t'(b) - (2t(b) - 2v_i t(b) - t(b)^2 + 2v_i t(b)^2) - b * t'(b)(2 - 2v_i - 2t(b) + 4v_i t(b)) = 0$$

- in symmetric equilibrium: $t(b) = v_i$, $t'(b) = 1/s'(v_i)$, $b_i = s(v_i)$, $s(0) = 0$