Decision making under uncertainty

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Introduction

- so far:
  - preference aggregation:
    - what if preferences are private information and have to be elicited?
    - possibilities for gaming the system
    - proper analysis: incomplete information
  - market equilibrium:
    - auction metaphor
    - auction: game with incomplete information
- today:
  - how to model decision making under uncertainty

Motivation: game theory

	C	D
C	2,2	0,3
D	3,0	1,1

Table: prisoner's dilemma

- What do the numbers in the game table actually mean?
- What if the other player plays C and D with 50% probability? How to evaluate that?
- can we model a rational decision maker as expected utility maximizer?

Setup I

- today: no game, just decision problem of 1 decision maker under uncertainty
- basic setup: a decision maker has to choose among lotteries over outcomes in a set C
  - set of *outcomes* $C = \{c_1, c_2 \dots c_n\}$
  - a *simple lottery* L is a probability distribution $(p_1, p_2 \dots p_n)$ with $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$ where p_i is the probability of outcome c_i

vacation lottery

You book a vacation in the south. Depending on the weather your vacation has the outcomes
$C = \{\text{lying on the beach}, \text{ stuck in the hotel room}\}$.
Given the weather forecast you assign probabilities $(0.9, 0.1)$ to the two possible outcomes.

Setup II

- we start from preferences
- the decision maker has a *complete and transitive* preference relation $\succeq$ on the set of all simple lotteries

Compound lotteries I

vacation lottery II

- third outcome: “being stuck at home”, i.e. $C = \{\text{lying on the beach, stuck in hotel room, stuck at home}\}$
- probabiltiy 0.2 that your tour operator goes bankrupt before you go on holidays (and 0.8 that your holiday goes through)
- compound lottery: with probability $\alpha_1 = 0.8$ you get the vacation lottery; with probability 0.2 you get the “lottery” that puts all probability on the outcome “stuck at home”

Compound lotteries II

A *compound lotteries* $(L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K)$ yields with probability α_k the simple lottery L_k ($\alpha_k \geq 0$ and $\sum_{k=1}^{K} \alpha_k = 1$)

- What is the probability that you lie on the beach?
- Is there a simple lottery that is similar to the compound lottery (same outcome probabilities)? ("reduced lottery")

Assumption

The decision maker evaluates compound lotteries like their *reduced lotteries*, i.e. the decision maker is indifferent between a compound lottery and the corresponding reduced lottery.

axioms for preference relation $\succeq$: continuity

continuity axiom:

for all lotteries L, L', L'', the sets

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succeq L''\}$$

and

$$\{\alpha \in [0,1] : L'' \succeq \alpha L + (1-\alpha)L'\}$$

are closed.

- no sudden jumps in preferences
- best understood as (mild) mathematical regularity assumption

axioms for preference relation $\succeq$: independence

independence axiom

for all lotteries L, L', L'' and $\alpha \in (0, 1)$ we have

$$L \succeq L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''$$

- main assumption for what follows
- appealing but some experimental violations are known

Example

There are three prices:

1. 2.500.000 \$
2. 500.000 \$
3. 0 \$

An individual prefers the lottery $L_1 = (0.1, 0.8, 0.1)$ to the lottery $L_1' = (0, 1, 0)$.
If the independence axiom is satisfied (as well as transitivity and monotonicity), can we say which of the following lotteries the individual prefers?
$L_2 = (0.55, 0.4, 0.05)$ $\qquad$ $L_2' = (0.5, 0.5, 0)$

Some implications I

Lemma

Assume the independence axiom holds for the preference relation $\succeq$ on the set of lotteries $\mathcal{L}$. Then the following holds:

$$L \sim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''$$

$$L \succ L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$$

Proof (indifference)

- let $L \sim L'$
  - then $L \succeq L'$: by independence axiom equivalent to $\alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''$
  - then $L' \succeq L$: by independence axiom equivalent to $\alpha L' + (1-\alpha)L'' \succeq \alpha L + (1-\alpha)L''$

combined: $\alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''$

Some implications II

Lemma

If $L \sim L'$ and $L'' \sim L'''$ and the independence axiom holds, then $\alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L'''$ where $\alpha \in [0,1]$.

Proof

By the independence axiom, $L \sim L'$ implies

$$\alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''.$$

Also by the independence axiom, $L'' \sim L'''$ implies

$$\alpha L' + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L'''.$$

Finally, use transitivity to get the result.

Utility representation

Definition

A utility function representing the preferences $\succeq$ on $\mathcal{L}$ is a function $U : \mathcal{L} \rightarrow \Re$ such that $U(L) \geq U(L')$ whenever $L \succeq L'$ for $L, L' \in \mathcal{L}$.

von Neumann-Morgenstern utility

Definition (von Neumann-Morgenstern utility)

The utility function $U : \mathcal{L} \to \Re$ has expected utility form if there is an assignment of numbers $(u_1, \ldots, u_n)$ to the n outcomes in C such that for any simple lottery $(p_1, \ldots, p_n)$

$$U(L) = u_1 p_1 + \cdots + u_n p_n.$$

Such a utility function U is called von Neumann-Morgenstern utility function.

The idea is that outcome (with certainty) c_i yields utility u_i. To evaluate lotteries, we take the expected utility (i.e. expectation over those u_i).

Expected utility theorem

Theorem

Assume that the preference relation $\succeq$ satisfies transitivity, completeness, the continuity axiom and the independence axiom. Then $\succeq$ can be represented by a von Neumann-Morgenstern utility function $U : \mathcal{L} \rightarrow \Re$, i.e. there exists a utility function of the form $U(L) = \sum_{i=1}^{n} u_i p_i$ such that

$$L \succeq L' \quad \text{if and only if } U(L) \geq U(L').$$

Proof

somewhat lengthy, see ch. 6B in Mas-Colell, Whinston and Green (1995) or Jehle and Reny (2011) ch. 2.4.2

- under our assumptions a decision maker maximizes expected utility
- U: "von Neumann-Morgenstern utility function"
- u_i: "Bernoulli utilities"

Risk preferences

- suppose the outcomes are amounts of money
- instead of u_i, function $u : \Re \to \Re$
- risk preferences
  - take an arbitrary lottery L with expected payout μ
    - risk aversion: decision maker prefers getting μ (for sure!) to L
    - risk love: decision maker prefers L to μ

Proposition

A decision maker is risk averse if and only if his Bernoulli utility function u is concave.
A decision maker is risk loving if and only if his Bernoulli utility function u is convex.

Risk preferences: graph

- let L pay x_1 with probability α and x_2 with $1 - \alpha$
- expected payout $\mu = \alpha x_1 + (1 - \alpha)x_2$
- line connecting $(x_1, u(x_1))$ and $(x_2, u(x_2))$ contains point $(\mu, \alpha u(x_1) + (1 - \alpha)u(x_2))$