Imperfect Information in Health Care Markets

Exercise Session 2 - Introduction

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Questions?

Exercise 1

Assume that the utility function u_i represents i's preferences over a set of alternatives $X = \{x_1, x_2, \quad ..., x_n\}$. Show that

a) i's preferences are transitive;

b) the function v_i denoted by $v_i(x) = f(u_i(x))$ also represents i's preferences if f is a strictly increasing function.

c) Assume now that there are only 2 alternatives, i.e. $X = \{x_1, x_2\}$. Assume that there are 2 people in the society and person 1 prefers x_1 over x_2 while person 2 prefers x_2 over x_1. Choose some utility functions u_1 and u_2 to represent their preferences. Assume that society chooses the alternative x maximizing $u_1(x) + u_2(x)$.

- Which alternative does society choose with the utility functions you chose?
- Show that a transformation as in the previous subquestion can change society's choice. What is the problem and how does it come about?

1a)

What does the utility function u_i represents i's preferences mean?

u_i represents individual i's preferences means

$$x_1 \succsim x_2 \iff \underbrace{u_i(x_1)}_{\text{is a number in } \mathbb{R}} \geq u_i(x_2) \text{ for all } x_1, x_2 \text{ in } X$$

x_1 is weakly prefered over x_2

Example:

Person i weakly prefers an apple over an orange.

$\iff$ Person i's utility from an apple is at least as high as from an orange

e.g. $u_i(\text{apple}) = 10$, $u_i(\text{orange}) = 5$ $\quad u_i(\text{apple}) > u_i(\text{orange})$

What are transitive preferences?

i's preferences are transitive, if

$x_1 \succsim x_2$ and $x_2 \succsim x_3$ then $x_1 \succsim x_3$

for all x_1, x_2, x_3 in X

↳ set of alternatives

We want to show:

If i's preferences can be represented by a utility function, then the preferences are transitive.

Proof:

Let us assume that $x_1 \succsim x_2$ and $x_2 \succsim x_3$ for some x_1, x_2, x_3 in X.

As the preferences can be represented by u_i, we know that

$$u_i(x_1) \geq u_i(x_2) \quad \text{and} \quad u_i(x_2) \geq u_i(x_3)$$

$$\Rightarrow u_i(x_1) \geq u_i(x_2) \geq u_i(x_3)$$

→ all of these are numbers

and in particular $u_i(x_1) \geq u_i(x_3)$

as the $\geq$ relation on the real numbers is (naturally) transitive

→ This implies that $x_1 \succsim x_3$. $\square$

$\boxed{1b)}$

$V_i(X) = f(u_i(X))$

We want to show:

$u(x) = x^2$ $f(x) = 2x$ chained functions

$V(x) = f(u(x)) = 2 \cdot (x^2)$

e.g. $V(3) = f(u(3)) = 2 \cdot 3^2 = 18$

$v_i(X) = f(u_i(X))$ also represents i's preferences for f being a $\underbrace{\text{strictly increasing function}}_{f'(X) > 0 \text{ for all } X}$

$$x_1 \succsim x_2 \iff f(u_i(x_1)) \geq f(u_i(x_2)) \text{ for all } x_1, x_2 \text{ in } X$$

We have to show:

(i) If $x_1 \succsim x_2$ then $f(u_i(x_1)) \geq f(u_i(x_2))$

(ii) If $f(u_i(x_1)) \geq f(u_i(x_2))$ then $x_1 \succsim x_2$

(i) We know that $x_1 \succsim x_2$:

As u_i represents i's preferences, this means that $u_i(x_1) \geq u_i(x_2)$

→ This implies $f(u_i(x_1)) \geq f(U_i(x_2))$ as f is strictly increasing

(ii)

We know that $f(u_i(x_1)) \geq f(u_i(x_2))$.

We can apply the inverse function f^{-1} of f to this inequality.

$f^{-1}(f(x)) = x$ An inverse is a function that serves to "undo" another function.

$$\Rightarrow f^{-1}(f(u_i(x_1))) \geq f^{-1}(f(u_i(x_2)))$$

$$\Leftrightarrow u_i(x_1) \geq u_i(x_2) \Leftrightarrow x_1 \succsim x_2$$

f^{-1} is also strictly increasing due to the derivative of the inverse function $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$

$\boxed{1c)}$

Assume:

Person 1: $u_1(x_1) = 1$, $u_1(x_2) = 0$

$\rightarrow$ Person 1 prefers x_1

Person 2: $u_2(x_1) = 0$, $u_2(x_2) = 0{,}1$

$\rightarrow$ Person 2 prefers x_2

What would society choose? (maximize $u_1(x) + u_2(x)$)

$u_1(x_1) + u_2(x_1) = 1 + 0 = 1$

$u_1(x_2) + u_2(x_2) = 0 + 0{,}1 = 0{,}1 < 1$

$\rightarrow$ Society would choose x_1

Show that a transformation as in 1b) can change society's choice.

Now assume that person 2 reports the utility function

$\tilde{U}_2$ with $\tilde{U}_2(x_1) = 0$, $\tilde{U}_2(x_2) = 10$

→ Transformation by $f(x) = 100 \cdot x$

→ Person 2 still prefers x_2

What would society choose now?

$$u_1(x_1) + \tilde{U}_2(x_1) = 1 + 0 = 1$$

$$u_1(x_2) + \tilde{U}_2(x_2) = 0 + 10 = 10 > 1$$

Utility is an ordinal concept

→ Society will choose x_2

Exercise 2

Assume that there are m people in society and society has to choose an option from $X = \{x_1, x_2, \quad ..., x_n\}$. The preferences of each member of society can be represented by a utility function u_i. Society chooses the alternative $x \in X$ maximizing $\sum_{i=1}^{m} u_i(x)$. Show that the chosen alternative is Pareto efficient.

$\boxed{2}$

Pareto efficient:

There exists no alternative that could make one individual better off without making at least one individual wore off

We want to show:

If society chooses the alternative $x \in X$ maximizing $\sum_{i=1}^{m} U_i(x)$, then the chosen alternative is Pareto efficient

Proof:

Assume society chose a state $y \in X$ maximizing $\sum_{i=1}^{n} u_i(x)$

We want to show that y is Pareto efficient.

Proof by contradiction:

We assume that y is not Pareto efficient, this means that there exists some alternative $\tilde{x} \in X$ that makes at least one person strictly better off than y and that makes all the other persons not worse off.

This means that one person has a higher utility from $\tilde{x}$ and all others have at least the same utility.

$$\Rightarrow \sum_{i=1}^{n} u_i(\tilde{x}) > \sum_{i=1}^{n} u_i(y) \quad \lightning$$

This is a contradiction, since y was supposed to maximize $\sum_{i=1}^{n} u_i(x)$

$\Rightarrow$ y has to be Pareto efficient. $\square$

Exercise 3

Assume i's preferences over lotteries on the set of outcomes $\{healthy, ill, dead\}$ satisfy the assumptions of the von Neumann-Morgenstern expected utility theorem and can therefore be represented by three numbers $u^{healthy}, u^{ill}$ and u^{dead}. Assume that $u^{healthy} = 1$, $u^{ill} = 0.75$ and $u^{dead} = 0$.

a) Treatment 1 leads to the probability distribution (0.3, 0.5, 0.2) (over $\{healthy, ill, dead\}$) while treatment 2 leads to the probability distribution (0.4, 0.3, 0.3). Which treatment does i prefer?

b) Show that i's preferences over lotteries can also be represented by the three numbers $v^healthy = a \cdot u^{healthy} + b$, $v^{ill} = a \cdot u^{ill} + b$ and $v^{dead} = a \cdot u^{dead} + b$ where $a > 0$ and $b \in \mathbb{R}$ are some real numbers.

$\boxed{3a)}$ Set of outcomes: {healthy, ill, dead}

$u^{healthy} = 1$, $u^{ill} = 0{,}75$, $u^{dead} = 0$

Treatment 1:

Probability distribution $(0{,}3, 0{,}5, 0{,}2)$

$$(\underset{healthy}{0{,}3},\ \underset{ill}{0{,}5},\ \underset{dead}{0{,}2})$$

Expected utility of treatment 1:

$0{,}3 \cdot u^{healthy} + 0{,}5 \cdot u^{ill} + 0{,}2 \cdot u^{dead} = 0{,}3 \cdot 1 + 0{,}5 \cdot 0{,}75 + 0{,}2 \cdot 0 = 0{,}675$

Treatment 2: Probability distribution $(0{,}4, 0{,}3, 0{,}3)$

Expected utility of treatment 2:

$0{,}4 \cdot u^{healthy} + 0{,}3 \cdot u^{ill} + 0{,}3 \cdot u^{dead} = 0{,}4 \cdot 1 + 0{,}3 \cdot 0{,}75 + 0{,}3 \cdot 0 = 0{,}625 < 0{,}675$

$\rightarrow$ Person i would choose treatment 1

3b)

$$V^{healthy} = a \cdot U^{healthy} + b$$

$$V^{ill} = a \cdot U^{ill} + b \qquad \text{where } a > 0 \text{ and } b \in \mathbb{R}$$

$$V^{dead} = a \cdot U^{dead} + b$$

We want to show that $V^{healthy}$, V^{ill}, and V^{dead} also represent i's preferences over lotteries on the set of outcomes.

Compute expected utility from some random lottery $(p, q, 1-p-q)$ with $p, q, 1-p-q$ in $[0,1]$

prob. of healthy, prob. of ill, prob. of dead

$$\mathbb{E}(v(p, q, 1-p-q)) = p \cdot v^{healthy} + q \cdot v^{ill} + (1-p-q) \cdot v^{dead}$$

$$= p \cdot (a \cdot u^{healthy} + b) + q \cdot (a \cdot u^{ill} + b) + (1-p-q) \cdot (a \cdot u^{dead} + b)$$

$$= \underbrace{p \cdot b + q \cdot b + (1-p-q) \cdot b}_{(\underbrace{p+q+1-p-q}_{=1}) \cdot b = b} + a \cdot \underbrace{\left(p \cdot u^{healthy} + q u^{ill} + (1-p-q) u^{dead} \right)}_{\text{old utility see a)}}$$

$$= b + a \cdot \mathbb{E}(u(p, q, 1-p-q))$$

We can see that we just applied the transformation function $f(x) = a \cdot x + b$ to the old utility.

Since $f'(x) = a > 0$ (by assumption), this is a positive monotone transformation and results in the same preferences

↳ a seen in exercise 1b)

c) Show by means of an example that i's preferences are not necessarily represented by $v^{healthy} = f(u^{healthy}), v^{ill} = f(u^{ill})$ and $v^{dead} = f(u^{dead})$ for some strictly increasing function f. Why does this not contradict our result from exercise 1 above?

3c)

Example: $f(x) = \sqrt{x}$, which is $\underbrace{\text{strictly increasing}}_{f'(x) > 0}$ on $(0, \infty)$

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}} > 0$ for $x > 0$

Before: $u^{healthy} = 1$, $u^{ill} = 0{,}75$, $u^{dead} = 0$

Now: $v^{heatlhy} = f(u^{healthy}) = \sqrt{1} = 1$

$v^{ill} = \sqrt{0{,}75} \approx 0{,}866$

$v^{dead} = \sqrt{0} = 0$

Compare the two lotteries:

Lottery 1: $(0, 1, 0)$, Lottery 2: $(0{,}4 \,,\, 0{,}5 \,,\, 0{,}1)$

$v^{healthy} = 1$
$v^{ill} \approx 0{,}866$
$v^{dead} = 0$

$u^{healthy} = 1$
$u^{ill} = 0{,}75$
$u^{dead} = 0$

Before the transformation:

Lottery 1:

$$\mathbb{E}(u(0,1,0)) = 0 \cdot u^{healthy} + 1 \cdot u^{ill} + 0 \cdot u^{dead} = 0 \cdot 1 + 1 \cdot 0{,}75 + 0 \cdot 0 = 0{,}75$$

Lottery 2:

$$\mathbb{E}(u(0{,}4,\ 0{,}5,\ 0{,}1)) = 0{,}4 \cdot u^{healthy} + 0{,}5\, u^{ill} + 0{,}1\, u^{dead} = 0{,}4 \cdot 1 + 0{,}5 \cdot 0{,}75 + 0{,}1 \cdot 0 = 0{,}775 > 0{,}75$$

→ Here, the person chooses lottery 2

After the transformation:

$$\mathbb{E}(v(0,1,0)) = 0 \cdot v^{healthy} + 1 \cdot v^{ill} + 0 \cdot v^{dead} = 0 \cdot \sqrt{1} + 1 \cdot \sqrt{0{,}75} + 0 \cdot \sqrt{0} = 0{,}866$$

$$\mathbb{E}(v(0{,}4,\ 0{,}5,\ 0{,}1)) = \underbrace{0{,}4 \cdot v^{healthy} + 0{,}5 \cdot v^{ill} + 0{,}1 \cdot v^{dead}}_{\neq \sqrt{0{,}4 \cdot 1 + 0{,}5 \cdot 0{,}75 + 0{,}1 \cdot 0} = f(\mathbb{E}(u(0{,}4,\ 0{,}5,\ 0{,}1)))} = 0{,}4 \cdot 1 + 0{,}5 \cdot 0{,}75 + 0{,}1 \cdot 0 = 0{,}833 < 0{,}866$$

→ Here, the person chooses lottery 1

The transformation was not applied to the whole utility function, but just to each of the u_i's seperately.