Bayes Nash equilibrium

Christoph Schottmüller

Introduction

- so far
  - need to look at games of incomplete information (preference aggregation when preferences are private, auctions)
  - under certain assumptions decision makers can be modeled as expected utility maximizers
- still missing
  - how to react to information?
  - strategic interaction under uncertainty

Bayes' rule: A simple example

- I know someone who lives in Munich. What is the probability that this person is male?
- I know someone who lives in Munich and who is 1.90 m tall. What is the probability that this person is male?
- I know someone who lives in Munich and has green eyes. What is the probability that this person is male?

Bayes' rule

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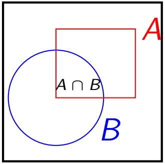
For two events A and B

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

- easier to remember as $P(A|B)P(B) = P(B|A)P(A)$ which is also equal to $P(A \cap B)$
- hence, $P(A|B) = P(A \cap B)/P(B)$



Bayes' rule: example

- an antigen test for a certain virus is 70% reliable at detecting an illness and 99.5% reliable at correctly reporting that somebody is healthy
- suppose about 80.000 people are currently infected with the virus
- suppose 80 million people live in Germany
- if a random person is takes a test and the test is positive, what is the probability that this person is infected?

Bayes' rule: comments

- calculations are reasonably simple
- intuition often goes wrong when the prior is extremely skewed
- make sure to understand it as it will often loom in the background

Independence

Independence

Two random variables X and Y are independent if

$$P(X = x, Y = y) = P(X = x)P(Y = y).$$

- by Bayes' rule, $P(X = x|Y = y) = P(X = x)$ if X and Y are independent
  - knowing Y does not affect my belief about X
- independence will often be assumed to keep the models simple

Games of incomplete information I: an example

- an incumbent decides whether to build a new plant (I for invest) at cost c
- entrant simultaneously decides whether to enter (E)
- entrant does not know whether c is "low" (l) or "high" (h)

Table: Payoffs with $c = h$

	E	NE
I	0,-1	2,0
NI	2,1	3,0

Table: Payoffs with $c = l$

	E	NE
I	1.5,-1	3.5,0
NI	2,1	3,0

- how to solve this game?

Games of incomplete information II: an example

- entrant has to think about
  - how likely is it that incumbent has low cost or high cost?
  - what will incumbent do if he has high cost? what if he has low cost?
  - what should I do?
- incumbent with low cost has to think about
  - what will entrant do?
    - partly depends on what he thinks I would do if I had high costs...
- we will return to this example later on!

Games of incomplete information III: general thoughts

- say two firms do not know the cost of the respective other firm
- the main trick:
  - add beliefs about costs of other firm (i.e. a probability distribution over possible costs)
  - maximize expected utility
- we might want to allow this belief to depend on own costs
  - e.g. a high cost firm may think it is more likely that the other firm has also high costs

Games of incomplete information IV: formal description

- finite set of players: $i = 1, \ldots, N$
- each player has a set of pure strategies S_i
- to capture uncertainty of other players:
  - player i has a type t_i from a set T_i
  - player i knows his own type t_i but other players do not
- player i maximizes expected utility with Bernoulli utility function $u_i : S \times T \to \Re$
  - $T = \times_{i=1}^{N} T_i$ is set of all type profiles
  - $S = \times_{i=1}^{N} S_i$ is set of all strategy profiles
  - actions and types of all players can affect i's payoff
- each type of each player has a belief $p_i(t_{-i}|t_i)$ about other players' types
  - $p_i(t_{-i}|t_i) \in [0, 1]$
  - $\sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) = 1$ where T_{-i} is the set of type profiles of all players but i

Games of incomplete information V: formal description (short)

A N-player game of incomplete information can be denoted as $G = (S_i, T_i, p_i, u_i)_{i=1}^N$ where

- S_i is the strategy set of player i
- T_i is the type set of player i
- p_i assigns to each $t_i \in T_i$ a belief over T_{-i}
- $u_i : S \times T \to \Re$ is player i's utility function.

If all S_i and T_i are finite, the G is called a *finite game of incomplete information.*

Games of incomplete information VI: assumptions on beliefs

- usually, it is assumed that types have a joint distribution p (over T) and beliefs are derived using Bayes' rule:

$$p_i(t_{-i}|t_i) = \frac{p(t_i, t_{-i})}{\sum_{t'_{-i} \in T_{-i}} p(t_i, t'_{-i})}$$

  then p is called the *common prior*

- often we assume independence of types, i.e. the belief $p_i(t_{-i}|t_i)$ is the same for all t_i

Bayesian Nash equilibrium I

each player maximizes expected utility given his type and others strategies
$\rightarrow$ trick:

- think of each type of every type as an own player maximizing expected utility (with utility function u_i and beliefs $p_i(t_{-i}|t_i)$)
- a Bayesian Nash equilibrium consists of one strategy for each type of each player such that
    - the strategy of type t_i maximizes expected utility of player i given the strategies of the others and the belief $p_i(\cdot|t_i)$

Bayesian Nash equilibrium II (formal)

For Bayesian game $G = (S_i, T_i, p_i, u_i)_{i=1}^N$, define the auxiliary game of complete information G':

- set of players is $T_1 \cup T_2 \cup \cdots \cup T_N$
- strategy set of player t_i is S_i
- von Neumann-Morgenstern utility $v_{t_i}(s) = \mathbb{E}_{t_{-i} \in T_{-i}}[u_i(s(t_1), \ldots, s(t_N), t, \ldots, t_N)]$
  - where $s(t_i)$ is the strategy of player t_i and $s = (s(t_1), \ldots, s(t_N))$
  - where expectation is take using the belief $p_i(\cdot|t_i)$

Definition: Bayesian Nash equilibrium (BNE)

A (mixed) Bayesian Nash equilibrium of game G is a (mixed) Nash equilibrium of the corresponding auxiliary game G'.

Bayesian Nash equilibrium III: back to example

- assume the belief $p_E(l) = p_E(h) = 1/2$

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Table: Payoffs with $c = l$

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- what is the optimal strategy for type h?
- if type l invests with probability $s(l)$, what is the entrant's best response?
- if the entrant enters with probability $s(e)$, what is type l's best response?

public good example I

- N guests at a garden party
- each guest has to decide whether to bring a speaker to play music, $S_i = \{0, 1\}$
- payoff of player i:
  - zero if no one brings a speaker
  - t_i if someone else brought a speaker
  - $t_i - 1/2$ if person i brought a speaker
- t_i are independently distributed and 1 (high) with probability $2/3$ and 0 (low) with probability $1/3$
- we want to find a *symmetric BNE*, i.e. one where all high types use one strategy and all low types use one other strategy

public good example II

- what is the optimal strategy of a low type?
- suppose all high types bring a speaker with probability α
  - for player i: what is the probability that no one else brings a speaker?
  - what is the expected payoff for a high type of player i when bringing the speaker?
  - what is the expected payoff for a high type of player i when not bringing the speaker?
- which value of α gives a BNE?