Mechanism Design

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Outline

1. Efficiency and optimal mechanisms
2. Optimal non-linear pricing
3. Revision questions and exercises etc.

Optimal mechanisms

- What is ”optimal”?
- Pareto efficiency is probably a good starting point
- ”optimal” might depend on the perspective

Efficiency in Bayesian environments

f is a Pareto-efficient social choice function if it is feasible and there is no other feasible scf $\hat{f}$ that makes everyone better off (and some agent strictly better off)

- *two things to clarify*
    - *what does "feasible" mean?*
    - *what does "making an agent better off" mean?*

Efficiency in Bayesian environments: Feasibility

- f has to be implementable
- by revelation principle: f has to be incentive compatible (truthfully implementable)

$F_{BIC} = \{f : \Theta \to X : \ f \text{ is Bayesian incentive compatible } \}$

- agents must be willing to participate

$$F_{IR} = \{f : \Theta \to X : \ f \text{ is individually rational}\}$$

- we say f is *feasible* if $f \in F^* = F_{IR} \cap F_{BIC}$

Efficiency in Bayesian environments: "better off"

- timing matters
  - ex ante: before knowing the types
    higher expected utility where expectation is taken over own and other agents' types
  - interim: when knowing own type but not other agents' types
    higher expected utility where expectation is taken over other agents' types
  - ex post: when all types are known
    higher utility given own and other types

Efficiency in Bayesian environments: ex ante efficiency

- denote by $U_i(\theta_i|f)$ the expected utility of agent i with type θ_i under the mechanism f (interim expected utility)
- denote by $U_i(f) = E_{\theta_i} U_i(\theta_i|f)$ the ex ante expected utility of agent i

Definition (ex ante efficient)

A social choice function $f \in F^*$ is *ex ante efficient* in F^* if there is no $\hat{f} \in F^*$ such that $U_i(\hat{f}) \geq U_i(f)$ for all $i = 1, \ldots, I$ and $U_i(\hat{f}) > U_i(f)$ for some i.

Efficiency in Bayesian environments: interim efficiency

Definition (interim efficient)

A social choice function $f \in F^*$ is *interim efficient* in F^* if there is no $\hat{f} \in F^*$ such that $U_i(\theta_i|\hat{f}) \geq U_i(\theta_i|f)$ for all $\theta_i \in \Theta_i$ and all $i = 1, \ldots, I$ and $U_i(\theta_i|\hat{f}) > U_i(\theta_i|f)$ for some i and some $\theta_i \in \Theta_i$.

Ex ante and interim efficiency

Lemma (ex ante efficiency more demanding than interim efficiency).

If f is ex ante efficient (and ϕ has full support), then f is also interim efficient. □

Proof.

Let f be ex ante efficient. Suppose that f was not interim efficient, i.e. suppose there exists an $\hat{f}$ such that $U_i(\theta_i|\hat{f}) \geq U_i(\theta_i|f)$ for all i and strict inequality for some i. But then $U_i(\hat{f}) = E_{\theta_i}\left[U_i(\theta_i|\hat{f})\right] \geq E_{\theta_i}\left[U_i(\theta_i|f)\right] = U_i(f)$ for all i with strict inequality for some i. This contradicts that f is ex ante efficient. □

Optimal mechanisms

- whether ex ante or interim efficiency is appropriate depends on context
  - do the agents already know their types at the time of the welfare analysis?
- many mechanisms are ex ante/interim efficient
- we will usually take the perspective of one player and choose the efficient (and feasible) mechanism that maximizes this player's utility
  - determine the pricing scheme that maximizes the profits of a seller
  - the regulatory scheme that maximizes consumer surplus (or a mix of producer and consumer surplus)
  - the income tax scheme that maximizes the payoff of a utilitarian government

Optimal non-linear pricing by a monopolist I

- Monopolist sells a good and has costs $c * q$, i.e. constant marginal costs c
- there is a single consumer with quasilinear utility $v(q, \theta) - p$ where q is the quantity the consumer gets and p is the price paid
- we assume $v_q > 0$, $v_{qq} < 0$ and $v(0, \theta) = 0$
- θ is the consumer's type which is private information
- we assume $v_{q\theta} > 0$
  ("Spence-Mirrlees condition", "single crossing")
  a higher type has a higher marginal utility
- from the monopolist's point of view, θ is distributed on $[0, 1]$ with density function ϕ (and corresponding distribution function Φ) and we assume $\phi(\theta) > 0$ for all $\theta \in [0, 1]$

Optimal non-linear pricing by a monopolist I

- monopolist can charge a non-linear pricing schedule $p(q)$ (like ice cream in CPH)
- Our question:
  What is the expected profit maximizing pricing scheme for the monopolist?
  (i.e. this mechanism will be ex ante efficient)

Optimal non-linear pricing by a monopolist III

- revelation principle:
  - we can concentrate on incentive compatible direct revelation mechanisms
- $q(\theta)$ tells which quantity type θ gets
- $t(\theta)$ tells how much type θ has to pay
- incentive compatibility: $\theta = arg\max_{\hat{\theta}} v(q(\hat{\theta}), \theta) - t(\hat{\theta})$
- individual rationality: $U(\theta) = v(q(\theta), \theta) - t(\theta) \geq 0$

Optimal non-linear pricing by a monopolist IV

Theorem (envelope theorem and monotonicity)

A mechanism is incentive compatible if and only if

- $U(\theta) = U(0) + \int_0^\theta v_\theta(q(x), x)\, dx$
- $q(\theta)$ *is non-decreasing.*

Optimal non-linear pricing by a monopolist V

- expected profits

$$\pi = \int_0^1 (t(\theta) - cq(\theta))\, \phi(\theta)\, d\theta$$

substitute $U(\theta) = v(q(\theta), \theta) - t(\theta)$ for $t(\theta)$

$$\pi = \int_0^1 (v(q(\theta), \theta) - cq(\theta) - U(\theta))\, \phi(\theta)\, d\theta$$

substitute $U(\theta)$ from the envelope theorem

$$\pi = \int_0^1 (v(q(\theta), \theta) - cq(\theta))\, \phi(\theta)\, d\theta$$

$$- U(0) - \int_0^1 \int_0^\theta v_\theta(q(x), x)\, dx\, \phi(\theta)\, d\theta$$

Optimal non-linear pricing by a monopolist VI

use integration by parts to simplify the last term (first "part" is $\int_0^\theta v_\theta \, dx$ and second "part" is $\phi(\theta)$)

$$\pi = \int_0^1 (v(q(\theta), \theta) - cq(\theta))\, \phi(\theta)\, d\theta - U(0) \\ - \left(\int_0^1 v_\theta(q(x), x)\, dx - \int_0^1 v_\theta(q(\theta), \theta)\Phi(\theta)\, d\theta \right)$$

taking all the integrals together gives

$$\pi = \int_0^1 [v(q(\theta), \theta) - cq(\theta)]\, \phi(\theta) - [1 - \Phi(\theta)] v_\theta(q(\theta), \theta)\, d\theta - U(0)$$

or equivalently

$$\pi = \int_0^1 \left[v(q(\theta), \theta) - cq(\theta) - \frac{1 - \Phi(\theta)}{\phi(\theta)} v_\theta(q(\theta), \theta) \right] \phi(\theta)\, d\theta - U(0)$$

Optimal non-linear pricing by a monopolist VII

- the monopolist's maximization problem

$$\max_{q(\theta)} \int_0^1 \left[v(q(\theta),\theta) - cq(\theta) - \frac{1-\Phi(\theta)}{\phi(\theta)} v_\theta(q(\theta),\theta) \right] \phi(\theta)\, d\theta - U(0)$$

subject to

- individual rationality $U(\theta) \geq 0$ for all θ
- incentive compatibility: $q(\theta)$ is non-decreasing
- individual rationality is satisfied for all types if $U(0)≥ 0$
  $\rightarrow$ optimal to have $U(0) = 0$
- we will ignore the constraint "$q(\theta)$ non-decreasing" and check later whether it is satisfied

Optimal non-linear pricing by a monopolist VIII

- then we are left with

$$\max_{q(\theta)} \int_0^1 \left[v(q(\theta),\theta) - cq(\theta) - \frac{1-\Phi(\theta)}{\phi(\theta)} v_\theta(q(\theta),\theta) \right] \phi(\theta)\, d\theta$$

- this can be maximized "pointwise" (separately for each type) leading to the first order condition

$$v_q(q(\theta),\theta) - c - \frac{1-\Phi(\theta)}{\phi(\theta)} v_{q\theta}(q(\theta),\theta) = 0$$

- comparison to first best: downward distortion but no distortion at the top
- interpretation and rent extraction effect

Optimal non-linear pricing by a monopolist IX

- still to check: is the constraint "$q(\theta)$ non-decreasing" satisfied?
  - monotone hazard rate assumption (MHR): $\frac{1-\Phi(\theta)}{\phi(\theta)}$ is non-increasing in θ (satisfied by uniform, normal and most other commonly used distributions)
  - under (MHR) and $v_{q\theta\theta} \leq 0$, q is increasing

An example I

- say $v(q,\theta) = (\theta + 1)\sqrt{q}$ and ϕ is the uniform distribution
- first order condition

$$\frac{\theta + 1}{2\sqrt{q(\theta)}} - c - (1-\theta)\frac{1}{2\sqrt{q(\theta)}} = 0$$

$$\Leftrightarrow q(\theta) = \left(\frac{\theta}{c}\right)^2$$

- check: q is increasing in θ
- rents under the optimal pricing scheme

$$U(\theta) = \int_0^\theta v_\theta(q(x), x)\, dx = \int_0^\theta \sqrt{q(x)}\, dx = \frac{1}{2}\frac{\theta^2}{c}$$

An example II

- from rents we can calculate optimal transfers t because $U(\theta) = v(q(\theta), \theta) - t(\theta)$

$$t(\theta) = \sqrt{q(\theta)}(\theta + 1) - U(\theta) = \frac{\theta^2 + \theta}{c} - \frac{\theta^2}{2c} = \frac{\theta^2 + 2\theta}{2c}$$

- bringing $t(\theta)$ and $q(\theta)$ together gives the optimal price as function of quantity
  - quantity q is sold to type $\theta = \sqrt{q}c$ who pays $t(\theta) = (\theta^2 + 2\theta)/(2c)$
  - hence, the price of q is $p(q) = \frac{(\sqrt{q}c)^2 + 2\sqrt{q}c}{(2c)} = \frac{qc + 2\sqrt{q}}{2}$
  - offering this price schedule is equivalent to the optimal direct revelation mechanism and therefore optimal

The option of not selling:

- we had the maximization problem

$$\max_{q(\theta)} \int_0^1 \left[v(q(\theta),\theta) - cq(\theta) - \frac{1-\Phi(\theta)}{\phi(\theta)} v_\theta(q(\theta),\theta) \right] \phi(\theta)\, d\theta$$

- this can be maximized "pointwise" (separately for each type) where the optimal q is characterized by a first order condition
- there is also the option of not selling to type θ:
  - if you do not sell to θ your profit for this type is 0
  - if $\left[v(q(\theta),\theta) - cq(\theta) - \frac{1-\Phi(\theta)}{\phi(\theta)} v_\theta(q(\theta),\theta) \right] < 0$, it is better not to sell to this type at all
    (where $q(\theta)$ is the q determined by the first order condition)

The option of not selling: An example

- say $v(q,\theta) = log(q)(\theta + 1)$ with and ϕ is the uniform distribution on $[0,1]$ and $c = 1/4$
- first order condition

$$\frac{1+\theta}{q} - \frac{1}{4} - (1-\theta)\frac{1}{q} = 0$$

$$\Leftrightarrow q(\theta) = 8\theta$$

- If we now calculate $v(q(\theta),\theta) - cq(\theta) - (1-\Phi(\theta))v_\theta(q(\theta),\theta)$ we get

$$(1+\theta)log(8\theta) - 2\theta - (1-\theta)log(8\theta) = 2\theta\,(log(8\theta) - 1)$$

which is negative for all $\theta < e/8 \approx 0.34$.

- the optimal selling mechanism does not sell to types $\theta < 0.34$ and sells $q(\theta) = 8\theta$ to all types above 0.34

Interpretation and economics

- the model can be reinterpreted:
  - not one consumer but a continuum of consumers with different tastes (i.e. types) and the monopolist cannot tell them apart
- a monopolist distorts the quantity for all types but the highest type downward
  - reason: "rent extraction"
  - a lower quantity reduces the slope of the rent function $U(\theta)$ and therefore the rent of all higher types

Teaser

Suppose a teacher wants his student to study. Studying is hard for the student. The teacher can "pay" good grades to his student which the student likes but the teacher dislikes (because it gives the school a bad reputation as being "easy"). The teacher sets up a difficulty level (how much you have to study to get grade x). When will the the student be optimally (from the teacher's point of view) incentivized to study harder: (i) the student has private information on how hard studying is for him or (ii) the teacher "knows the student well" (i.e. he knows how hard studying is for the student). In which of the two cases is the optimal difficulty level higher?

Revision questions

- What do we mean with ”ex ante”, ”interim” and ”ex post” efficiency?
- Write down the envelope theorem condition for the non-linear pricing problem!
- The monopolist sells a lower quantity than the first best to all types (apart from the top type).
  - Why does he do so?
  - What is the intuition behind this distortion?
  - Why is the quantity for the top type not distorted?

Exercise

- Assume $v(q,\theta) = q(2+\theta) - \frac{q^2}{4}$, let θ be uniformly distributed on $[0,1]$ and $c = 1$. Derive the optimal $q(\theta)$, the rents in the optimal mechanism $U(\theta)$ and the optimal transfers $t(\theta)$. Check whether it is optimal to sell to all types.
  From $t(\theta)$ and $q(\theta)$, derive the optimal price schedule, i.e. price as function of quantity.
  Derive the "first best", i.e. the quantity that maximizes $v(q,\theta) - cq$. Show that there exists pricing scheme such that every type will buy his first best quantity. Why is this pricing scheme not optimal for the monopolist?
- Go slowly through the whole derivation of the optimal selling mechanism. Do you understand each step? Can you do them on your own? Pay special attention to the envelope theorem, the integration by parts and taking the first order condition.