Mechanism Design

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Outline

1. How (not) to sell nuclear weapons
2. Optimal contests
3. Revision questions and exercises etc.

How (not) to sell nuclear weapons: Intro

- after break up of Soviet Union, the Ukraine inherits many old-fashioned nuclear weapons
- maintenance is rather expensive
- modern nuclear powers are not interested in buying (too old)
- some other states were interested but US/Russia did not want these states to get nuclear weapons
- what to do?

How (not) to sell nuclear weapons: Mechanism Design

- What is the optimal sales mechanism if there are externalities of trade?
    - externality: selling nuclear weapons to Lybia/Irak/North Korea etc. affects other countries' safety
- you can get payments from non-buyers (in exchange for not selling to their enemies)

How (not) to sell nuclear weapons: Setup

- if buyer $i \in B = \{1, ..., n\}$ buys the good at price p, i's payoff equals $\pi^i - p$
- payoff to buyer i if j gets the good: $-\alpha_{ji} \leq 0$
- if seller (buyer 0) keeps the good, utilities of all agents normalized to 0: $\alpha_{0j} = 0$ for all j
- let $\alpha^i = \max_j \alpha_{ji}$ denote i's worst outcome if the good is sold but not to i
- $v(i) = \{j | \alpha_{ji} = \alpha^i\}$: set of players j that (when one of them buys) leads to i's worst outcome
- to be able to break ties; let ϵ denote the smallest money unit
- for simplicity: all π^i and α_{ij} are commonly known (see section IV in the paper for private info case)

How (not) to sell nuclear weapons: Optimal mechanism I

- first, what is the maximum revenue R^* that the seller can get?

two cases:

- seller does not sell:
    - what is the maximum i might pay in this case?
    - what is the maximum revenue the seller can make?
- seller sells to i:
    - what is the maximum i might pay?
    - what is the maximum $j \neq i$ might pay?
    - what is the maximum revenue the seller can make?

- if we find a mechanism that achieves the maximum possible revenue we are done
- any suggestions?

How (not) to sell nuclear weapons: Optimal mechanism II

- case 1: $\sum_i \alpha^i \geq \max_i\{\pi^i + \alpha^i + \sum_{j\neq i}(\alpha^j - \alpha_{ij})\}$
  - if everyone participates, no sale and every i has to pay $\alpha^i - \epsilon$
  - if all but i participate, sell to $k \in v(i)$
  - some arbitrary rules for cases where less than $n-1$ players participate
  - check: participating is NE
- case 2: $\sum_i \alpha^i < \max_i\{\pi^i + \alpha^i + \sum_{j\neq i}(\alpha^j - \alpha_{ij})\}$
  - if all participate, sell to $j = \arg\max_i\{\pi^i + \alpha^i + \sum_{j\neq i}(\alpha^j - \alpha_{ij})\}$ and charge $\alpha^i - \alpha_{ij} - \epsilon$ to all $i \neq j$, charge $\pi^j + \alpha^j - \epsilon$ to j
  - if all but i participate, sell to $k \in v(i)$
  - some arbitrary rules for cases where less than $n-1$ players participate
  - check: participating is NE

How (not) to sell nuclear weapons: Optimal mechanism III

- in both cases revenues approach maximal revenues as $\epsilon \to 0$
- ”arbitrary rules” can be chosen such that participating is even a dominant strategy (see exercise)

Galton's problem (Biometrika, Vol. 1, 1902)

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A certain sum, say 100€, is available for two prizes to be awarded at a forthcoming competition; the larger one for the first of the competitors, the smaller one for the second. How should the 100€ be most suitably divided between the two? What ratio should a first prize bear to that of a second one? Does it depend on the number of competitors, and if so, why?

Contests

- sports contests
- R&D contests (many "best entry" prizes)
- examinations (at least when "grading on a curve")
- elections

How to interpret Galton's problem?

- "most suitably divided": what is the objective?
  - today: objective is to maximize overall effort of participants
- how to allocate prizes if
  - the total prize budget is fixed
  - total expected effort should be maximized
- for simplicity we assume
  - agents are risk neutral
  - we only consider whether one should award 1 or 2 prizes (and how much bigger the first prize should be if one awards 2)
  - there are at least 3 contestants

Setup

- two prizes V_1 and V_2
- prize budget is 1: $V_1 + V_2 = 1$
- $k \geq 3$ contestants
- a contestant i receiving prize V_j and exerting effort x_i has a payoff of

$$V_j - c_i x_i$$

where c_i is the contestant's type which is private information

- c_i are distributed according to strictly positive density f and distribution function F on $[m, 1]$ where $0 < m < 1$
- contestants maximize expected payoff
- the contestant with the highest (second highest) x_i wins V_1 (V_2)
- contest designer maximizes the expected value of $\sum_i x_i$ by choosing V_1 and V_2

How much effort to exert for given V_1 and V_2?

- similar to an auction problem
- we assume that the equilibrium effort is $b(c)$ which is strictly decreasing and differentiable and the same for each contestant (and based on this assumption derive an equilibrium that actually has these properties)
  - if you exert effort x, what is the probability of winning the first prize?
  - if you exert effort x, what is the probability of winning the second prize?
  - what is the optimal effort of the worst type, i.e. of $c = 1$?

How much effort to exert for given V_1 and V_2?

- maximization problem for contestant i

$$\max_x \left[V_1(1-F(b^{-1}(x)))^{k-1} + V_2(k-1)F(b^{-1}(x))(1-F(b^{-1}(x)))^{k-2} - cx\right]$$

- first order condition:
  - denote the inverse function of b as y, i.e. $y(x) = b^{-1}(x)$, and for brevity drop the argument (x), then the foc is:

$$c = -(k-1)V_1(1-F(y))^{k-2}f(y)y' + (k-1)V_2(1-F(y))^{k-2}f(y)y' - (k-1)(k-2)V_2F(y)(1-F(y))^{k-3}f(y)y'$$

- this is a differential equation in y with the boundary condition $y(0) = 1$

not easy to solve but doable (see appendix A of the paper)

- the solution is strictly decreasing and differentiable

How much effort to exert for given V_1 and V_2?

Proposition 1

Take V_1 and V_2 as given. There is a symmetric equilibrium in which each contestant exerts effort $b(c) = V_1 A(c) + V_2 B(c)$ where

$$A(c) = (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-2} f(a)\, da$$

$$B(c) = (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-3}\left[(k-1)F(a)-1\right] f(a)\, da.$$

The designer's problem I

- take $V_2 = \alpha$ and $V_1 = 1 - \alpha$ where $0 \leq \alpha \leq 1/2$
- the designer maximizes expected effort
- a contestant's effort is
  $b(c) = (1-\alpha)A(c) + \alpha B(c) = A(c) + \alpha(B(c) - A(c))$
- expected effort of one contestant is therefore

$$\int_m^1 [A(c) + \alpha(B(c) - A(c))] \, f(c) \, dc$$

- the designer's problem is therefore

$$\max_{\alpha \in [0,1/2]} k \int_m^1 [A(c) + \alpha(B(c) - A(c))] \, f(c) \, dc$$

The designer's problem II

- the designer's problem is equivalent to

$$\max_{\alpha \in [0,1/2]} \alpha \int_{m}^{1} (B(c) - A(c))\, f(c)\, dc$$

- which values of α could possibly be optimal?
- what are the benefits of a high α? which types might be motivated by this?
- what are the benefits of a low α?

The designer's problem III

Theorem (Optimal contest with linear costs)

It is optimal for the contest designer to award a single prize; i.e. $V_1 = 1$ and $V_2 = 0$.

- Proof: relatively lengthy in general (see paper)

simple example: $k = 3$, $m = 1/2$ and F is uniform distribution, i.e. $f = 2$ and $F(a) = 2a - 1$, then

$$A(c) = 2\int_c^1 \frac{2-2a}{a} 2\, da = -8 + 8c - 8log(c)$$

$$B(c) = 2\int_c^1 \frac{4a-3}{a} 2\, da = 16 - 16c + 12log(c)$$

$$\int_{1/2}^1 (B(c) - A(c)) f(c)\, dc = 2\int_{1/2}^1 24(1-c) + 20log(c)\, dc \approx -0.137$$

The designer's problem IV

- paper looks at convex cost functions, i.e. payoff of contestant receiveing prize V_j and exerting effort x_i is $V_j - c_i\gamma(x_i)$ where γ is increasing and convex with $\gamma(0) = 0$
- for convex γ it can be optimal to assign two (equal) prizes: $V_1 = V_2 = 1/2$
- do you have an explanation for this?

Revision questions (selling with externalities)

- What is the main research question of the paper?
- Why can it be most profitable not to sell a good? Under which circumstances will this be?
- What is the crucial threat that allows the seller to achieve the maximal possible revenues?

Revision questions (contests)

- What is the research question of the paper?
- What are the benefits of having a second prize? What are the benefits of allocating all prize money to the second prize?
- Can you give an intuition why several prizes can be optimal if the effort cost functions are convex (although a singel prize is optimal when costs are linear)?

Exercise (selling with externalities)

- we use the mechanism in the slides but specify the following additional rules (where B^* is the set of participating players)
    - if $B^* = \{\}$, seller keeps good and no payments
    - if $|B^*| \leq n-2$, then $j = min_i B^*$ gets the good. Define $h = min_i B^* \backslash \{j\}$. Then j pays $\pi^j + \alpha_{hj} - \epsilon$ and all other participating players receive payment ϵ
    - if $B^* = B \backslash i$, then $k \in v(i)$ gets the good and pays $\pi^k + \alpha_{hk} - \epsilon$ where $h = min\, B^* \backslash \{k\}$. All other participating players m pay $\alpha_{hm} - \alpha_{km} - \epsilon$
- show that it is a (weakly) dominant strategy to participate

Exercise (contests)

- Suppose the effort cost function γ is concave. Do you have an intuitive idea whether two or one prize will be optimal?
- We assumed that contestants are risk neutral. What do you think will happen concerning the number of optimal prizes if contestants are risk averse?