# Imperfect Information in Health Care Markets

## Exercise Session 1

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# Course Information

- **Material** can be found on GitHub
- **Prerequisites:** High school level math knowledge
  - Solve linear and quadratic equations
  - Take derivatives and know their interpretation
  - Integrate simple functions
  - What is a distribution
  - Calculation of expected value and variance
- Refresh your knowledge if necessary
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# Advice for Understanding the Material

- Work on the exercises **beforehand** (but do not be discouraged if you cannot solve them)
- Study in **groups** if that is beneficial to you
- Solve the exercises **yourself** after the exercise session
- Make sure you **understand** and can **apply** the **mathematics steps/tools** as well as the **economic concepts and intuition** taught in the lecture
- The exam will ask you to **transfer** your built up knowledge to new exercises
- Ask **questions** if anything is unclear

# Exercise 1

1. Recall the notion of a random variable and how to compute the expected value of discrete and continuous random variables.
2. Let X and Y be two random variables with $Y \sim U([0,1])$ and X taking a value of 1 with the probability 0.4 and a value of 0 with the probability 0.6. Calculate $\mathbb{E}(X)$ and $\mathbb{E}(2Y)$.
3. Consider the random experiment "rolling a regular die once" and define a random variable Z that gives the number that is rolled. How is Z distributed? Assume that you receive a payment of $z^2$ when the die shows the number $z$. What is the expected payment you get?

# Random Variable:

A variable whose possible values are numerical outcomes of a random phenomenon.

Typically represented by a capital letter (e.g. $X, Y$)

A random variable $X$ is a map $X: \Omega \to \mathbb{R}$

Set of events that can occur

$\rightarrow$ Two types: discrete and continuous random variables

# Discrete random variable

↳ number of possible outcomes / events is countable

→ finite number of distinct values

Example: number of patients in a doctor's office on a given day

## Expected value:

$$E(X) = \sum_{i=1}^{n} \overbrace{x_i}^{\text{values}} \cdot \underbrace{P(X = x_i)}_{\text{probability}}$$

where $x_1, \dots, x_n$ are the values that $X$ can take

## Example:

Suppose a variable $X$ can take the values 1, 2, 3 or 4.

The probabilities associated with each outcome are described by the following table:

| Outcome | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Probability | 0,1 | 0,3 | 0,4 | 0,2 |

What is the expected value?

$$\mathbb{E}(X) = 0{,}1 \cdot 1 + 0{,}3 \cdot 2 + 0{,}4 \cdot 3 + 0{,}2 \cdot 4 = 2{,}7$$

## Continuous random variable

↳ there are infinitely many possible outcomes

Example: height of a person or weight

A continuous random variable is not defined at specific values.

Instead, it is defined over an interval of values and is represented by the area under a curve (integral)

The probability of observing any single value is 0, since the number of values the random variable could take is infinite.

## Expected value:

$$\mathbb{E}(X) = \int_a^b x \cdot \underbrace{f(x)}_{\text{probability density function (pdf) of } X} dx$$

↳ gives the probability density at each point in the interval

# Example:

Suppose $X$ is a continuous random variable defined on the interval $[0,1]$

pdf: $f(x) = 2x$ for $0 \leq x \leq 1$

$f(x) = 0$ otherwise

What is the expected value?

$$\mathbb{E}(X) = \int_a^b x \cdot f(x)\,dx = \int_0^1 x \cdot 2x\,dx = \int_0^1 2x^2\,dx = \left[\frac{2}{3}x^3\right]_0^1$$

$$= \frac{2}{3}$$

integration:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}$$

$\boxed{1.2}$

$\mathbb{E}(X)$ :

$P(X=1) = 0{,}4$

$P(X=0) = 0{,}6$

$$\mathbb{E}(X) = \sum_{i=0}^{1} x_i \cdot P(X=x_i) = 0 \cdot 0{,}6 + 1 \cdot 0{,}4 = 0{,}4$$

$\mathbb{E}(2Y)$:

$Y \underset{\text{is distributed as}}{\sim} U([0,1]) \rightarrow$ Uniform distribution

cumulative distribution function: cdf

Gives the answer to:

"What is the probability that $Y$ is weakly less than some value $y$?"

$F(y) = 0{,}3$

↳ The probability of $Y$ being less than $y$ is $0{,}3$.

The derivative of $F$ is called the probability density function pdf ($f$)

For $Y \sim U([0,1])$

cdf:

$$F(y) = \begin{cases} 0 & \text{for } y < 0 \\ y & y \in [0,1] \\ 1 & y > 1 \end{cases}$$

pdf:

$$f(y) = \begin{cases} 1 & y \in [0,1] \\ 0 & \text{else} \end{cases}$$

Any interval of numbers of equal length has an equal probability of being observed.

Example: Fix an interval of length 0.1.

The probability that $y$ falls in the subinterval $[0.2, 0.3]$ is the same as the probability that $y$ falls in $[0.55, 0.65]$.


red and green area are of equal size

orange area is equal to 1

$$\mathbb{E}(Y) = \int_a^b y \, f(y) \, dy = \int_0^1 y f(y) \, dy = \int_0^1 y \cdot 1 \, dy = \left[ \frac{1}{2} y^2 \right]_0^1$$

$$= \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 0 = \frac{1}{2}$$

$$\mathbb{E}(2Y) = \int_0^1 2y \cdot f(y) \, dy = \int_0^1 2y \cdot 1 \, dy = \left[ y^2 \right]_0^1 = 1^2 - 0^2 = 1$$

$\boxed{1.3}$ Random experiment: roll a regular die once

What is the set of outcomes?

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Define a random variable $Z$:

$$Z : \Omega \to \mathbb{R}$$

How is $Z$ distributed?

The die is regular

$$\Rightarrow Z \sim \mathrm{Unif}(\{1, 2, 3, 4, 5, 6\})$$

$\rightarrow$ Uniform distribution in the discrete case

$Z^2$ is the random variable that defines our payment.

Expected payment:

$$\mathbb{E}(Z^2) = \sum_{i=1}^{6} i^2 \cdot P(Z=i)$$

$$= 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6}$$

$$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{1}{6} \cdot 91$$

$$= 15{,}1667$$

# Exercise 2

1. Let $f(x, y) = y^2 \ln(x) - y$. Compute $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$.
   What is the geometric interpretation $\frac{\partial f(x,y)}{\partial x}$?
2. Recall the definition of concave functions in one real variable.
3. Compute $\max_{x \in \mathbb{R}} g(x)$ with $g(x) = -2x^2 + 32x + 7$.

$\boxed{2.1}$ Partial derivatives:

$$f(x,y) = y^2 \ln(x) - y$$

$$\frac{\partial f(x,y)}{\partial x} = y^2 \cdot \frac{1}{x} - 0 = \frac{y^2}{x}$$

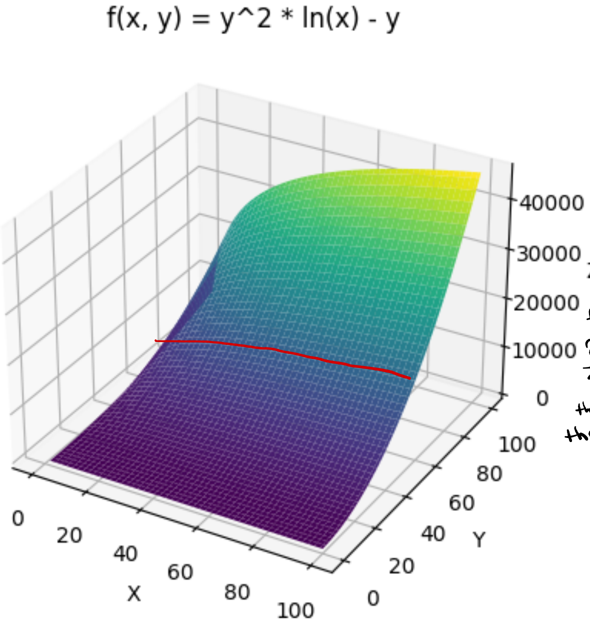
$$\frac{\partial f(x,y)}{\partial y} = 2y \ln(x) - 1$$

## Geometric interpretation:

How does a function change with respect to one of its independent variables while keeping the other independent variables constant.

For $\dfrac{\partial f(x, y)}{\partial x}$ ?

# 2.1: 3D Plot



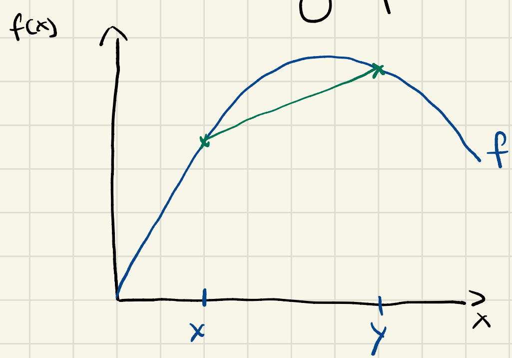

For a fixed $y$, $\frac{\partial f(x,y)}{\partial x}$ is just the slope of the red line

## 2.2

**Definition:** A function $f$ is called concave, if

$$f(\alpha x + (1-\alpha)y) \geq \alpha \cdot f(x) + (1-\alpha) f(y)$$

→ If you draw a straight line connecting any two points on the graph of the function, the line will always be below the graph.




For differentiable functions: $f$ concave $\Leftrightarrow f'' \leq 0$

## 2.3

$$\max_{x \in \mathbb{R}} g(x) \quad \text{with} \quad g(x) = -2x^2 + 32x + 7$$

How do we derive the maximum?

First-order-condition (FOC):

$$g'(x) = 0$$

$$g'(x) = -4x + 32 \overset{!}{=} 0$$

$$\Leftrightarrow \quad 32 = 4x$$

$$\Leftrightarrow \quad 8 = x$$

Second-order-condition (SOC): $g''(x) < 0$

$g''(x) = -4 < 0$ ✓ $\Rightarrow$ the maximum is attained at $x = 8$

the maximum value is: $g(8) = -2 \cdot 8^2 + 32 \cdot 8 + 7 = -128 + 256 + 7 = 135$

# Some helpful notes on mathematical tools

## Types of numbers

Natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

Integers: $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \ldots\}$

Rational numbers: $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$

Real numbers: $\mathbb{R}$ all rational and irrational numbers

## Summation sign:

→ abbreviation of a sum

$$\sum_{i=m}^{n} a_i := a_m + a_{m+1} + a_{m+2} + \cdots + a_n \qquad \text{where } a_i \in \mathbb{R} \text{ for all } i$$

# Calculations with fractions

$\frac{p}{q}$ $\begin{array}{l} \to \text{numerator} \\ \to \text{denominator} \end{array}$

Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$

Addition: $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

Addition without the same denominator: $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$

Division: $\frac{a}{b} : \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{c \cdot d}$

# Power Laws

$$a^0 = 1$$

$$a^1 = a$$

$$a^m \cdot a^n = a^{m+n}$$

$$(a^m)^n = a^{m \cdot n}$$

$$a^n \cdot b^n = (ab)^n$$

$$a^{-n} = \frac{1}{a^n}$$

$$\frac{a^n}{a^m} = a^{n-m}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

# Solving equations

## p-q-Formula:

$$x^2 + px + q = 0$$

$$\Rightarrow x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

## Binomial formula:

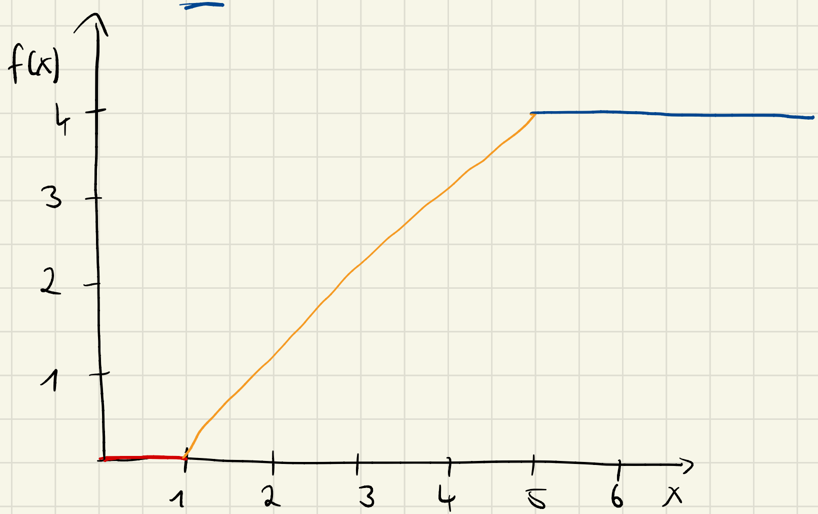
$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b) \cdot (a-b) = a^2 - b^2$$

# Piecewise Functions

$$f(x) = \begin{cases} 0, & x \leq 1 \\ x - 1, & 1 < x \leq 5 \\ 4, & 5 < x \end{cases}$$



# Differentiation Rules

| $f(x)$ | $f'(x)$ |
|---|---|
| $c$ | $0$ |
| $x^{\alpha}$ | $\alpha x^{\alpha-1}$ |
| $e^x$ | $e^x$ |
| $\ln(x)$ | $\frac{1}{x}$ |

$c$ is a constant $(c \in \mathbb{R})$

$\alpha \neq 0$

Examples:

$f(x) = x^4$ , $f'(x) = 4x^3$

$f(x) = \frac{1}{x^5} = x^{-5}$ , $f'(x) = -5x^{-6} = -\frac{5}{x^6}$

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ , $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Sum / Difference Rule: $(f \pm g)' = f' \pm g'$

Example: $(x^2 + 3x) = 2x + 3$

Constant Multiplication: $(\alpha f)' = \alpha f'$ for all $\alpha \in \mathbb{R}$

Example: $(2x^2)' = 4x$

Product Rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$

Example: $(\ln(x) \cdot 2x)' = \frac{1}{x} \cdot 2x + \ln(x) \cdot 2$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$ for $g \neq 0$

Example: $\frac{\ln(x)}{x^2} = \frac{\frac{1}{x} \cdot x^2 - \ln(x) \cdot 2x}{(x^2)^2} = \frac{x - \ln(x) \cdot 2x}{x^4} = \frac{1 - \ln(x) \cdot 2}{x^3}$

**Chain Rule:** $(g(f(x)))' = \underbrace{g'(f(x))}_{\text{exterior derivative}} \cdot \underbrace{f'(x)}_{\text{inner derivative}}$

Example: $(\ln(x^2))' = \frac{1}{x^2} \cdot 2x = \frac{2x}{x^2} = \frac{2}{x}$

# Integration : Power Rule

**Power Rule:** $\int x^n \, dx = \frac{1}{n+1} x^{n+1}$

Example: $\int x^3 \, dx = \frac{1}{4} x^4$

Example: $\int \sqrt{x} \, dx = \frac{1}{1+\frac{1}{2}} x^{\frac{1}{2}+1} = \frac{2}{3} x^{\frac{3}{2}}$