Mechanism Design

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Matching: Introduction

- matching problems
- examples:
  - labor markets
  - college admission
  - kindergarten and child care
  - kidney donors
  - marriage (?)
- Gale, Shapley (1962) "College admissions and the stability of marriage " (Nobel prize 2012)
- the focus is stability

Model I

- n men and n women on the "marriage market"
- each man/woman has preferences over the women/men (who he likes best, second best etc.)
- for simplicity: no indifference
- what are we looking for?
  a matching of each man with exactly one woman that is "stable"

Definition (stability)

A matching is stable if there is no man α and woman A such that α prefers A to his wife and A prefers α to her husband.

Model II

Example

3 men (α, β, γ) and 3 women (A-C) with preferences (first number is the rank of the woman in the men's preference and the second number vice versa)

	A	B	C
α	1,3	2,2	3,1
β	3,1	1,3	2,2
γ	2,2	3,1	1,3

There are 3 stable matchings.

Model III

Example

	A	B	C	D
α	1,3	2,3	**3,2**	4,3
β	1,4	4,1	3,3	**2,2**
γ	**2,2**	1,4	3,4	4,1
δ	4,1	**2,2**	3,1	1,4

- The only stable matching in this example is $\gamma - A, \delta - B, \alpha - C, \beta - D$.
- For example, $\alpha - A, \beta - B, \gamma - C, \delta - D$ is not stable as, for example, $\gamma - A$ want to run off together.

Deferred acceptance/Gale-Shapley algorithm I

Theorem

There always exists a stable set of marriages.

Proof. We describe an algorithm to get a matching and show that this matching is stable.

1. each man proposes to his most preferred woman
2. each woman starts a relationship with the most preferred proposer (and remains single if no proposals)
3. all rejected men propose to their second most preferred woman
4. each woman gets into a relationship with her preferred man from the set of (i) the proposers from the previous step plus (ii) the man she was previously in a relationship with (if any)

Deferred acceptance/Gale-Shapley algorithm II

5. every single man proposes to the most preferred woman he has not proposed to previously
6. each woman gets/stays in a relationship with her most preferred man from the set of (i) the men proposing in the previous step plus (ii) the man she is currently in a relationship with (if any)
7. repeat the previous 2 steps until no one is single
8. all couples get married

To show: algorithm produces a stable matching

- The algorithm ends after a finite number of steps:...

Deferred acceptance/Gale-Shapley algorithm III

- the outcome is stable: Suppose, John and Mary are not married but John prefers Mary to his own wife... Hence,

  Mary prefers her husband over John and the marriage market from the algorithm is stable. □

Deferred acceptance/Gale-Shapley algorithm IV

Example

Let's try the algorithm on the following example:

	A	B	C	D
α	1,3	2,2	3,1	4,3
β	1,4	2,3	3,2	4,4
γ	3,1	1,4	2,3	4,2
δ	2,2	3,1	1,4	4,1

Table: example ranking of potential partners

Deferred acceptance/Gale-Shapley algorithm V

- let's move to college admission
- m colleges and n potential students
- each college j has q_j places
- each college has a strict preference ordering on the set of potential students
- each student has a strict preference ordering on the set of colleges

Deferred acceptance/Gale-Shapley algorithm VI

deferred acceptance algorithm:

1. each student proposes to his most preferred college
2. each college keeps the q_i most preferred proposers on its waiting list
3. each student without a place proposes to his second most preferred college
4. each college puts/keeps on its list: the q_i most preferred students from the set of (i) students that applied in the last step plus (ii) students that were already on its list
5. proceed until every student has a place or was rejected from all colleges

Deferred acceptance/Gale-Shapley algorithm VII

Example

3 colleges (A-C) and 5 students (1-5) with the following preferences and capacities

- A: $1 \succ 2 \succ 3 \succ 4 \succ 5$ and $q_A = 2$
- B: $1 \succ 4 \succ 3 \succ 2 \succ 5$ and $q_B = 2$
- C: $3 \succ 5 \succ 1 \succ 2 \succ 4$ and $q_C = 1$
- 1: $C \succ B \succ A$
- 2: $B \succ A \succ C$
- 3: $A \succ B \succ C$
- 4: $B \succ A \succ C$
- 5: $B \succ C \succ A$

use the student proposing Gale Shapley algorithm to derive a stable matching

Deferred acceptance/Gale-Shapley algorithm VIII

Proposition

The deferred acceptance algorithm produces a stable matching, i.e. there is no student-college pair such that the student prefers the college over his assigned college and the college prefers the student over some of its assigned students.
Furthermore, every applicant is at least as well off under the algorithm matching as under any other stable matching.

Proof.

- The stability proof is as in the marriage problem.
- Let's focus on the optimality part.
  - We call a college "possible" for an applicant if there is a stable matching in which the student is at this college.

Deferred acceptance/Gale-Shapley algorithm IX

- Suppose there is a stable matching M2 that is better for applicant i than the deferred acceptance algorithm matching M1. Say in this better stable matching M2 i is matched with college j while the deferred acceptance algorithm matches i with k where $j \succ_i k$
- In particular, let i be the first student who is turned down by possible college in the deferred acceptance algorithm. (without loss of generality)

… □

Deferred acceptance/Gale-Shapley algorithm X

- note that a similar proposition holds for the marriage market:
  The deferred acceptance procedure leads to the best stable matching for the proposing group.
  It is better to be on the active side of the market!

Matching in practice I

- additional problems in practice
- example: spouse problem implies that algorithm outcome is not stable:
  - round 1: student A, B and C apply at university j
    - j keeps its most prefered student A but rejects B and C
    - say B is the spouse of A
  - round 2:
    - B is admitted at university k (far away)
    - therefore, A gives up his place at university j
  - j regrets that it has rejected C who will not propose again to j in the algorithm
  - final matching unstable as j wants to take C and C wants to take j
- “solution” to spouse problem: change algorithm
  - whenever someone gives up place because of spouse, all previously rejected candidates apply to j again

Matching in practice II

- bigger problem: algorithm is not "strategy-proof" (= it is not a dominant strategy to reveal your true preferences)

Example

men (greek letters) propose

	A	B	C	D
α	2,1	3,1	4,2	**1,4**
β	4,2	**1,3**	3,4	2,1
γ	3,4	1,4	**2,1**	4,2
δ	**1,3**	2,2	4,3	3,3

Table: example ranking of potential partners

- check: outcome of algorithm is $\delta - A$, $\beta - B$, $\gamma - C$, $\alpha - D$
- suppose B misrepresents her preferences as $\alpha \succ \delta \succ \gamma \succ \beta$

Matching in practice III

Example

men (greek letters) propose

- suppose B misrepresents her preferences as $\alpha \succ \delta \succ \gamma \succ \beta$ leading to the table

	A	B	C	D
α	**2,1**	3,1	4,2	1,4
β	4,2	1,4	3,4	**2,1**
γ	3,4	1,3	**2,1**	4,2
δ	1,3	**2,2**	4,3	3,3

Table: example ranking of potential partners

- check: outcome of algorithm is $\alpha - A$, $\beta - D$, $\delta - B$, $\gamma - C$
- B gets δ instead of β who she prefers!

Matching in practice IV

- general results:
  - the proposing side does not want to misrepresent its preferences (dominant strategy to reveal true preferences!)
  - the non-proposing side might want to misrepresent if the outcome under the men-proposing mechanism is different from its outcome under the women-proposing mechanism

Matching in practice V

- stability vs. strategy-proofness
  - do you know preferences of other participants?
  - if not, can you nevertheless know how to misrepresent profitably? how relevant are these cases?
  - in practice:
    - simulations (past data)
    - lab experiments

Review questions

- What is the basic problem matching mechanisms try to solve?
- What is “stability” of a matching?
- Explain the Gale-Shapley algorithm!
- Explain the deferred acceptance algorithm.
- What is the main (theoretical) disadvantage of the Gale-Shapley algorithm?

Teaser

At your work place, you are part of the organization committee of the anual department retreat. Part of the program is that people are split into work-groups and each work-group has to discuss a different question. You have to assign your co-workers into the different groups. The groups should have equal size and the preferences of your coworkers should be respected as much as possible (every of your colleagues has his own preferences concerning which questions he finds interesting). You have the possibility to ask your coworkers for their preferences.
How is this similar to the college matching problem? How is it different?
In last year's retreat, a similar situation occurred and your boss thinks that those who did not get their preferred working-group last year should have priority this year. Can you satisfy all these demands?

Exercises I

- Use the deferred acceptance algorithm to derive a stable matching with the following preferences:

	A	B	C	D
α	2,3	3,2	1,2	4,3
β	1,4	2,3	3,1	4,4
γ	2,1	4,4	1,3	3,2
δ	2,2	3,1	1,4	4,1

Table: ranking of potential partners

- Prove the stability part of the proposition (college admission problem).
- Bonus exercise (hard): can you construct another example where one of the participants wants to misrepresent her true preferences?