Math for Mechanism Design

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1. Taking expectations for continuous random variables

Here I just want to remind you how to take expectations of continuous random variables. Let θ be uniformly distributed on $[0,1]$. Intuitively, you know that the expectation is $1/2$. But how would you calculate this? A uniform ditribution on $[0,1]$ has the density function

$$\phi(\theta) = 1 \qquad \text{if } \theta \in [0,1]$$

and 0 if $\theta \notin [0,1]$. The expected value can now be computed as

$$\int_0^1 \theta * 1 \; d\theta = \left[\frac{\theta^2}{2}\right]_0^1 = \frac{1}{2} - 0 = 1/2.$$

In general, if a random variable θ is distributed with density ϕ on $[\underline{\theta}, \bar{\theta}]$, the expected value of θ is given by

$$\int_{\underline{\theta}}^{\bar{\theta}} \theta\phi(\theta) \; d\theta.$$

Let's get back to our uniform example. Now suppose you have to pay 1 to participate in a lottery that pays you 3θ. What is your expected payoff of this lottery? Again this is intuitively $3/2 - 1 = 0.5$ but how do we compute it?

$$\int_0^1 (3\theta - 1) * 1 \; d\theta = \left[3\theta^2/2 - \theta\right]_0^1 = 3/2 - 1 = 0.5.$$

What if the lottery pays $2\theta^2$ instead of 3θ? Then the general formula

$$\mathbf{E}[u(\theta)] = \int_{\underline{\theta}}^{\bar{\theta}} u(\theta)\phi(\theta) \; d\theta$$

helps us (here u is some payoff function that tells us how much money you will have in state θ). In our example, $\phi(\theta) = 1$ and the payoff is $u(\theta) = 2\theta^2 - 1$ (i.e. in state θ you get $2\theta^2$ but you had to pay 1 to participate). This means your expected payoff from this lottery is

$$\int_0^1 (2\theta^2 - 1) * 1 \, d\theta = \left[2\theta^3/3 - \theta\right]_0^1 = 2/3 - 1 - 0 = -1/3.$$

2. Leibniz rule

The Leibniz rule tells you how to take the derivative of a function that is an integral. We will only use one very special case of it. Therefore, I show you this special case and give the complete rule only for sake of completeness below.

What we want to look at are functions like the following

$$f(x) = \int_0^x g(y) \, dy.$$

What is the derivative of f? The Leibniz rule says that[1]

$$f'(x) = g(x).$$

The way you should think about this is that the integral is the area below the curve. So, draw some (continuous) function g now. Seriously, do it! The integral $\int_0^x g(y)\, dy$ is the area between the axis and the function g from 0 to some x (take some $x > 0$ and shade this area in the graph you just drew). The derivative of f with respect to x gives the answer to the following question: How does the size of the shaded area change if you make x a bit bigger. From the graph, it should be clear that the area gets approximately $g(x) * dx$ bigger if you increase x by dx. This means that $f'(x) = g(x)$ (one marginal unit increase in x increases f by $g(x)$).

With a similar intuition the derivative of the function

$$h(x) = \int_x^1 g(y) \, dy$$

is

$$h'(x) = -g(x).$$

[1]Strictly speaking, we should assume for this that g is continuous at x. If you go through the next paragraph, you should understand why.

The complete Leibniz rule (that we won't need) uses a function

$$f(x) = \int_{a(x)}^{b(x)} g(x,y)\, dy$$

and says (assuming that all the functions are nicely behaved, i.e. a, b and g are differentiable at x) that

$$f'(x) = -a'(x)g(x, a(x)) + b'(x)g(x, b(x)) + \int_{a(x)}^{b(x)} \frac{\partial g}{\partial x}(x,y)\, dy.$$

3. Integration by parts

This is a little trick that allows you to solve some seemingly tricky integration problems. We will use it a lot in the course.

You might remember how to integrate simple polynomial functions, e.g.

$$\int_2^3 x^2 + 1\, dx = \left[\frac{x^3}{3} + x\right]_2^3 = \frac{27}{3} + 3 - \frac{8}{3} - 2 = 8 - \frac{2}{3}.$$

However, constructing the "antiderivative" is not always this easy. Think for example of the following integral:

$$\int_4^5 xe^x\, dx$$

Here the following little rule helps us:

Theorem 1. $\int_a^b u(x)v'(x)\, dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\, dx$

Here u and v are arbitrary functions of x (strictly speaking u and v should be differentiable "almost everywhere"). How does this help us? Let's take the example above and say $v'(x) = e^x$ and $u(x) = x$. This implies that $v(x) = e^x$ because the exponential function is its own derivative and $u'(x) = 1$. Then the formula above gives us

$$\int_4^5 xe^x\, dx = [xe^x]_4^5 - \int_4^5 1 * e^x\, dx.$$

This last integral is something we can solve: The antiderivative of e^x is e^x and therefore the last expression can be rewritten as

$$= 5e^5 - 4e^4 - [e^x]_4^5 = 4e^4 - 5e^5 - e^5 + e^4 = 5e^4 - 6e^5.$$

Another example:

$$\int_a^b x^2 log(x)\ dx$$

Here we use $u(x) = log(x)$ and $v'(x) = x^2$. Therefore, we get $u'(x) = 1/x$ and $v(x) = x^3/3$. Plugging this into our rule gives

$$\int_a^b x^2 log(x)\ dx = \left[log(x)x^3/3\right]_a^b - \int_a^b \frac{1}{x}\frac{x^3}{3}\ dx = log(b)b^3/3 - log(a)a^3/3 - \int_a^b \frac{x^2}{3}\ dx$$
$$= log(b)b^3/3 - log(a)a^3/3 - \left[\frac{x^3}{9}\right]_a^b = log(b)b^3/3 - log(a)a^3/3 - \frac{b^3}{9} + \frac{a^3}{9}.$$

A double integral example:

Integration by parts can sometimes be used to transform double integrals to simple integrals:

$$\int_0^1 \left(2x \int_a^x f(y)\ dy\right)\ dx$$

Here $f(y)$ is some arbitrary function. We choose $u(x) = \int_a^x f(y)\ dy$ and $v'(x) = 2x$. This implies that $u'(x) = f(x)$ (this uses the Leibniz rule above!) and $v(x) = x^2$. Integration by parts gives now

$$\int_0^1 \left(2x \int_a^x f(y)\ dy\right)\ dx = \left[x^2 \int_a^x f(y)\ dy\right]_0^1 - \int_0^1 f(x)x^2\ dx = \int_a^1 f(y)\ dy - \int_0^1 f(x)x^2\ dx.$$

If you check the last expression, you see that we simplified the complicated double intergral we started out with into two simpler "normal" integrals.

3.1. Why does integration by parts work? (for experts only)

Integration by parts is actually a rule that you already know (but in disguise). You might remember the differentiation rule:

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$$

Now rearrange this last equation by subtracting $u'(x)v(x)$ on both sides. This gives

$$(u(x)v(x))' - u'(x)v(x) = u(x)v'(x).$$

Now integrate both sides from a to b:

$$\int_a^b \{(u(x)v(x))' - u'(x)v(x)\}\ dx = \int_a^b u(x)v'(x)\ dx$$

Now we can split up the integral on the left hand side and get

$$\int_a^b (u(x)v(x))' \; dx - \int_a^b u'(x)v(x) \; dx = \int_a^b u(x)v'(x) \; dx.$$

Taking derivatives and integrating are inverse operations ("they cancel each other out"). Therefore, the first term on the left hand side can be rewrritten $\int_a^b (u(x)v(x))' \; dx = [u(x)v(x)]_a^b$. If you plug this in, you get exactly the integration by parts formula!

4. Fundamental Theorem of Calculus

This theorem states a fact that you probably knew already: Integration and taking derivatives are opposite operations. Formally stated the theorem comes in two parts: The first part roughly says that "the derivative of the integral of an integrable function is the function itself". The second part says that "the integral of the derivative of a differentiable function is the function itself".

Theorem 2 (Fundamental Theorem of Calculus)**.** *First part: Let $f : [a, b] \to \mathbb{R}$ be a bounded function which is continuous almost everywhere. Let $F : [a, b] \to \mathbb{R}$ satisfy*

$$F(s) = F(a) + \int_a^s f(t)\, dt \qquad \text{for } a \leq s \leq b. \tag{1}$$

Then, F is continuous on $[a, b]$ and $F'(s)$ exists and equals $f(s)$ for every s at which f is continuous.

*Second part: Let $F : [a, b] \to \mathbb{R}$ be a continuously differentiable function and let $f : [a, b] \to \mathbb{R}$ be a bounded and Riemann integrable function such that $F' = f$ almost everywhere on $[a, b]$. Then (**??**) holds.*

5. Implicit Function Theorem

It is enough to read this part of the handout around lecture 6.

Often we just know that the optimal decision satisfies a first order condition. However, we cannot solve this first order condition explicitly. In the non-linear pricing example that we cover in the lecture, for example, we arrive at the first order condition

$$v_q(q, \theta) - c - \frac{1 - \Phi(\theta)}{\phi(\theta)} v_{q\theta}(q, \theta) = 0$$

which had to be satisfied by the optimal solution q. Let us make the assumptions $v_{qq} < 0$ and $v_{qq\theta} \geq 0$. Then the left hand side of the previous equation is strictly decreasing in q. (Yes?) This means that there is one unique $q = q(\theta)$ solving this equation for a given θ. In this sense, the first order condition defines the optimal quantity $q(\theta)$ as a function of type θ.

While determining the first order condition, we had assumed that the optimal solution is monotone (i.e. we had neglected the monotonicity constraint). Is the $q(\theta)$ defined by the first order condition actually monotone? The implicit function theorem allows us to compute the derivative of $q(\theta)$ although we cannot really solve the first order condition for $q(\theta)$ (all we know is that this first order condition uniquely defines $q(\theta)$ but we cannot explicitly say what $q(0.3)$ is because we use general functions v and ϕ).

Before looking at the theorem let us get intuitively why the implicit function theorem works. We will assume here that (i) $\phi(\theta)/(1 - \Phi(\theta))$ is non-decreasing, (ii) $v_{q\theta} > 0$ ("single-crossing") and (iii) $v_{q\theta\theta} \leq 0$. This implies that the following for the left hand side of the first order condition: If we hold $q(\theta)$ fix but increase θ, then the left hand side increases. (Yes?) So if we increase θ from θ to $\theta + \varepsilon$ while holding q fixed at $q(\theta)$, the left hand side increases and therefore is positive (by the first order condition it is 0 at our starting point θ and $q(\theta)$). Now the question is: In order for the first order condition to hold at $\theta + \varepsilon$, does $q(\theta + \varepsilon)$ have to be higher or lower than $q(\theta)$? Remember that the left hand side of the first order condition is decreasing in q. To make it smaller, we therefore get that $q(\theta + \varepsilon)$ has to be higher than $q(\theta)$. Hence, $q(\theta)$ is indeed increasing!

Now (a very much simplified but for our case completely sufficient version of) the implicit function theorem itself. The function V below stands in our example for the left hand side of the first order condition.

Theorem 3 (Simple Implicit Function Theorem)**.** *Let* $V : \mathbb{R}_+ \times [\underline{\theta}, \bar{\theta}] \to \mathbb{R}$ *be a continuously differentiable function. Assume that* $q(\theta)$ *is such that* $V(q(\theta), \theta) = 0$*. If* $\partial V(q(\theta), \theta)/\partial q \neq 0$*, then* $q(\theta)$ *is differentiable and*

$$q'(\theta) = -\frac{\frac{\partial V}{\partial \theta}(q(\theta), \theta)}{\frac{\partial V}{\partial q}(q(\theta), \theta)}.$$

Hence, we can even determine a bit more than just the sign of the derivative of $q(\theta)$ if we like to: In our example, we have $V(q,\theta) = v_q(q,\theta) - c - \frac{1-\Phi(\theta)}{\phi(\theta)} v_{q\theta}(q,\theta)$ and therefore we get

$$q'(\theta) = -\frac{v_{q\theta}(q(\theta),\theta) - c - \frac{1-\Phi(\theta)}{\phi(\theta)} v_{q\theta\theta}(q(\theta),\theta) + \frac{\phi^2(\theta)+(1-\Phi(\theta))\phi'(\theta)}{\phi^2(\theta)} v_{q\theta}(q(\theta),\theta)}{v_{qq}(q(\theta),\theta) - c - \frac{1-\Phi(\theta)}{\phi(\theta)} v_{qq\theta}(q(\theta),\theta)}.$$

If you check the assumptions we made above, you will realize that the numerator is strictly positive and the denominator is strictly negative. Together with the "$-$" in front this implies that q is strictly increasing.