Mechanism Design

Envelope and monotonicity condition in the non-linear pricing model

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1 Deriving the envelope and monotonicity condition

We assume that v is twice continuously differentiable. Furthermore, we assume $v_{q\theta} > 0$. This assumption is well known under different names ("single crossing condition", "Spence-Mirrlees condition", "constant sign condition"). This condition will play a major role in the proof of lemma 1 below.

Now, take an incentive compatible direct revelation mechanism consisting of the two functions $q(\theta)$ and $t(\theta)$. We define $U(\theta) = v(q(\theta), \theta) - t(\theta)$. As the mechanism is incentive compatible, the following holds for any two types θ' and θ'':

$$\begin{aligned} v(q(\theta'), \theta') - t(\theta') &\geq v(q(\theta''), \theta') - t(\theta'') \\ v(q(\theta''), \theta'') - t(\theta'') &\geq v(q(\theta'), \theta'') - t(\theta'). \end{aligned}$$

These two conditions can be rewritten as

$$\begin{aligned} U(\theta') &\geq U(\theta'') - v(q(\theta''), \theta'') + v(q(\theta''), \theta') \\ U(\theta'') &\geq U(\theta') - v(q(\theta'), \theta') + v(q(\theta'), \theta''). \end{aligned}$$

Rearranging these two inequalities gives

$$\begin{aligned} U(\theta') - U(\theta'') &\geq -v(q(\theta''), \theta'') + v(q(\theta''), \theta') \\ U(\theta') - U(\theta'') &\leq v(q(\theta'), \theta') - v(q(\theta'), \theta''). \end{aligned}$$

Assume $\theta' > \theta''$. Then combining these two inequalities gives

$$\frac{v(q(\theta''), \theta') - v(q(\theta''), \theta'')}{\theta' - \theta''} \leq \frac{U(\theta') - U(\theta'')}{\theta' - \theta''} \leq \frac{v(q(\theta'), \theta') - v(q(\theta'), \theta'')}{\theta' - \theta''}. \tag{1}$$

First we derive the monotonicity constraint:

Lemma 1. *In every incentive compatible mechanism, $\theta' > \theta''$ implies $q(\theta') \geq q(\theta'')$.*

Proof. (1) implies that

$$v(q(\theta''), \theta') - v(q(\theta''), \theta'') \leq v(q(\theta'), \theta') - v(q(\theta'), \theta'')$$

(recall that $\theta' - \theta'' > 0$ by assumption). The last inequality can be rewritten as

$$\int_{\theta''}^{\theta'} v_\theta(q(\theta''), x)\, dx \leq \int_{\theta''}^{\theta'} v_\theta(q(\theta'), x)\, dx$$

which in turn is equivalent to

$$0 \leq \int_{q(\theta'')}^{q(\theta')} \int_{\theta''}^{\theta'} v_{q\theta}(y, x)\, dx\, dy.$$

By the assumption $v_{q\theta} > 0$, the integrand of the previous expression is strictly positive everywhere. But then the previous inequality can (by $\theta'' < \theta'$) hold only if $q(\theta'') \leq q(\theta')$. (Remember that $\int_a^b \cdots = -\int_b^a \ldots$.)

Since the types θ' and θ'' were arbitrary we get that q has to be monotone, i.e. higher types lead to (weakly) higher decision. □

Second, we derive the envelope condition. In (1), the right hand side converges to $v_\theta(q(\theta'), \theta')$ as $\theta'' \to \theta'$. If q is continuous at θ', then the left hand side converges to $v_\theta(q(\theta'), \theta')$ as well. This, of course, implies that the middle term also converges to $v_\theta(q(\theta'), \theta')$. As the middle term is $U'(\theta)$, we then get $U'(\theta) = v_\theta(q(\theta'), \theta')$.

As q is monotone (see the lemma above), q is continuous almost everywhere.[1] Consequently, we have shown the following result:

Lemma 2 (envelope theorem)**.** *In an incentive compatible mechanism, $U'(\theta) = v_\theta(q(\theta), \theta)$ for almost all $\theta \in \Theta$ and therefore*[2]

$$U(\theta) = U(0) + \int_0^\theta v_\theta(q(x), x)\, dx.$$

[1] This useful fact follows form the following idea: At every discontinuity q has to "jump" over a rational number. If a monotone, say increasing, function had an uncountable number of discontinuities we would therefore have to conclude that there are uncountably many rational numbers (as the function is monotone it has to jump over a different rational at every discontinuity). The rational numbers, however, are countable.

[2] Here we use the fundamental theorem of calculus. Strictly, speaking we have not checked its conditions properly. In particular, we have to verify that $v_\theta(q(\theta), \theta)$ is bounded and integrable. Without going into details, let me just say that given the assumptions made this can be shown by showing that for q a bounded domain can be used without loss of generality.

We have now established that incentive compatibility implies both the monotonicity and the envelope condition. Now we want to show the reverse: If a mechanism satisfies both the monotonicity and the envelope condition, then this mechanism is incentive compatible.

Let t and q be such that the envelope contition and monotonicity are satisfied. Take arbitrary types θ' and θ''. Because $v_{q\theta} > 0$, monotonicity of q implies that

$$\int_{\theta''}^{\theta'} \int_{q(\theta'')}^{q(x)} v_{q\theta}(y,x)\, dy\, dx \geq 0. \tag{2}$$

(The trick is that monotonicity implies $q(x) \geq q(\theta'')$ for all $x \in [\theta'', \theta']$ if $\theta' > \theta''$. If $\theta' < \theta''$, then we can rewrite the expression above as $\int_{\theta'}^{\theta''} \int_{q(x)}^{q(\theta'')} v_{q\theta}(y,x)\, dy\, dx \geq 0$ and monotonicty implies then that $q(x) \leq q(\theta'')$ for all $x \in [\theta', \theta'']$.)

Rewriting (2) gives

$$\int_{\theta''}^{\theta'} v_\theta(q(x),x) - v_\theta(q(\theta''),x)\, dx \geq 0.$$

By the envelope condition $\int_{\theta''}^{\theta'} v_\theta(q(x),x)\, dx = U(\theta') - U(\theta'')$, and therefore the previous expression is equivalent to

$$U(\theta') - U(\theta'') - v(q(\theta''),\theta') + v(q(\theta''),\theta'') \geq 0.$$

Rearranging gives

$$U(\theta') \geq U(\theta'') + v(q(\theta''),\theta') - v(q(\theta''),\theta'') = v(q(\theta''),\theta') - t(\theta'').$$

But this simply says that θ' does not want to misrepresent as θ''. Since θ' and θ'' were arbitrary, the mechanism is incentive compatible which is what we wanted to show.

2 "Single crossing" and monotonicity: A graphical explanation

The utility of type θ when getting quantity q and paying price t is $v(q,\theta) - t$. Think of the consumer's indifference curves in a q,t diagram. Recall that an indifference curve are all the points (q,t) such that $v(q,\theta) - t = \bar{U}$ for some constant $\bar{U}$. The slope of the indifference curve in a point (q,t) is

$$\left.\frac{dt}{dq}\right|_{U=\bar{U}} = v_q(q,\theta)$$

as can be seen from rearranging the equation defining the indifference curve as $t = v(q,\theta) - \bar{U}$. Intuitively, if we give the consumer one (marginal) unit more, then he is willing to pay $v_q(q,\theta)$ for this additional unit. Hence, we have to charge him $v_q(q,\theta)$ more if we want to keep him indifferent to the starting point.

The single crossing assumption $v_{q\theta} > 0$ states that higher types have steeper indifference curves. That is, if the indifference curves of types θ' and $\theta'' < \theta'$ intersect in one point (q', t') then the curve of θ' will be steeper and therefore intersect the curve of θ'' "from below" (that is the θ' curve is below (above) the curve of θ'' for lower (higher) q than q'). Since this is true for any intersection, a given indifference curve of type θ' can intersect a given indifference curve of type θ'' only once: Suppose it intersected twice. By what we just said it would have to intersect both times "from below". But as indifference curves are continuous,[3] this implies that in between there must be an intersection "from above". An intersection from above is, however, impossible by $v_{q\theta} > 0$. This contradicts that there is more than one intersection. This is where the name "single crossing" comes from.

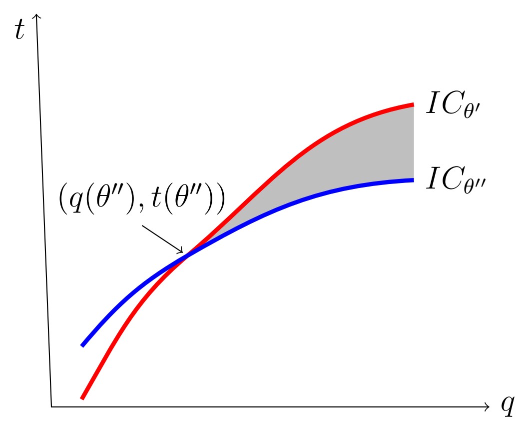

Figure 1: Single crossing and monotonicity: Indifference curves of θ' and $\theta'' < \theta'$ through the contract $(q(\theta''), t(\theta''))$

How does this help us to obtain our monotonicity result? In figure 1 I draw the indifference curves of two types, θ'' and θ' with $\theta' > \theta''$, passing through the contract $(q(\theta''), t(\theta''))$ intended for θ''. As $\theta' > \theta''$, we have drawn the indifference curve of θ' – labeled $IC_{\theta'}$ – steeper than the indifference curve of θ'' – labeled $IC_{\theta''}$. (This is the role of the single crossing condition $v_{q\theta} > 0$ in the figure.)

Where can the contract of type θ' be if we want the menu to be incentive compatible? In order to be incentive compatible, θ' has to prefer his own contract $(q(\theta'), t(\theta'))$ to $(q(\theta''), t(\theta''))$. Hence, $(q(\theta'), t(\theta'))$ has to be below $IC_{\theta'}$ (recall that right bottom is the direction which

[3]Continuity of the indifference curves follows from the assumed continuity of v.

the consumer likes better in the q, t diagram: get more, pay less). Incentive compatibility furthermore requires that θ'' prefers $(q(\theta''), t(\theta''))$ over $(q(\theta'), t(\theta'))$. Hence, $(q(\theta'), t(\theta'))$ has to be above $IC_{\theta''}$. Taking these two requirements together, $(q(\theta'), t(\theta'))$ has to be in the shaded area in figure 1. Note that all points in the shaded area have a q above $q(\theta'')$. Hence, $q(\theta') \geq q(\theta'')$ for any incentive compatible menu – we obtain the monotonicity condition.