Mechanism Design

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Outline

1. Optimal contracts in a principal agent setting
2. Revenue equivalence in auctions
3. Review questions etc.

Principal agent problems

- Setting
  - ”agent” has to work for a ”principal”
  - agent can decide himself how much effort he puts into his work
  - effort is costly but increases the principal’s (expected) payoff
- Applications
  - labor contracts
  - regulation of natural monopolists
  - outsourcing
  - teacher-student?
  - sponsor/spectator-professional athlete?

Formal Setting

- agent exerts effort $e \geq 0$ and receives transfer t from principal
- agent's type θ is in $[\underline{\theta}, \bar{\theta}]$ where $\bar{\theta} < 0$
- θ is distributed according to a distribution Φ with strictly positive density function ϕ
- assumption: $\phi(\theta)/(1 - \Phi(\theta))$ is non-decreasing in θ
- agent's utility is $u(e, \theta, t) = t + \theta g(e)$ where g is disutility of effort
- assumption: $g(0) = 0$, $g'(0) = 0$, $g'(e) > 0$ for $e > 0$, $g''(e) \geq 0$
- principal's utility is $v(e) - t$ with $v'(e) > 0$ and $v''(e) < 0$

Benchmarks: The first best

- What would be the contract if the principal knew the agent's type?
  - agent can "walk away": utility of agent must be at least 0

$$\max_{e(\cdot),t(\cdot)} v(e(\theta)) - t(\theta)$$

$$s.t.: \; t(\theta) + \theta g(e(\theta)) \geq 0$$

The direct revelation mechanism

- social choice function: assigns to each type a transfer and an effort level $f(\theta) = (e(\theta), t(\theta))$
- revelation principle says we can concentrate on direct revelation mechanisms
  - principal searches for the social choice function $(e(\theta), t(\theta))$ that
    - is incentive compatible
    - is individually rational (i.e. agent cannot get a utility below 0 for any type)
    - maximizes his expected payoff.

$$\max_{e(\cdot),t(\cdot)} E_\theta \left[v(e(\theta)) - t(\theta) \right]$$

$$s.t.: \; e(\cdot), t(\cdot) \text{ is ic and ir}$$

Envelope theorem

- denote the agent's "rent" as $U(\theta) = t(\theta) + \theta g(e(\theta))$
- incentive compatibility requires according to the envelope theorem that

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} g(e(s))\, ds$$

- if $e(\cdot)$ is nondecreasing, the envelope condition is also sufficient for incentive compatibility (check MWG)

our strategy:

- we use the envelope condition
- we check later whether the resulting $e(\cdot)$ is nondecreasing
- if yes, incentive compatibility is satisfied
- individual rationality: $U(\theta) \geq 0$ for all types
  - check: if $U(\underline{\theta}) \geq 0$, then $U(\theta) \geq 0$ for all types

The relaxed problem

$$\max_{e(\cdot),t(\cdot)} \; E_\theta \left[v(e(\theta)) - t(\theta) \right]$$

$$s.t.: U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} g(e(s))\, ds$$

$$U(\underline{\theta}) \geq 0$$

Solving the relaxed problem I

- substitute $U(\theta)$ for $t(\theta)$ in the objective giving

$$E_\theta\left[v(e(\theta)) + \theta g(e(\theta)) - U(\theta)\right]$$

$$= \int_{\underline{\theta}}^{\bar{\theta}} \left[v(e(\theta)) + \theta g(e(\theta)) - U(\theta)\right] \phi(\theta)\, d\theta$$

- substitute the envelope condition for $U(\theta)$
- use integration by parts to get rid of the double integral leading to

$$\int_{\underline{\theta}}^{\bar{\theta}} \left[v(e(\theta)) + g(e(\theta))\left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)}\right)\right] \phi(\theta)\, d\theta - U(\underline{\theta})$$

- clearly $U(\underline{\theta}) = 0$ maximizes the principal's payoff (under the constraint $U(\underline{\theta}) \geq 0$)

Solving the relaxed problem II

- the previous expression can be maximized pointwise over $e(\theta)$ giving the first order condition defining the optimal $e^*(\cdot)$

$$v'(e^*(\theta)) + g'(e^*(\theta)) \left(\theta - \frac{1-\Phi(\theta)}{\phi(\theta)}\right) = 0$$

- check: under our assumptions, $e^*(\theta)$ as defined by the first order condition is increasing
  - incentive compatibility is fine if envelope condition holds
- use envelope condition (and optimal $e^*(\cdot)$) to define optimal $t(\theta)$

$$t^*(\theta) = U(\theta) - \theta g(e^*(\theta)) = \int_{\underline{\theta}}^{\theta} g(e^*(s))\, ds - \theta g(e^*(\theta))$$

- $(e^*(\cdot), t^*(\cdot))$ is the optimal contract

Interpretation

- how does $e^*(\cdot)$ compare to the first best effort level?
- what is the intuition?
- do you see the similarity to last lecture?

Auction setting

- you have one indivisible good that you want to sell
- I potential buyers (bidders) with valuation $(\theta_1, \theta_2, \ldots, \theta_I)$
- the valuation of bidder i is distributed on $[\underline{\theta}_i, \bar{\theta}_i]$ with distribution Φ_i with strictly positive density ϕ_i
- types of the different bidders are independent
- the utility of bidder i if he gets the good with probability y_i and receives a transfer t_i is $y_i\theta_i + t_i$

Direct revelation mechanism

- by the revelation principle, we can concentrate on direct revelation mechanisms
- an (auction) mechanism conists of
  - $0 \leq y_i(\theta_1, \theta_2, \ldots, \theta_I) \leq 1$ is the probability that i gets the good
  - $t_i(\theta_1, \theta_2, \ldots, \theta_I)$ is the transfer i gets
- the direct mechanism must be incentive compatible:

$$E_{\theta_{-i}}\left[y_i(\theta_i, \theta_{-i})\theta_i + t_i(\theta_i, \theta_{-i})\right] \geq E_{\theta_{-i}}\left[y_i(\hat{\theta}_i, \theta_{-i})\theta_i + t_i(\hat{\theta}_i, \theta_{-i})\right]$$

Envelope theorem

- define the expected rent of type θ_i as
  $U_i(\theta_i) = E_{\theta_{-i}}\left[y_i(\theta_i, \theta_{-i})\theta_i + t_i(\theta_i, \theta_{-i})\right]$
- by the envelope theorem, incentive compatibility implies:

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} E_{\theta_{-i}}[y_i(s, \theta_{-i})]\, ds$$

Revenue equivalence theorem

Theorem (Revenue equivalence theorem)

Any two auction formats such that

- *for all i and every type vector $(\theta_1, \theta_2, \ldots, \theta_I)$ the probability of bidder i to get the good is the same under both auction formats*
- *for all i, the expected utility of type $\underline{\theta}_i$ is the same under both formats*

lead to the same expected revenue for the seller.

- e.g. compare English auction and Vickrey auction (but also all-pay-auction etc.)
- intuition: envelope condition

Revenue equivalence theorem: Proof I

- take any incentive compatible direct mechanism
- the expected seller revenue is $\sum_{i=1}^{I} E_\theta[-t_i(\theta)]$
- denote by $Y_i(\theta_i) = E_{\theta_{-i}}[y_i(\theta_i, \theta_{-i})]$
- using the envelope theorem

$$
\begin{aligned}
E_\theta[-t_i(\theta)] &= E_{\theta_i} E_{\theta_{-i}}[-t_i(\theta)] \\
&= \int_{\underline{\theta}_i}^{\bar{\theta}_i} [Y_i(\theta_i, \theta_{-i})\theta_i - U_i(\theta_i)]\phi_i(\theta_i)\, d\theta_i \\
&= \int_{\underline{\theta}_i}^{\bar{\theta}_i} \left[Y_i(\theta_i)\theta_i - U_i(\underline{\theta}_i) - \int_{\underline{\theta}_i}^{\theta_i} Y_i(s)\, ds \right] \phi_i(\theta_i)\, d\theta_i \\
&= \int_{\underline{\theta}_i}^{\bar{\theta}_i} Y_i(\theta_i) \left[\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right] \phi_i(\theta_i)\, d\theta_i - U_i(\underline{\theta}_i)
\end{aligned}
$$

where the last step uses integration by parts

Revenue equivalence theorem: Proof II

this is equivalent to

$$E_\theta[-t_i(\theta)] = \int_{\underline{\theta}_1}^{\bar{\theta}_1} \dots \int_{\underline{\theta}_I}^{\bar{\theta}_I} \left\{ y_i(\theta_1, \dots, \theta_I) \left[\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right] \phi_1(\theta_1) \dots \phi_I(\theta_I) \right\} d\theta_I \dots d\theta_1 - U_i(\underline{\theta}_i)$$

the seller's revenue is therefore

$$\sum_{i=1}^{I} E_\theta[-t_i(\theta)] = \int_{\underline{\theta}_1}^{\bar{\theta}_1} \dots \int_{\underline{\theta}_I}^{\bar{\theta}_I} \left\{ \left(\sum_{i=1}^{I} y_i(\theta_1, \dots, \theta_I) \left[\theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right] \right) \phi_1(\theta_1) \dots \phi_I(\theta_I) \right\} d\theta_I \dots d\theta_1 - \sum_{i=1}^{I} U_i(\underline{\theta}_i)$$

- hence, if $y_i(\theta_1, \dots, \theta_I)$ and $U_i(\underline{\theta}_i)$ is the same for two auction mechanisms, the seller's revenue is the same □

Interpretation

- it can be shown that the English auction with a reserve price is the mechanism that maximizes the seller's revenues (if bidders are symmetric)
  - what does this imply for the Vickrey auction with reserve price?
- many different auction formats are optimal
  - however, revenue equivalence relies on the following implicit assumptions
    - risk neutrality
    - independent, private value
    - (payments are conditional on bids only)

Review questions

- Why is the optimal effort below first best effort in the optimal contract?
- Why is the optimal effort level equal to the first best effort level for $\bar{\theta}$?
- Why do we make the assumption that $\phi/(1 - \Phi)$ is non-decreasing?
- What does the revenue equivalence principle state?
- What are the implications of the revenue equivalence principle?
- Why does the revenue equivalence principle hold?

Teaser

After the breakdown of the Sovietunion, the newly independent Ukraine inherited a number of old long range nuclear missiles. While Ukraine did not have a lot of use for these missiles (they are expensive to maintain), other governments (some of them quite wealthy) certainly were interested in having nucelar weapons. Ukraine ended up dismantling the weapons. Was this purely out of pacifism or could it be part of a clever "selling" scheme?

Exercise I

- derive the envelope and monotonicity condition in the principal agent problem (hint: it works pretty much similar to the derivation in last lecture's pricing problem)

Exercise II: Revenue maximizing auctions

In the auction setup assume there are 2 bidders. In the proof of the revenue equivalence principle, we have shown that the seller's expected revenue is

$$\int_{\underline{\theta}_1}^{\bar{\theta}_1}\int_{\underline{\theta}_2}^{\bar{\theta}_2}\left\{\left(\sum_{i=1}^{2} y_i(\theta_1,\theta_2)\left[\theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}\right]\right)\phi_1(\theta_1)\phi_2(\theta_2)\right\} d\theta_2\, d\theta_1 - U_1(\underline{\theta}_1) - U_2(\underline{\theta}_1).$$

The seller maximizes this over $y_i(\theta_1, \theta_2)$ and $U_i(\underline{\theta}_i)$. Note that $y_1(\theta_1, \theta_2) + y_2(\theta_1, \theta_2) \leq 1$ and $y_i(\theta_1, \theta_2) \geq 0$ as these are probabilities of getting the good.

What are the optimal $U_i(\underline{\theta}_i)$? When is it optimal to give the good to bidder 1? When is it optimal to give the good to bidder 2? When is it optimal for the seller to keep the good?

Exercise II: Revenue maximizing auctions (continued)

Bonus:

- If all bidders have the same Φ (and this distribution is such that $\phi/(1-\Phi)$ is non-decreasing), can you see that the Vickrey auction with reserve price is optimal?
- If bidders have different Φ_i, can you see that sometimes the bidder with the highest valuation does not get the good in the revenue maximizing mechanism? Do you have some intuition for this?