Nash Equilibrium

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Nash equilibrium (NE)

Definition (Nash equilibrium)

A strategy profile $(s_1, \ldots, s_I)$ is a *Nash equilibrium* of game $\Gamma = [I, \{S_i\}, \{u_i\}]$ if for every $i = 1, \ldots, I$

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for all $s_i' \in S_i$.

- mutual best response: let $b_i(s_{-i})$ be $\{s_i \in S_i : \; u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i\}$; then NE is an action profile $(s_1, \ldots, s_I)$ such that $s_i \in b_i(s_{-i})$ for all i

Nash equilibrium: interpretations

- non-paternalistic prediction
- self enforcing agreement
- stable steady state (convention)

mixed strategy Nash equilibrium

Definition (mixed strategy Nash equilibrium)

A mixed strategy Nash equilibrium of the game $\Gamma = [I, \{S_i\}, \{u_i\}]$ is a Nash equilibrium $(\sigma_1, \ldots, \sigma_I)$ in the game $\Gamma = [I, \{\Delta S_i\}, \{u_i\}]$, i.e. for every $i = 1, \ldots, I$

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta S_i$.

- by linearity of expected utility, $(\sigma_1, \ldots, \sigma_I)$ is a mixed Nash equilibrium if and only if every action in the support of σ_i is a best response to σ_{-i}

Existence of Nash equilibrium

- not all games have a Nash equilibrium

Example (non-existence of NE)

- 2 players ($I = \{P1, P2\}$)
- each player says a number ($S_i = \Re$)
- player that says the higher number wins
  (e.g. winner has payoff 1 while loser has payoff 0, if both say the same number each has payoff $1/2$)

Existence

Why are we interested in existence of NE?

- worthwhile to search for an equilibrium
- steady state interpretation of equilibrium, existence of NE says that the process might become stable
- sometimes possible to do comparative statics without computing the NE (only sensible if NE exists)

Fixed point theory I

Definition (Fixed point)

Let f be a real function, i.e. $f : S \to \Re$ where $S \subseteq \Re$. A point $s \in S$ is a *fixed point* of f if $f(s) = s$.

- Does $f(x) = 1$ have a fixed point? What about $f(x) = x + 1$? What about $f(x) = x^2$?
- fixed point theory gives general conditions under which functions have fixed points
- fixed points for functions from $f : \ S \to \Re^n$ where $S \subseteq \Re^n$ are defined analogously

Fixed point theory II

Theorem (Brouwer's fixed point theorem (1-dimensional))

Let $f : [0, 1] \to [0, 1]$ *be a continuous function. Then* f *has a fixed point.*

Fixed point theory III

Example (Cournot)

- two firms each choose a quantity q_i
- both firms have costs $c(q_i) = cq_i$ for some $c > 0$
- inverse demand is $P(q_1 + q_2)$ where we assume that P is two times continuously differentiable with $P' < 0$ and $P'' \leq 0$
- assume that $P(1) < c$
  $\Rightarrow$ a firm will never offer a quantity greater than 1
- firm 1 chooses a quantity from $[0, 1]$ to maximize profits

$$\max_{q_1}(P(q_1 + q_2) - c)q_1$$

we get the first order condition

$$P'(q_1 + q_2)q_1 + P(q_1 + q_2) - c = 0$$

Fixed point theory IV

Example (Cournot (continued))

- the second order condition holds by assumption

$$P''(q_1 + q_2)q_1 + 2P'(q_1 + q_2) < 0$$

- the first order condition defines a best response function $q_1(q_2)$
- the best response function is continuous because P and P' are continuous by assumption
- Brouwer: best response function has a fixed point!
- game is symmetric $\Rightarrow$ fixed point is an equilibrium $\Rightarrow$ equilibrium exists
- We showed this without being able to actually calculate the equilibrium!

Fixed point theory V

Theorem (Brouwer fixed point theorem)

Let S be a convex and compact set in $\Re^n$ and let $f : S \to S$ be a continuous function. Then there exists an $s^* \in S$ such that $f(s^*) = s^*$.

Nash theorem

Theorem (Nash theorem)

A strategic game $[I, \{S_i\}, \{u_i\}]$ *with*

- *a finite number of players*
- *a finite number of actions for each player*

has a Nash equilibrium in mixed strategies.

- proof uses Brouwer's fixed point theorem
- to simplify notation, we prove the theorem for 2×2 games

Proof of Nash theorem for 2×2 games I

- 2×2 game:

	L	R
U	a,b	c,d
D	e,f	g,h

- mixed strategy of P1: probability $\alpha \in [0, 1]$ of playing U
- mixed strategy of P2: probability $\beta \in [0, 1]$ of playing L
- idea of proof:
  - define a function $f : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ such that
    - f is continuous
    - if $f(\alpha^*, \beta^*) = (\alpha^*, \beta^*)$, then (α^*, β^*) is a Nash equilibrium of the game
  - use Brouwer's theorem to establish that f has a fixed point

Proof of Nash theorem for 2×2 games II

- $u_1(U, \beta) = \beta a + (1 - \beta)c$ is the expected utility of P1 when playing U and P2 uses the mixed strategy β
- $u_1(U, \beta)$ is linear and therefore continuous in β
- define

$$g(\alpha, \beta) = max \left\{ 0, \frac{\alpha + u_1(U, \beta) - u_1(D, \beta)}{1 + |u_1(U, \beta) - u_1(D, \beta)|} \right\}$$

  - $g(\alpha, \beta)$ is higher than α if U is the best response to β and lower than α if D is best response
  - g is continuous because u_1 is continuous in β
- define

$$h(\alpha, \beta) = max \left\{ 0, \frac{\beta + u_2(L, \alpha) - u_2(R, \alpha)}{1 + |u_2(L, \alpha) - u_2(R, \alpha)|} \right\}$$

  - $h(\alpha, \beta)$ is higher than β if L is best response to α and lower than β if R is best response
  - h is continuous because u_2 is continuous in α

Proof of Nash theorem for 2×2 games III

let f be defined by

$$f(\alpha, \beta) = (g(\alpha, \beta), h(\alpha, \beta))$$

- if $f(\alpha^*, \beta^*) = (\alpha^*, \beta^*)$ then
  - $g(\alpha^*, \beta^*) = \alpha^* \Rightarrow \alpha^*$ is best response to β^*
  - $h(\alpha^*, \beta^*) = \beta^* \Rightarrow \beta^*$ is best response to α^*
  - (α^*, β^*) is Nash equilibrium

  every fixed point of f is Nash equilibrium
- f is continuous because g and h are continuous
- Brouwer: f has a fixed point

Generalization of Nash's theorem

Theorem (Nash theorem)

A strategic game $[I, \{S_i\}, \{u_i\}]$ *with*

- *a finite number of players*
- *a convex and compact action set* S_i *(for all* i*)*
- *continuous utility functions* u_i

has a Nash equilibrium in mixed strategies (and in pure strategies if all u_i *are quasi-concave in* s_i*).*