Mechanism Design: Dominant Strategy Equilibrium

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Dominant strategy equilibrium in mechanism design

- last time: mechanism $(S_1, \dots, S_I, g)$ implements scf f if game induced by mechanism has an equilibrium $(s_1^*, \dots, s_I^*)$ such that $f(\theta) = g(s_1^*(\theta_1), \dots, s_I^*(\theta_I))$
- normally: equilibrium = Bayesian Nash equilibrium
- today: equilibrium = dominant strategy equilibrium

Definition (Dominant strategy equilibrium)

The strategy profile $(s_1^*, \dots, s_I^*)$ is a dominant strategy equilibrium in the game induced by the mechanism $(S_1, \dots, S_I, g)$ iff for each player i and type $\theta_i \in \Theta_i$

$$u_i(g(s_i^*(\theta_i), s_{-i}), \theta_i) \geq u_i(g(s_i, s_{-i}), \theta_i)$$

for all $s_i \in S_i$ and all $s_{-i} \in S_{-i}$.

- dominant strategy (in mechanism design): a strategy that is weakly (!) optimal no matter what the other players are doing
- examples for games with dominant strategy equilibrium?

Dominant strategy equilibrium

- every dominant strategy equilibrium is also a BNE (not vice versa)
- very robust equilibrium concept
  - beliefs about others' play irrelevant
  - knowledge of others' payoffs or rationality irrelevant

Revelation principle for dominant strategy implementation

If f is implementable in dominant strategy equilibrium by some mechanism, then f is truthfully implementable in dominant strategy equilibrium by the direct revelation mechanism. (proof: see MWG)

- dominant strategy incentive compatibility: for all $\theta_i, \theta_i' \in \Theta_i$ and $\tilde{\theta}_{-i} \in \Theta_{-i}$

$$u_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) \geq u_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i)$$

- any drawbacks of using dominant strategy equilibrium?

Towards Gibbard Satterthwaite

- BIG QUESTION:

Which social choice functions are incentive compatible in dominant strategies?

Towards Gibbard Satterthwaite

- One class of ic social choice functions that are however *not so nice* are dictatorial choice functions:

Definition (dictatorial social choice function)

The social choice function is dictatorial if there is an agent i (the dictator) such that for all $\theta \in \Theta$

$$f(\theta) \in \{x \in X : u_i(x, \theta_i) \geq u_i(y, \theta_i) \text{ for all } y \in X\}.$$

- roughly: if the social choice function always picks the alternative that i loves most, then i is a dictator
- Check: a dictatorial social choice function is incentive compatible and Pareto efficient

Gibbard Satterthwaite Theorem (informal)

- Assumptions
- X is a finite set with at least 3 elements, say $X = \{x_1, x_2, \ldots, x_n\}$
- preferences are strict, i.e. no agent is indifferent between two alternatives x_m and x_k
- all preferences over X are possible; e.g. for $n = 3$ this means that for each player i there is
  - a type θ_i^1 such that $u_i(x_1, \theta_i^1) > u_i(x_2, \theta_i^1) > u_i(x_3, \theta_i^1)$
  - a type θ_i^2 such that $u_i(x_1, \theta_i^2) > u_i(x_3, \theta_i^2) > u_i(x_2, \theta_i^2)$
  - a type θ_i^3 such that $u_i(x_2, \theta_i^3) > u_i(x_1, \theta_i^3) > u_i(x_3, \theta_i^3)$
  - a type θ_i^4 such that $u_i(x_2, \theta_i^4) > u_i(x_3, \theta_i^4) > u_i(x_1, \theta_i^4)$
  - a type θ_i^5 such that $u_i(x_3, \theta_i^5) > u_i(x_2, \theta_i^5) > u_i(x_3, \theta_i^5)$
  - a type θ_i^6 such that $u_i(x_3, \theta_i^6) > u_i(x_1, \theta_i^6) > u_i(x_2, \theta_i^6)$
- Result: Only dictatorial social choice functions are truthfully implementable in dominant strategies.

Gibbard Satterthwaite Theorem (formal)

Theorem (Gibbard Satterthwaite Theorem)

Suppose X is finite and contains at least three elements. Suppose further that all preferences on X are possible for all agents i. A social choice function f that maps onto X is then truthfully implementable in dominant stategies if and only if it is dictatorial.

Proof.

(skipped; see, for example, Lars-Gunnar Svensson, Alexander Reffgen, The proof of the Gibbard–Satterthwaite theorem revisited, Journal of Mathematical Economics, Volume 55, December 2014, Pages 11-14, `http://dx.doi.org/10.1016/j.jmateco.2014.09.007`.) □

Gibbard Satterhwaite Theorem: Interpretation and economics

- in connection with revelation principle:
  only dictatorial social choice functions can be implemented by any mechanism
- quite demoralizing!
- comment: similar result holds for infinite X

Gibbard Satterhwaite Theorem: What now?

- two ways to get out of this negative result:
  - don't allow all possible preferences
  - don't use dominant strategy implementation; i.e. use Bayesian Nash equilibrium instead of dominant strategy equilibrium (see the following lectures)
- both ways out have their drawbacks!!!

Quasi-linear preferences

- consider setups where outcome consists of one decision $y \in \Re$ and transfer payments $t_1, \ldots, t_I$
  - e.g. public good example last time with $c = 0$ where $X = \{(y, t_1, \ldots, t_I) \in \Re^{I+1} : y \in \{0, 1\}, \sum_i t_i \geq 0\}$
- restrict preferences to quasi-linear preferences: $u_i(x, \theta_i) = v_i(y, \theta_i) - t_i$
- denote by y^* efficient decision, i.e.

$$y^*(\theta) \in argmax_y \sum_i v_i(y, \theta_i)$$

  - e.g. public good example: $y^*(\theta) = \begin{cases} 1 & \text{if } \sum_i \theta_i > 0 \\ 0 & \text{else} \end{cases}$
- denote by y^*_{-i} the efficient decision if "i was not there", i.e.

$$y^*_{-i}(\theta_{-i}) \in argmax_y \sum_{j \neq i} v_j(y, \theta_j)$$

  - e.g. public good example: $y^*_{-i}(\theta_{-i}) = \begin{cases} 1 & \text{if } \sum_{j \neq i} \theta_j > 0 \\ 0 & \text{else} \end{cases}$

Pivot mechanisms

Theorem (Pivot mechanism)

Let

$$t_i^*(\theta) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(y^*(\theta), \theta_j)$$

where $h_i : \Theta_{-i} \to \Re$ *is*

$$h_i(\theta_{-i}) = \sum_{j \neq i} v_j(y_{-i}^*(\theta_{-i}), \theta_j).$$

Then the social choice function

$$f(\theta) = (y^*(\theta), t_1^*(\theta), \ldots, t_I^*(\theta))$$

is dominant strategy incentive compatible.

- f is not dictatorial!
- f implements efficient project choice

Proof theorem

To show: for all $\tilde{\theta}_{-i} \in \Theta_{-i}$ and all $\theta_i, \theta_i' \in \Theta_i$

$$\theta_i \in argmax_{\hat{\theta}_i \in \Theta_i} v_i(y^*(\hat{\theta}_i, \tilde{\theta}_{-i}), \theta_i) - h_i(\tilde{\theta}_{-i}) + \sum_{j \neq i} v_j(y^*(\hat{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_j)$$

$$\Leftrightarrow \theta_i \in argmax_{\hat{\theta}_i \in \Theta_i} v_i(y^*(\hat{\theta}_i, \tilde{\theta}_{-i}), \theta_i) + \sum_{j \neq i} v_j(y^*(\hat{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_j)$$

- true as $y^*(\theta_i, \tilde{\theta}_{-i}) \in argmax_y v_i(y, \theta_i) + \sum_{j \neq i} v_j(y, \tilde{\theta}_j)$ by definition of y^* □

Example Pivot: public good

- public good example with $c = 0$ (note: θ_i may be negative)

$$t_i(\theta) = \sum_{j \neq i} y^*_{-i}(\theta_{-i})\theta_j - \sum_{j \neq i} y^*(\theta)\theta_j = \left(y^*_{-i}(\theta_{-i}) - y^*(\theta)\right) \sum_{j \neq i} \theta_j$$

- say $\theta_1 = 2$, $\theta_2 = -3$, $\theta_3 = 2$, calculate the Pivot transfers!

Example Pivot: private value auction of an indivisible object

- y_i be probability that i gets good
- $v_i(y, \theta_i) = y_i \theta_i$
- y^*: assign good to person with highest value
- what is $h_i(\theta_{-i})$?
- what is $t_i^*(\theta)$?
- reminds you of anything?

Some comments

- externality transfers
- $t_i^* \geq 0$ (strict inequality for pivotal players, equality for non-pivotal)
- budget balance? efficiency?
- Vickrey-Clarke-Groves (VCG) mechanisms

Dealing with positive costs

- if alternatives come with costs then adopt a default sharing of costs
  - e.g. public good example with $c > 0$; equal cost sharing as default:

$$\tilde{v}_i(y, \theta_i) = v_i(y, \theta_i) - c(y)/I$$

- use Pivot mechanism with $\tilde{v}_i$ instead of v_i (leading to Pivot transfers $\tilde{t}_i^*$) and set

$$t_i^*(\theta) = \tilde{t}_i^*(\theta) + c(y^*(\theta))/I$$

Example: positive costs

- say $\theta_1 = 2$, $\theta_2 = 0.9$, $\theta_3 = 2$, $c = 4.5$
- calculate the Pivot transfers!
- participation?