Mechanism Design

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Outline

1. Introduction
2. Example: Discrete Auction
3. Type spaces
4. ex post BNE and robust implementation
5. Revision questions and exercises etc.

"Robust Mechanism Design"

- caveat: somewhat misleading name
  - does *not* mean "robust to small changes in the environment"
  - we will allow for more complex belief structure
- research in progress

Type and Belief

- ”type” contains private information on
  - payoff
  - beliefs
- so far: relatively simple types
  - types drawn independently from common prior (e.g. auctions): belief by Bayes’ rule
  - types correlated: Cremer/McLean condition says that there is a one-to-one relation between private information on payoff and beliefs
- can/should we allow for other types?
  - two types might have the same utility function but differ in beliefs about other agent’s utility
  - common prior, correlated types where Cremer/McLean condition does not hold
  - different agents might have different priors

Aside: Simplicity

- another research objective
- restrict oneself to simple mechanisms
  - (much) worse outcome?
  - what is "simple"?

Example: Discrete Auction I

- seller with valuation 0 for one good
- two potential buyers whose valuations are distributed:

valuation	0	3
1	$\frac{1}{12}$	$\frac{3}{12}$
5	$\frac{2}{12}$	$\frac{2}{12}$
6	$\frac{2}{12}$	$\frac{2}{12}$

- common prior but not independent
- Cremer/McLean condition violated

Example: Discrete Auction II

- our strategy:
  - Cremer/McLean: we can extract buyers' *beliefs* at no costs
  - only care about ic between types 6 and 5 (same beliefs $\rightarrow$ Cremer/McLean cannot determine who is who $\rightarrow$ usual information rents etc. for those two types)
  - focus on types 5 and 6 (ignore all other types)
  - which type's IR and which type's IC will bind?
  - denote interim probability of getting the good for type i p_i (similarly transfers t_i)
  - what is t_5 (depending on p_5)?
  - what is t_6 (depending on p_6 and p_5)?
  - what is the seller's (interim) revenue from type 6 (depending on p_6 and p_5)? what should p_6 be?
  - what should p_5 be?

Example: Discrete Auction III

- this gives the optimal allocation rule

allocation	0	3
1	$(1, 0)$	$(0, 1)$
5	$(1, 0)$	$(1, 0)$
6	$(1, 0)$	$(1, 0)$

and interim payments (without Cremer-McLean belief elicitation transfers!) of $t_5 = t_6 = 5$, $t_1 = 1/4$, $t_0 = 0$ and $t_3 = 3 * 3/7$

Example: Discrete Auction IV

- next step is to elicit beliefs as in Cremer-McLean
  check that with the following transfers it is optimal for all types of player 1 to reveal his true belief

elicitation payments	0	3
1	9	-3
5	-5	5
6	-5	5

note: elicitation payments are zero in expectation for each type

- total payments: add up belief elicitation payments and t_i

elicitation payments	0	3
1	$(9.25, 0)$	$(-2.75, 3)$
5	$(0, 0)$	$(10, 0)$
6	$(0, 0)$	$(10, 0)$

check: elicitation payments are high enough to satisfy IC

Example: Discrete Auction V

- these payments are not entirely intuitive
- the "robust mechanism" is very much tailor made to the specific example
- note: type 6 has a rent (unlike in Cremer/McLean)

Belief Types

- we will split up a "type" τ_i into two components:
  - payoff type $\theta_i(\tau_i)$: payoff relevant information $u_i : X \times \Theta \rightarrow \Re$, i.e. utility depends on outcome and payoff types
  - belief type $\beta_i(\tau_i)$: *first- and higher-order* belief of player i of type
    - what does i believe about the distribution of the other players' payoff type
    - what does i believe about j 's beliefs over $-j$'s payoff types
    - what does i believe about j 's beliefs about k 's beliefs over $-k$'s payoff types
    - etc.

Belief Types (Example) I

Example (payoff and belief type)

type 5 above:

- payoff type: 5
- belief type:

1. P2's type is 0 with prob 1/2 and 3 with prob 1/2
2. P2 believes with prob 1/2 that my type is distributed (1/5,2/5,2/5) and P2 believes with prob 1/2 that my type is distributed (3/7,2/7,2/7)
3.

$$\text{with prob } 1/2 \text{ P2 believes that } \begin{cases} \text{I believe with prob } 1/5 \text{ that P2's} \\ \quad \text{type is distributed } (1/4,3/4). \\ \text{I believe with prob } 4/5 \text{ that P2's} \\ \quad \text{type is distributed } (1/2,1/2). \end{cases}$$

Belief Types (Example) II

Example (payoff and belief type (continued))

3. (continued)

$$\text{with prob } 1/2 \text{ P2 believes that } \begin{cases} \text{I believe with prob } 3/7 \text{ that P2's} \\ \quad \text{type is distributed } (1/4, 3/4). \\ \text{I believe with prob } 4/7 \text{ that P2's} \\ \quad \text{type is distributed } (1/2, 1/2). \end{cases}$$

4. ...

...

Type Space I

Definition (Type Space)

A *type space* is a list $\left(T_i, \hat{\theta}_i, \hat{\beta}_i\right)_i$ where T_i is a set of types for agent i and $\hat{\theta}_i$ and $\hat{\beta}_i$ are functions that map each type into a payoff and a belief type:

$$\hat{\theta}_i : \; T_i \to \Theta_i$$

$$\hat{\beta}_i : \; T_i \to \Delta T_{-i}$$

- note: instead of defining the belief type as an infinite hierarchy of beliefs we can also define it as a distribution over other players' type (usually more convenient in terms of notation)

Type Space II

- a type space describes all the types of all the players that are considered by the modeler
- some special (classes of) type spaces
  - universal type space: $\sim$ all beliefs are allowed
  - finite type space
  - common prior type spaces: beliefs are derived from a common prior using Bayes' rule
  - large variety of certainties: for any given payoff type vector θ, each player i has a type τ_i such that (i) $\hat{\theta}(\tau_i) = \theta_i$ and (ii) τ_i believes with probability 1 that the vector of payoff types is θ

Type Space III

- type spaces we considered so far
  - independently distributed types drawn from a common prior (as in Myerson-Satterthwaite or lecture on revenue equivalence in auctions)
    - what is $\hat{\theta}$?
    - what is $\hat{\beta}$?
  - correlated types drawn from a common prior that satisfies the Cremer-McLean condition
    - what is $\hat{\theta}$?
    - what is $\hat{\beta}$?

Mechanisms

- revelation principle still holds
  - we can concentrate on direct mechanisms where $S_i = T_i$
- if T_i is large, direct mechanisms are still quite complicated
  - sometimes we restrict ourselves to *reduced direct mechanisms* where $S_i = \Theta_i$
  - outcome depends then on payoff types only

Solution concept I

Definition (ex post Bayesian equilibrium)

A Bayesian equilibrium $\sigma^* = (\sigma_1^*, \ldots, \sigma_N^*)$ of a reduced direct mechanism $(\Theta_1, \ldots, \Theta_N, F)$ is an *ex post Bayesian equilibrium* if for every agent i and payoff type vector θ,

$$u_i(F(\theta), \theta) \geq u_i(F(\theta_i', \theta_{-i}), \theta)$$

for all $\theta_i' \in \Theta_i$.

- strong equilibrium concept
    - if u_i does not depend on θ_{-i}, then same as dominant strategy

Solution concept II

- check: every ex post Bayesian equilibrium is also an interim Bayesian equilibrium
- what about the opposite?
  - can every SCF that is implementable in interim BNE also be implemented in ex post BNE?
- if a SCF was implementable (through a direct revelation mechanism) no matter what the beliefs are, we would not have to worry about beliefs $\rightarrow$ robustness

Main result I

- we focus on social choice functions $F: \Theta \to X$
  - only payoff types matter for F
  - function maps each payoff type vector in a unique outcome
  - we are interested whether we can implement F for all possible beliefs (keeping the space of payoff types fix)

Theorem (Robustness (Bergemann and Morris 2005))

A social choice function $F: \Theta \to X$ that is interim BNE implementable on every type space (with payoff type space Θ) is ex post BNE implementable.

Main result II

- interpretation
  - we want to implement F in a simple way:
    - ask for payoff type
    - it is optimal to say true payoff type whatever your belief is
  - we want robustness in the following sense: mechanism should work on all belief type spaces, i.e. irrespective of beliefs
  - theorem above tells us that we can concentrate on social choice functions that are ex post implementable

1. F is interim BNE implementable on all type space $\rightarrow$
2. F is ex post implementable $\rightarrow$
3. F is interim BNE implementable on all type spaces

Proof of the theorem I

we show a stronger result which will imply the theorem

Lemma

If F is interim BNE implementable on all type spaces (with payoff type space Θ) that have a large variety of certainties, then F is ex post BNE implementable.

Proof:

- take an arbitrary belief type space with large variety of certainties
- revelation principle and the fact that F depends only on payoff type: reduced direct mechanism is interim ic
- take an arbitrary player i and payoff type vector θ
- we show that i does not want to misrepresent ex post in the reduced direct mechanism when the payoff vector is θ

Proof of the theorem II

- consider type τ_i whose belief puts probability 1 on θ and whose payoff type is θ_i (how do I know τ_i exists?)
- as F is BNE interim implementable, τ_i does not want to misrepresent as τ_i' where τ_i' has payoff type θ_i' and his belief puts probability 1 on (θ_i', θ_{-i}) (at interim stage) (why does τ_i' exist?)
- hence, $u_i(F(\theta), \theta) \geq u_i(F(\theta_i', \theta_{-i}), \theta)$ and this inequality holds for all $\theta \in \Theta$ and all θ_i, $\theta_i' \in \Theta_i$
- truth telling is ex post BNE $\square$

Discussion

- robustness here: implementable on all (belief) type spaces (by reduced direct mechanism)
  - leads us to ex post BNE
    - for private payoff types (u_i does not depend on θ_{-i}), we are back to dominant strategy implementation
  - only one way to think about robustness
  - alternative suggestions?

Revision questions

- What is a payoff/belief type?
- What was the strategy in the example? Why does the mechanism leave some type with rents and some types without?
- What is a type space? When does a type space contain a "large variety of certainties"?
- Explain ex post Bayesian equilibrium!
- What did our main result on robustness state? How did we interpret robustness in this result? Is this the only way in which one can think of robustness?

Teaser

Students have preferences over what they want to study and about where they want to study. Universities also have preferences about students. Universities have also capacity constraints, i.e. they cannot enroll all students that want to study there. How can students be matched with universities? What criteria should such a matching mechanism satisfy?

Exercises I

The exercise[1] shows how the theorem depends on the assumption that F picks exactly one outcome in X. We have 2 players with 2 payoff types each: $\Theta_1 = \{\theta_1, \theta_1'\}$, $\Theta_2 = \{\theta_2, \theta_2'\}$. Assume $X = \{a, b, c\}$. The table entries below give both players' payoffs for a given allocation and payoff type profile.

a	θ_2	θ_2'
θ_1	1,0	-1,2
θ_1'	0,0	0,0

b	θ_2	θ_2'
θ_1	-1,2	1,0
θ_1'	0,0	0,0

c	θ_2	θ_2'
θ_1	0,0	0,0
θ_1'	1,1	1,1

[1] The exercise is based on an example in Bergemann and Morris 2005.

Exercises II

Consider the social choice correspondence F:

F	θ_2	θ_2'
θ_1	$\{a, b\}$	$\{a, b\}$
θ_1'	$\{c\}$	$\{c\}$

We consider F to be implemented if for every type profile θ one of the altenatives in $F(\theta)$ results.

- Show that F maximizes the sum of the players' payoffs.
- Consider the following mechanism: Player 1 chooses (at the interim stage) the outcome. Show that this mechanism interim BNE implements F no matter what the belief type space is.

Exercises III

- Now we show that F is not ex post implementable.
  - What has to be the choice of the mechanism when payoff types are (θ_1, θ_2) in order to satisfy ex post incentive compatibility for player 1?
  - What is the outcome of the mechanism for (θ_1, θ_2') in order to satisfy ex post incentive compatibility of player 1?
  - Show that such a mechanism violates ex post incentive compatibility of some type of player 2.

2.) Consider the Myerson-Satterthwaite setup (seller and buyer with independently distributed cost/value). Which mechanisms are ex post incentive compatible? (hint: go back to the midterm question 3)

Exercises IV

- bonus exercise (will not be discussed in class): Take the example in Figure 10.1 (page 182) of Börgers' book. It is basically the same example that we had in the lecture ("discrete auction") with different payoffs and probabilities. Derive an optimal selling mechanism for this example. You can check the book (pages 183-186) for a solution. In addition, consider the following questions about the example:
  - why does the example not satisfy the Cremer-McLean condition?
  - how many payoff types do the players have?
  - how many belief types do the two players have?
  - Is the type space: finite? universal? a common prior type space? are types independent? does it have large variety of certainties?