Mechanism Design

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Outline

Introduction

- we turned to Bayesian Nash equilibrium to overcome the problem of budget balance (this was the problem of the Clarke-Groves mechanism)
- the expected externality mechanism ensures budget balance but has another problem: participation
  - if agents cannot be forced to participate (as in bilateral trade), the expected externality mechanism does not work
- today:
  is there a mechanism that is
  1. efficient,
  2. has a balanced budget and
  3. every agent of every type is willing to voluntarily participate

Introduction

- we analyze this question in a trade setting
- 1 buyer with value v which is uniformly distributed on $[0, 1]$
- 1 seller with costs c that are uniformly distributed on $[0, 1]$
- Bayesian Nash equilibrium implementation
- using the revelation principle we concentrate on direct mechanisms, i.e. buyer announces $\hat{v}$, seller announces $\hat{c}$ and it must be incentive compatible to announce the true value/costs
- direct mechanism: after announcements $(\hat{v}, \hat{c})$ trade occurs with probability $y(\hat{v}, \hat{c})$ and a transfer $t(\hat{v}, \hat{c})$ is paid from buyer to seller
- utility buyer: $y(\hat{v}, \hat{c})v - t(\hat{v}, \hat{c})$
- utiltiy seller: $t(\hat{v}, \hat{c}) - cy(\hat{v}, \hat{c})$

Possible mechanism

Example (Seller posted price)

- seller sets a price
- buyer decides to buy at this price or not to buy
- which price maximizes expected profits?
- is this mechanism efficient?
- how would a direct revelation mechanism look like that gives the same outcome as the seller posted price mechanism?

Incentive compatibility

- revelation principle says that we can concentrate on direct mechanisms that are *incentive compatible* (announcing true value is equilibrium)
- which direct mechanisms are incentive compatible?
- for which functions $y(\hat{v}, \hat{c})$ and $t(\hat{v}, \hat{c})$ is announcing your true type an equilibrium?

Incentive compatibility buyer I

- we use capital letters for expected transfers/trade probabilities/utility
- $T(v) = E_c[t(v,c)]$
- $Y(v) = E_c[y(v,c)]$
- $U_b(v) = vY(v) - T(v)$
- if the mechanism with y and v is incentive compatible, then for all v and v'

$$vY(v) - T(v) \geq vY(v') - T(v')$$
$$v'Y(v') - T(v') \geq v'Y(v) - T(v)$$

- this is the same as writing

$$U_b(v) \geq U_b(v') + (v - v')Y(v')$$

$$U_b(v') \geq U_b(v) + (v' - v)Y(v)$$

Incentive compatibility buyer II

- if $v - v' > 0$ this can also be written as (if $v - v' < 0$, the inequalities hold in the opposite direction)

$$Y(v) \geq \frac{U_b(v) - U_b(v')}{v - v'} \geq Y(v')$$

- Implications:

if the direct mechanism consisting of the functions y and t is incentive compatible, then

- Y is increasing in v (as $v > v'$ implies $Y(v) \geq Y(v')$)
- the derivative of U_b has to be $Y(v)$: $U_b'(v) = Y(v)$ (take the limit $v' \to v$)
- do these implications intuitively make sense?

Incentive compatibility buyer III

Theorem (Envelope theorem and monotonicity)

A social choice function $(t(v, c), y(v, c))$ *is incentive compatible for the buyer if and only if*

- $U_b'(v) = Y(v)$ *and*
- $Y(v)$ *is increasing in* v
- *note: we have shown the "only if" part but we have not shown the "if" part (see MWG p. 888-889 for this)*

Incentive compatibility seller I

- we now do the same for the seller:
- $T(c) = E_v[t(v, c)]$
- $Y(c) = E_v[y(v, c)]$
- $U_s(c) = T(c) - cY(c)$
- if the mechanism with y and v is incentive compatible, then for all c and c'

$$T(c) - cY(c) \geq T(c') - cY(c')$$

$$T(c') - c'Y(c') \geq T(c) - c'Y(c)$$

- this is the same as writing

$$U_s(c) \geq U_s(c') + (c' - c)Y(c')$$

$$U_s(c') \geq U_s(c) + (c - c')Y(c)$$

Incentive compatibility seller II

- if $c - c' > 0$ this can also be written as (if $c - c' < 0$, the inequalities hold in the opposite direction)

$$Y(c) \leq \frac{U_s(c') - U_s(c)}{c - c'} \leq Y(c')$$

- Implications:

if the direct mechanism consisting of the functions y and t is incentive compatible, then

- Y is decreasing in c (as $c > c'$ implies $Y(c) \leq Y(c')$)
- the derivative of U_s has to be $Y(c)$: $U_s'(c) = -Y(c)$ (take the limit $c' \to c$)
- do these implications intuitively make sense?

Incentive compatibility seller III

Theorem (Envelope theorem and monotonicity)

A social choice function $(t(v,c), y(v,c))$ is incentive compatible for the seller if and only if

- $U_s'(c) = -Y(c)$ *and*
- $Y(c)$ *is decreasing in* c
- *note: we have shown the "only if" part but we have not shown the "if" part (see MWG p. 888-889 for this)*

Incentive compatibility: comments

- there are many functions $y(v, c)$ that lead to the same $Y(c)$ or the same $Y(v)$
- U and T are almost interchangeable:
  - if we know U_s and Y, we can always determine T as
$$T(c) = U_s(c) + cY(c)$$
- same for the buyer
$$T(v) = -U_b(v) + vY(v)$$
- in most mechanism design application, people search for the optimal y, U_s and U_b instead of searching for the optimal y and t

Participation constraints

- we wanted to search for an incentive compatible direct mechanism that
  1. is efficient,
  2. has a balanced budget and
  3. every agent of every type is willing to voluntarily participate
- ad 1.: $y(v, c) = 1$ if $v > c$ and $y(v, c) = 0$ otherwise; this implies $Y(c) = 1 - c$ and $Y(v) = v$
- ad 2.: as we only looked at a transfer paid from buyer to seller, this is fine in our formulation
- ad 3.: Participation constraints

$$U_b(v) \geq 0 \qquad \text{for all } v \in [0, 1]$$

$$U_s(c) \geq 0 \qquad \text{for all } c \in [0, 1]$$

Myerson-Satterthwaite Theorem

Theorem (Myerson-Satterthwaite)

No incentive compatible direct revelation mechanism satisfying budget balance and the participation constraints yields the efficient outcome. Therefore, by the revelation principle, no mechanism can achieve the ex post efficient outcome in the bilateral trade setting.

Myerson-Satterthwaite Theorem: Intuition

Example (efficient, buyer-friendly mechanism)

- buyer announces v; seller announces c
- if $v > c$, trade takes place at price c
- is this incentive compatible? (i.e. do both have incentives to announce their true type?)

Example (fair, efficient mechanism)

- buyer announces v; seller announces c
- if $v > c$, trade takes place at price $\frac{v+c}{2}$
- is this incentive compatible? (i.e. do both have incentives to announce their true type?)

Myerson-Satterthwaite Theorem: Proof I

- The proof is by contradiction. Suppose there was an incentive compatible (IC) direct mechanism satisfying participation constraints (PC) and efficiency (E).
- (E) implies that $Y(c) = 1 - c$ and $Y(v) = v$.
- (IC) implied $U_b'(v) = Y(v)$ (envelope theorem), therefore

$$\begin{aligned} U_b(v) &= U_b(0) + \int_0^v U_b'(\tilde{v})\, d\tilde{v} && \text{(fundamental thm of calculus)} \\ &= U_b(0) + \int_0^v Y(\tilde{v})\, d\tilde{v} && \text{(envelope thm)} \\ &= U_b(0) + \int_0^v \tilde{v}\, d\tilde{v} && \text{(efficiency)} \\ &= U_b(0) + \frac{v^2}{2} \end{aligned}$$

- as $T(v) = -U_b(v) + vY(v)$ we get $T(v) = -U_b(0) - \frac{v^2}{2} + v^2 = -U_b(0) + \frac{v^2}{2}$

Myerson-Satterthwaite Theorem: Proof II

- (IC) implies by the envelope theorem that $U_s'(c) = -Y(c)$, therefore

$$
\begin{aligned}
U_s(c) &= U_s(1) - \int_c^1 U_s'(\tilde{c})\, d\tilde{c} && \text{(fundamental thm of calculus)} \\
&= U_s(1) + \int_c^1 Y(\tilde{c})\, d\tilde{c} && \text{(envelope thm)} \\
&= U_s(1) + \int_c^1 1 - \tilde{c}\, d\tilde{c} && \text{(efficiency)} \\
&= U_s(1) + \frac{1}{2} - c + \frac{c^2}{2}
\end{aligned}
$$

- as $T(c) = U_s(c) + cY(c)$ we get $T(c) = U_s(1) + \frac{1}{2} - c + \frac{c^2}{2} + c(1-c) = U_s(1) + \frac{1}{2} - \frac{c^2}{2}$

Myerson-Satterthwaite Theorem: Proof III

- by budget balance, the expected transfer payment made by the buyer has to equal the expected transfer payment received by the seller:

$$\int_0^1 T(c)\,dc - \int_0^1 T(v)\,dv = 0$$

- Plugging in the expressions we derived for T gives

$$\int_0^1 U_s(1) + \frac{1}{2} - \frac{c^2}{2}\,dc - \int_0^1 -U_b(0) + \frac{v^2}{2}\,dv = 0$$

$$U_s(1) + U_b(0) + \frac{1}{3} - \frac{1}{6} = 0$$

$$U_s(1) + U_b(0) + \frac{1}{6} = 0$$

but this is impossible because $U_s(1) \geq 0$ and $U_b(0) \geq 0$ by (PC). □

Myerson-Satterthwaite Theorem: Take aways and economics

- same theorem applies more generally (other distributions, non-identical type spaces etc.)
- contrast to "Coase theorem"
- it can be shown that a double auction is the most efficient mechanism in our setting
- economic implication:
  - "market failure" alone does not justify government intervention
  - no "government intervention mechanism" can improve on the double auction (which is a private market mechanism) in this example!

Teaser

Have you noticed that ice cream parlors in Copenhagen usually have a very non-linear pricing policy, e.g. 1 scoop 20kr, 2 scoops 30 kr, 3 scoops 35 kr, 4 scoops 38 kr.
Assume that people differ in how much they like ice cream. If you own an ice cream parlor and you want to make as much profits as possible, how should you set your prices?

Review questions

- Why does the expected externality mechanism not work in our trade setting?
- Define incentive compatibility (losely) and describe what role it plays in the revelation principle!
- Check the envelope theorem and its derivation again!
- What does the Myerson-Satterthwaite theorem state? What are the economic implications?

Exercises I

This exercise gives you an alternative way to arrive at the envelope theorem. Incentive compatibility implies that stating the true type is the utility maximizing type announcement for every agent. The buyer's expected utility when being type v and announcing $\hat{v}$ is $vY(\hat{v}) - T(\hat{v})$.

Write the first order condition of the maximization problem of the buyer $\max_{\hat{v}} vY(\hat{v}) - T(\hat{v})$. Remember that we defined $U_b(v) = vY(v) - T(v)$. Take the derivative of U_b to get $U_b'(v)$. Use the first order condition from the last step and incentive compatibility to derive the envelope condition $U_b'(v) = Y(v)$.

Derive the monotonicity condition from the second order condition of $max_{\hat{v}} vY(\hat{v}) - T(\hat{v})$, i.e. show that monotonicity holds in every incentive compatible mechanism (you will have to use the fact that the first order condition holds for all types).

Exercises II

Show that envelope theorem and monotonicity constraint are (together!) sufficient for incentive compatibility.