Exam Adv. Micro II

In all exercises players are assumed to be expected utility maximizers, i.e. all players act according to a von Neumann-Morgenstern utility function.

1. There are two players whose valuation θ_i for an indivisible object are independently and identically distributed according to the uniform distribution on $[0, 1]$. The players participate in the following auction format: Both bidders submit simultaneously a bid in a sealed envelope. The higher bidder gets the good and pays his bid, the lower bidder does not get the good but *has to pay half his bid.* (You can ignore the case of a tie.) Players have quasi linear utility, i.e. their payoff is $\theta_i - t$ when getting the good and paying t and their payoff is $-t$ when paying t while not getting the good.

   (a) Let $U_i(\theta_i)$ be the expected utility of player i with type θ_i and $Y_i(\theta_i)$ be i's expected probability of winning the auction in some Bayesian Nash equilibrium of the auction. Show that $U_i(\theta_i) = \int_0^{\theta_i} Y_i(s)\,ds$. Show also that Y_i is weakly increasing. *(15 points)*

   (b) Derive a symmetric Bayesian Nash equilibrium in strictly increasing strategies. *(10 points)*

   (c) Based on the results we derived in the lecture, is this auction format revenue maximizing for the seller? *(10 points)*
   (Hint: You do not have to derive the revenue maximizing auction format yourself to solve this exercise!)

2. Two players play a strategic form game. The monetary payoffs the two players get when choosing their strategies are given in table 1. (note: the numbers in the table are not utilities but amounts of money!) Assume that players are only interested in their own monetary payoffs and not in the monetary payoffs for the other player and that they like more money better.

	L	R
A	9,1	0,0
B	4,0	4,1
C	6.25,0	1,1
D	0,1	9,0

Table 1: Monetary payoff

   (a) Assume for this subquestion that players value money linearly, i.e. $u_i(x) = x$ for any monetary amount x. Which actions are rationalizable? *(10 points)*

   (b) How does the set of rationalizable actions change if players are (very) risk averse? Explain this intuitively and show the result in an example using the payoffs above. *(10 points)*

   (c) Assume for this subquestion again $u_i(x) = x$. Is there a correlated equilibrium that does not lead to the same distribution over outcomes as a Nash equilibrium of the game? *(5 points)*

Solution

1. Exercise 1:

   (a) Fix some BNE. Let $T_i(\theta_i)$ be the expect payment by player i of type θ_i. Then $U_i(\theta_i) = \theta_i Y_i(\theta_i) - T_i(\theta_i)$. In equilibrium, type θ_i prefers his bid to the bid of type θ_i' which is equivalent to

   $$U_i(\theta_i) \geq U_i(\theta_i') + Y_i(\theta_i')(\theta_i - \theta_i').$$

   Similarly, θ_i' prefers his bid to the bid of θ_i and therefore

   $$U_i(\theta_i') \geq U_i(\theta_i) - Y_i(\theta_i)(\theta_i - \theta_i').$$

   Without loss of generality let $\theta_i > \theta_i'$. Taking the two inequalities together yields

   $$Y_i(\theta_i') \leq \frac{U_i(\theta_i) - U_i(\theta_i')}{\theta_i - \theta_i'} \leq Y_i(\theta_i).$$

   Consequently, $\theta_i > \theta_i'$ implies $Y_i(\theta_i) \geq Y_i(\theta_i')$ which shows that Y_i is weakly increasing. This monotonicity implies that Y_i is continuous almost everywhere. At every point of continuity, taking the limit $\theta_i' \to \theta_i$ in the sandwich-inequality above implies $U_i'(\theta_i) = Y_i(\theta_i)$. Integrating up yields $U_i(\theta_i) = U_i(0) + \int_0^{\theta_i} Y_i(s)\, ds$. Clearly, it is optimal for a type 0 to bid 0 (as a positive bid leads to positive payments for sure while he does not value the good), i.e. $U_i(0) = 0$.

   (b) In such an equilibrium $Y(\theta_i) = prob(\theta_j \leq \theta_i) = \theta_i$. By the previous subquestion $U(\theta_i) = \int_0^{\theta_i} s\, ds = \theta_i^2/2$. By the auction rules, $T(\theta_i) = Y(\theta_i)b(\theta_i) + (1 - Y(\theta_i))b(\theta_i)/2 = (1+\theta_i)b(\theta_i)/2$ where b is the equilibrium bidding function. Hence, $U(\theta_i) = Y(\theta_i)\theta_i - T(\theta_i) = \theta_i\theta_i - (1+\theta_i)b(\theta_i)/2$. Equating this with the envelope condition above yields $b(\theta_i) = \theta_i^2/(1+\theta_i)$.

   (c) The revenue equivalence theorem states that two auction formats yield the same revenue if the bidder type with he lowest valuation, i.e. 0, has the same payoff and the probability of assigning the good to bidder i is, conditional on (θ_1, θ_2), the same in the two formats. As the higher type gets the good in both and a zero type has zero expected payoff, our auction is revenue equivalent to a Vickrey auction without reserve price. We have shown that a Vickrey auction with reserve price is revenue maximizing in this IPV setting. Consequently, the auction is not revenue maximizing because the optimal reserve price is strictly positive with a uniform distribution of types (the virtual valuation $\theta_i - (1-\theta_i)$ is negative for low types).

2. Exercise 2:

   (a) We can obtain the set of rationalizable actions by iterative elimination of strictly dominated strategies. The mixed strategy 0.5 on A and 0.5 on D strictly dominates B. The mixed strategy 0.7 on A and 0.3 on D strictly dominates C. Given that B and C are eliminated, L strictly dominates R. Given this, A strictly dominates D. Hence, A and L are the only rationalizable actions.

   (b) If players are risk averse, then "insurance actions", i.e. actions that give a moderate but not very low payoff against any choice by the opponent, become best responses against mixed strategies and therefore rationalizable. Hence, the set of rationalizable actions is getting bigger. Another way to look at this is that mixed strategies that yield high and low payoffs with some probabilities, e.g. the mix 0.5 A and 0.5 B, are less attractive if players are risk averse and therefore no longer able to dominate insurance actions like B. Take for example the utility function $u(x) = \sqrt{x}$ for both players. Then the payoffs become as depicted in table 2 and this means that B is a best response to a uniformly

	L	R
A	3,1	0,0
B	2,0	2,1
C	2.5,0	1,1
D	0,1	3,0

Table 2: Utilities with $u(x) = \sqrt{x}$

mixed strategy by player 2 and C is a best response against a mixed strategy with 2/3 on L and 1/3 on R mixed strategy. Action A is a best response against L and D is a best response against R. L is a best response against A and R is a best response against B. Hence, all actions are rationalizable as a best response cannot be strictly dominated.

(c) The support of a correlated equilibrium consists of rationalizable actions. As we showed above, that A and L are the only rationalizable actions, the unique correlated equilibrium of the game is the distribution putting probability 1 on (A, L).