Auctions

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[1] I want to thank Ole Jann (CERGE EI) for giving me access to his lecture slides. Many of the following slides are based on his material.

Auctions are used in many contexts I

- privatization of companies, drilling rights, mining rights, mobile phone spectrum, ...
- art, furniture, fresh flowers, fish, houses,...
- treasury bills, bonds, other debts,...
  $\Rightarrow$ billions of dollars every week!
- public and private procurement (reverse auction: lowest price wins)
- historical examples: auction of the Roman empire in 193 AD, bidding for wives in Babylonia[2],...
- Google (and others) ad auctions ($\approx$ 150 billion \$ in 2020)
- electricity wholesale markets

[2] Though famous, it is controversial whether Herodotus' account is historically accurate, see, for instance, https://www.jstor.org/stable/4436038.

Auctions are used in many contexts II

- many other situations can be thought of as auctions:
  - take-over battles for a company
  - queuing for something, lobbying politicians, advertising ("all-pay auction" or "war of attrition")
  - markets. . .

Auction design matters

- most visible application of auction theory in recent decades: Mobile spectrum auctions
- starting in the late 90s, all developed countries sold off mobile spectrum to companies to be used for mobile internet
- countries differed in size, number of incumbent companies, number of available licenses ...
  $\Rightarrow$ all used slightly different auction formats
- per capita revenue 3G spectrum auctions:
  - €650 in the UK
  - €615 in Germany
  - €170 in the Netherlands
  - €20 in Switzerland
- "economic theorists deserve some of the blame" (Klemperer 2003)

Auctions: A theory and a tool

- auctions are both a theoretical concept and a practical tool
  - thinking about their theoretical properties has larger benefits for our understanding of economics ...
  - ... but it also helps inform their practical use
  - practical tool that has been used for millennia to allocate goods
  - thinking about their practical use requires theory, but also detailed knowledge and "street smarts"
- our plan:
  1. What are the most common auction formats? Which one maximizes revenue?
  2. What are the advantages and disadvantages of different auction formats?

Private values framework

- I bidders and one object to be sold
- bidder i has valuation v_i for the object
- v_i is private information of bidder i (his *type*)
- all v_i are identically and independently distributed on $[\underline{v}, \bar{v}]$ with cdf F
  - for simplicity: uniform distribution on $[0, 1]$
- bidder i's payoff is
  - valuation v_i minus payment if he gets object
  - minus payment (if any) if he does not get the object
- we denote i's bid as b_i
  - i's bidding strategy is a function $b_i : [\underline{v}, \bar{v}] \to \Re_+$

Common auction formats for a single object

- sealed-bid auctions:
  - first-price sealed-bid: bidders submit their offer, the highest bidder wins and pays his bid
  - second-price sealed-bid: bidders submit their offer, the highest bidder wins and pays the second-highest bidder's bid
- open auctions:
  - descending ("Dutch"): The auctioneer announces a high price and then slowly decreases it, until a bidder says that he wants to buy at that price
  - ascending ("English"): The auctioneer starts with a low price and announces higher prices, until only one bidder remains and pays the last announced price

Which one will bring the highest revenue?
We need to understand how bidders will behave!

First-price sealed-bid auction (FPA)

- Bidders submit their offer, the highest bidder wins and pays his bid.
- If your valuation is v_i, should you bid v_i?

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- Bidders submit their offer, the highest bidder wins and pays his bid.
- If your valuation is v_i, should you bid v_i?
- basic trade-off for any bidder:
  - higher bid increases chance of winning...
  - ...but decreases surplus in case of winning
- $\Rightarrow$ "bid shading": In equilibrium, every bid b_i will be below bidder's value v_i.

Descending ("Dutch") auction

- The auctioneer announces a high price and then slowly decreases it, until a bidder says that he wants to buy at that price.
- Once announced price p is below v_i, there is a basic trade-off:
  - agreeing to pay earlier increases chance of winning (i.e. nobody else agrees before)...
  - ...but decreases surplus in case of winning
- $\Rightarrow$ "Bid shading"
- FPA and Dutch auction require the same considerations
- FPA and Dutch auction are actually strategically equivalent:
  - same strategy set $S_i = \Re_+$
  - same payoff $u_i(s)$ for every given strategy profile s
  - same Bayesian game!

Second-price sealed-bid auction (SPA)

- Bidders submit their offer, the highest bidder wins and pays the second-highest bidder's bid.
- With value v_i , should you ever submit bid $b_i > v_i$ instead of bidding v_i?
  - only changes the outcome if second-highest bid b_j is between v_i and b_i
    - if $b_j > b_i$ you don't win either way
    - if $b_j < v_i$ you win and pay b_j either way
  - But then you win and pay $b_j > v_i$ and make negative surplus $\Rightarrow$ better to bid $b_i = v_i$

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  - But then you win and pay $b_j > v_i$ and make negative surplus $\Rightarrow$ better to bid $b_i = v_i$
- Should you ever bid $b_i < v_i$ instead of bidding v_i?
  - only changes the outcome if $b_j \in (b_i, v_i)$
    - if $b_j > v_i$ you don't win either way
    - if $b_j < b_i$ you win and pay b_j either way
  - But then you lose, whereas with $b_i = v_i$ you would have won and made positive surplus $\Rightarrow$ better to bid $b_i = v_i$
- $\Rightarrow$ It is a weakly dominant strategy to bid $b_i = v_i$

Ascending ("English") auction

- The auctioneer starts with a low price and announces higher prices, until only one bidder remains and pays the last announced price.
- Should you ever leave the bidding while announced price p is below v_i ?
  - No: stay and either win at some $p < v_i$ (positive surplus), or leave at v_i (zero surplus)

Ascending ("English") auction

- The auctioneer starts with a low price and announces higher prices, until only one bidder remains and pays the last announced price.
- Should you ever leave the bidding while announced price p is below v_i ?
  - No: stay and either win at some $p < v_i$ (positive surplus), or leave at v_i (zero surplus)
- Should you ever stay in the bidding when announced price p is above v_i ?
  - No: leaving guarantees zero surplus while staying in might lead to negative surplus (and never to positive)
- $\Rightarrow$ weakly dominant strategy to stay in the auction until $p = v_i$
- (with independent values) SPA and English auction are strategically equivalent

Optimal bidder behavior in the first-price auction I

- What is the equilibrium strategy $b_i : [\underline{v}, \bar{v}] \rightarrow \Re_+$ in a FPA (or in a Dutch) auction?
- We will consider a simple case:
  - two bidders have with v_i independently and uniformly distributed on $[0, 1]$
- Assume that there exists a symmetric bidding equilibrium where everybody follows the strictly increasing bidding strategy $\beta : [\underline{v}, \bar{v}] \rightarrow \Re_+$.
- If j follows strategy β, i's expected payoff from bidding b_i is

$$Pr[b_i > \beta(v_j)] * (v_i - b_i) \\ + Pr[b_i = \beta(v_j)] \frac{1}{2} * (v_i - b_i) + Pr[b_i < \beta(v_j)] * 0$$

Optimal bidder behavior in the first-price auction II

- If j follows strategy β, i's expected payoff from bidding b_i is

$$
\begin{aligned}
Pr[b_i > \beta(v_j)] * (v_i - b_i) &= Pr[v_j < \beta^{-1}(b_i)] * (v_i - b_i) \\
&= \beta^{-1}(b_i) * (v_i - b_i)
\end{aligned}
$$

- first order condition of maximizing payoff over bid b_i

$$
\begin{aligned}
&\beta^{-1\prime}(b_i) * (v_i - b_i) - \beta^{-1}(b_i) = 0 \\
\Leftrightarrow\ &\frac{1}{\beta'(\beta^{-1}(b_i))} * (v_i - b_i) - \beta^{-1}(b_i) = 0
\end{aligned}
$$

- if β is equilibrium, maximum is achieved at $b_i = \beta(v_i)$ and therefore

$$\frac{v_i - \beta(v_i)}{\beta'(v_i)} - v_i = 0$$

- this differential equation is solved by

$$\beta(v_i) = v_i/2$$

Optimal bidder behavior in the first-price auction III

- both players bidding according to the strategy $\beta(v_i) = v_i/2$ is equilibrium!
  - derivation maybe tricky but...
  - ...make sure you understand that bidding $v_i/2$ is type v_i's best response if j uses the strategy $b_j(v_j) = v_j/2$
- a lot of bid shading in equilibrium!
  - with I players (and iid uniformly distributed values) the equilibrium is

$$\beta(v_i) = \frac{I-1}{I} v_i$$

  - $\Rightarrow$ bid shading decreases with number of bidders

Which auction yields higher revenue?

- For v_1, v_2 uniformly iid on $[0, 1]$, which auction gives the higher revenue?
- revenue of the FPA
  - expected revenue from bidder i:

$$\int_0^1 Pr[v_j < v_i]\frac{v_i}{2}\,dv_i = \int_0^1 v_i\frac{v_i}{2}\,dv_i = \frac{1}{6}$$

  - total expected revenue: $2 * 1/6 = 1/3$
- revenue of the SPA
  - expected revenue from bidder i:

$$\int_0^1 Pr[v_j < v_i]\mathbb{E}[v_j | v_j < v_i]\,dv_i = \int_0^1 v_i\frac{v_i}{2}\,dv_i = \frac{1}{6}$$

  - total expected revenue: $2 * 1/6 = 1/3$

Conclusions so far

- with independent private values
    - first price sealed bid and Dutch auction are strategically equivalent
    - second price sealed bid and ascending auction are strategically equivalent
- with independent private values and uniformly distributed types
    - both auctions are efficient (bidder with highest valuation gets the good)
    - both auctions yield the same expected revenue

Aside: Information rents and commitment

- for a moment, suppose the seller knows the bidders' valuations
  - how will he maximize revenue?
  - what is the expected utility of a buyer?
- how does this compare to expected buyer utility when the seller does not observe buyer valuations?

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- for a moment, suppose the seller knows the bidders' valuations
  - how will he maximize revenue?
  - what is the expected utility of a buyer?
- how does this compare to expected buyer utility when the seller does not observe buyer valuations?
- private information creates an "information rent"
- the seller can infer valuations from bids in auction
  - incentive to cancel auction and switch to procedure above. . .
  - . . . but buyers anticipating would then not participate in auction (or bid lower)
  - importance of *commitment*
    - seller will not change auction rules midway

Revenue equivalence theorem (RET)

Revenue equivalence theorem

Suppose that valuations are identically and independently distributed with strictly positive density on $[\underline{v}, \bar{v}]$ and that bidders are risk neutral.
Then, every *(auction format, equilibrium)* pair such that in equilibrium

- the object is won by the bidder with the highest valuation and
- a bidder of type $\underline{v}$ has zero expected utility

gives the same expected revenue to the seller and the expected utility of a type v_i is the same in every such pair.

- revenue equivalence is much more general (regardless of distribution, auction type etc.)

Revenue equivalence theorem: theoretical relevance

- RET as a theoretical tool
  - can help us think through changes in auction format and whether they will have an effect or not
  - helps us to understand complicated auctions by relating strategies and outcomes to simpler auctions
  - can be used to derive equilibria in a simple manner (example for this later on)
  - is a starting point for further analysis
    - how does the comparison of revenue change if assumptions of RET fail?

Revenue equivalence theorem: economic relevance

- RET as an economic insight
  - Does it mean that in real life, all auction formats will have the same revenue? No!
  - But there is no a priori reason to expect that any auction format would always be better than others ...
  - ... or that any format would be better for some people than another format.
  - One weak formulation with real-life relevance:
    - If we change the rule to increase expected payment for a given bid b_i, bidders will lower their bids b_i to compensate for that
    - $\Rightarrow$ Under idealized conditions, these effects exactly cancel each other out.

Revenue equivalence theorem: proof I

- take an (equilibrium, auction format) pair and assume that assumptions of RET are satisfied
- define $U_i(v_i) = v_i P(v_i) - T_i(v_i)$ where
  - U_i is expected utility of player i in equilibrium
  - P is probability that all other bidders have a lower type than v_i, i.e. $P(v_i) = F(v_i)^{I-1}$
  - T_i is the expected amount of money i pays to the auctioneer in equilibrium if he has type v_i

Revenue equivalence theorem: proof II

- define $U_i(v_i) = v_i P(v_i) - T_i(v_i)$
- envelope theorem: $U_i'(v_i) = P(v_i)$
  - in equilibrium type v_i' prefers his bid to the bid of type v_i''

$$U_i(v_i') \geq v_i' P(v_i'') - T_i(v_i'') = U_i(v_i'') + (v_i' - v_i'')P(v_i'')$$

  - in equilibrium type v_i'' prefers his bid to the bid of type v_i'

$$U_i(v_i'') \geq v_i'' P(v_i') - T_i(v_i') = U_i(v_i') - (v_1' - v_i'')P(v_i')$$

  - taking these two inequalities together (and let $v_i' > v_i''$)

$$P(v_i') \geq \frac{U_i(v_i') - U_i(v_i'')}{v_i' - v_i''} \geq P(v_i'')$$

  - taking limit $v_i'' \to v_i'$ gives (as P is continuous)

$$U_i'(v_i') = P(v_i)$$

Revenue equivalence theorem: proof III

- envelope theorem: $U_i'(v_i) = P(v_i)$
- hence,

$$U_i(v_i) = U_i(\underline{v}) + U_i(v_i) - U_i(\underline{v}) = U_i(\underline{v}) + \int_{\underline{v}}^{v_i} U_i'(v)\,dv$$

$$= U_i(\underline{v}) + \int_{\underline{v}}^{v_i} P(v)\,dv = \int_{\underline{v}}^{v_i} P(v)\,dv = \int_{\underline{v}}^{v_i} F(v)^{I-1}\,dv$$

- expected utility of player i of type v_i does not depend on specific auction format or equilibrium!
- same is true for expected payment of type v_i of player i:

$$T_i(v_i) = v_i P(v_i) - U_i(v_i) = v_i F(v_i)^{I-1} - \int_{\underline{v}}^{v_i} F(v)^{I-1}\,dv$$

- expected revenue is just $\sum_{i=1}^{I} \mathbb{E}[T_i(v_i)]$ which therefore also does not depend on auction format or equilibrium □

Revenue equivalence theorem: proof idea

- the envelope theorem states that the derivative of bidder utility only depends on the type distribution (given that the highest type wins)
- assumption that the lowest type has zero utility
- together this implies that the utility of every type depends only on the type distribution (and not the auction format, equilibrium strategies etc.)
- bidder utility only depends on the type distribution!
- welfare (defined as expected valuation of the person who gets the good) is the same in all auctions as the highest bidder always gets the good
- revenue is difference between welfare and bidder utility... done!
- note: envelope theorem does all the real work!

Applying RET I

- Consider the following auction formats:
  - English (i.e. ascending) auction
  - The highest bidder gets the object and pays twice her bid
  - The highest bidder gets the object and pays the sum of her bid and the next-highest bid
  - An ascending auction is held until only two bidders remain. Then these two are asked to submit sealed bids; the highest bidder gets the object and pays her bid ("Anglo-Dutch Auction")
- Which one will give the highest expected revenue?

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- Which one will give the highest expected revenue?
  - It seems reasonable that there are symmetric equilibria in strictly increasing strategies.
  - They will all give the same expected revenue (and in expectation make the bidders equally well off)!

Applying RET IIa

- 2 companies are engaged in a legal battle
- each company chooses how much money to spend on lawyers
- company that invests more wins the battle and gets a value of v_i, where v_i are iid and uniform on $[0, 1]$
- This is an all-pay auction: Every company chooses a bid ($=$ how much to pay for lawyers) and pays this for sure, but only the one who pays more wins the prize.
- How much will each company spend on lawyers in equilibrium, if it only knows its own valuation v_i?
- Calculating this directly would be as complicated as solving a FPA ...
- ... but the RET can help us!

Applying RET IIb

- consider the corresponding SPA: equilibrium to bid own valuation
- if i wins with bid $b_i = v_i$, the expected bid of j is $\mathbb{E}[v_j | v_j < v_i] = v_i/2$
- expected payment of type v_i in SPA is probability of winning, i.e. v_i, times expected payment when winning

$$T_i(v_i) = \mathbb{E}[v_j | v_j < v_i] Pr[v_j < v_i] = v_i/2 * v_i = v_i^2/2$$

- assume the all pay auction has a symmetric equilibrium in which the highest type wins
  - wining probability same as in SPA for every type
  - expected payment must be the same for every type as in SPA by RET
  - equilibrium bid in all pay auction must be $v_i^2/2$! done!

Challenges in practical auction design

- theoretical:
  - understanding and solving the auction
- competition policy:
  - collusion among bidders
  - enough bidders must enter
  - predatory behavior by powerful/rich bidders
- behavioral:
  - bidding is done by actual humans (or companies) with image concerns etc.
  - solving for equilibria can be hard (cognitive constraints)
  - risk-aversion, or other non-standard utility functions
- context/institutional:
  - credibility and commitment of auctioneer
  - political pressures on auction designer
- We will today discuss some examples of the above.

Collusion in SPA

- consider a second-price auction
- If bidders can figure out that i has the highest valuation, the following is an equilibrium:
  - i bids v_i , everybody else bids 0
  - $\Rightarrow$ i gets the object at price 0
- Bidders can accomplish this by forming a bidding ring and running a pre-auction knock-out.

Pre-auction knockout I

- Consider the following mechanism:
  1. ring organizer asks each member to report their valuation v_i
  2. member with the highest announced valuation $\hat{v}_i$ "represents" the ring and submits his actual bid; everybody else drops out or bids 0
  3. if i wins the auction, he pays the amount (price without ring) - (price with ring) ≥ 0 to the ring organizer
  4. organizer pays t_i to all ring members, where $t_i =$ (i's expected payment without ring)-(i's expected payment with ring)

Pre-auction knockout II

- is efficient (i.e. bidder with highest valuation wins)
- makes ring members better off (as they sometimes get a positive payoff even when losing)
- participation in ring and truthful bidding is an equilibrium
  - implicit assumption: within ring transfers are enforceable
- divert expected revenue from the seller to the ring members
  - effect similar to reducing the numbers of bidders

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- divert expected revenue from the seller to the ring members
  - effect similar to reducing the numbers of bidders
- pre-auction knockouts exist(ed) in real-life and were one reason for the development of antitrust legislation
- FPA seems less vulnerable to collusion (via pre-auction knockout)
  - higher incentives to cheat on the ring if ring bidder bids very low
  - if ring bidder bids not very low, he has to pay his bid

Asymmetric bidders I

- two bidders
- bidder one: v_1 distributed uniformly on $[0, 1]$
- bidder two: v_2 distributed uniformly on $[1, 2]$
- what is outcome of second price auction? what is expected revenue?

- is the first price auction efficient (i.e. does the bidder with the highest valuation win in equilibrium)?
- does revenue equivalence hold between FPA and SPA?

Asymmetric bidders II

- one equilibrium of the first price auction:

$$b_1(v_1) = \begin{cases} v_1 & \text{if } v_1 < 2/3 \\ v_1/2 + 1/3 & \text{else} \end{cases}$$

$$b_2(v_2) = \begin{cases} v_2/2 + 1/6 & \text{if } v_2 < 4/3 \\ 5/6 & \text{else} \end{cases}$$

- check that this is an equilibrium (exercise!)
- note:
    - bidder 1 has zero probability of winning iff $v_1 \leq 2/3$
    - bidder 2 has probability 1 of winning iff $v_2 \geq 4/3$
    - $b_2(1) < 1$, i.e. all types of bidder 2 shade their bids!
- what happens if $v_1 = 1$ and $v_2 = 1.1$?
- is expected revenue lower or higher than in the second price auction?

Encouraging entry

- higher number of bidders increases revenue
  - higher probability of having one/two bidders with high values
  - less bid shading in first price auction
- sometimes argued that first price sealed bid auction encourages entry
  - low value bidders can win against high value bidders (when those sufficiently shade their bids)
  - impossible in an ascending auction (see "asymmetric bidders")
- encouraging even low value bidders to participate in auction can be important
  - design focus on low cost of entry
    - easy access, simple design, information provision, possibly bidding subsidy...

Behavioral considerations

- Many of our game-theoretic results have to be taken with a grain of salt.
- For example, the sealed-bid FPA and the Dutch auction often give different results if played with the same people (and the same v_i).
- Risk-aversion and similar assumptions also imply the RET does not apply.
- Another consideration:
  - People may derive utility from winning (or resent losing).
  - People may be embarrassed if they win an auction and overpay $\Rightarrow$ further bid shading.

Risk aversion

- bidder i derives utility $u(v_i - p)$ from the object when winning at price p (and $u(-p)$ if he does not win)
  - $u : \Re \to \Re$ is strictly increasing and strictly concave
  - assume $u(0) = 0$ and $u'(0) = 1$ for comparison with risk neutrality
- SPA:
  - still optimal to bid valuation (same argument as before combined with u strictly increasing)
- FPA:
  - expected utility: $Pr(b_i > b_j)u(v_i - b_i)$
  - effect of increasing bid
    - increase $Pr(b_i > b_j)$
    - decrease $v_i - b_i$ and therefore $u(v_i - b_i)$
  - by concavity of u and $u'(0) = 1$, $u'(v_i - b_i) < 1$ $\Rightarrow$ second effect is weaker than under risk neutrality
  - indeed risk aversion leads typically to higher equilibrium bids than risk neutrality in FPA
  - $\Rightarrow$ more revenue in FPA than in SPA

Example: Embarrassment from overbidding I

- sealed-bid FPA in a standard private values model
- after the auction ends all bids are revealed
- winning bidder could be said to be "overpaying" by $b_i - max_{j \neq i} b_j$
- assume that the winning bidder's utility is

$$v_i - b_i - \gamma(b_i - max_{j \neq i} b_j)$$

- How does this influence each bidder's expected revenue?
- How does it influence expected revenue?

Example: Embarrassment from overbidding II

$$v_i - b_i - \gamma(b_i - max_{j \neq i} b_j)$$

- "Embarrassment cost" is like an extra payment for the winning bidder.
- RET applies but "revenue" is overall expenditure by all bidders, including resources "spent" on embarrassment!
- RET: The expected utility of each bidder is the same as in an FPA (or SPA) without the embarrassment cost.
- RET: revenue including embarrassment costs is the same as in an FPA (or SPA) without the embarrassment cost.
- $\Rightarrow$ expected revenue for the seller is lower due to embarrassment costs
- Running an ascending auction (where there is no overbidding embarrassment) will increase seller revenue!

Common-value auctions

- So far, we have assumed that each bidder has their own privately known valuation v_i.
- alternative assumption: The object has the same value v for all, but v is unknown
- simple case: The wallet auction
  - two bidders; bidder i observes t_i
  - value of the object is $v = t_1 + t_2$. i.e. each bidder knows only part of the value (Examples: Oil fields, company takeovers, ...)
- equilibrium in an ascending auction:
  stay in the bidding until $2t_i$:
  - if other player does so and you stay in the bidding for $p > 2t_i$ and win, you overpay $p = 2t_j > t_j + t_i$
  - if you leave bidding at a lower price you might not win at $p = 2t_j < t_i + t_j$
  - (all other cases: deviation does not change payoff)

Common-value auctions: winner's curse

- calculating equilibrium of FPA is somewhat tricky (we will not do it!)
- with independent private values: bidding your valuation yields zero payoff in FPA
- what is your expected valuation of the good if...
  - ... you observe t_1 and
  - ... t_2 is distributed, independently from t_1, uniformly on $[0, 1]$?

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- what is your expected valuation of the good if...
    - ... you observe t_1 and
    - ... t_2 is distributed, independently from t_1, uniformly on $[0, 1]$?
- is your expected payoff also zero if you bid your expected valuation in a common value auction?

Almost common value auctions

- Now imagine that the object has a slightly higher value for 1 than for 2, i.e. $v_1 = t_1 + t_2 + \varepsilon$ and $v_2 = t_1 + t_2$.
- Is there an equilibrium in which 2 wins an ascending auction?
  - No: If 2 is willing to pay a price p, then 1 should also be willing to pay that price (and a bit more).
- In equilibrium, 2 never wins the auction!
- Slight modification completely changes the result $\Rightarrow$ 2 has no reason to even enter the auction!
- It can be shown that first price auctions are much less affected (i.e. a change in ε in 1's valuation leads to a similarly-sized change in bidding strategies winning probabilities do not change much).

Almost common value auctions: Application

- If two bidders compete to take over a company (= common value)...
    - ...but one of them already owns a small stake of the company...
    - ...we have precisely this situation!
- The problem: Takeover battles are by their nature ascending auctions.
- Example: In Britain in the late 1990s, BSkyB (a TV company) tried to buy a stake in Manchester United.
    - TV licenses (for football broadcasts) are sold in an auction and the money is then distributed to the football clubs.
    - If a TV station owns a bit of a football club, they have an ε advantage in bidding.
    - $\Rightarrow$ In an ascending auction, this can make all the difference!

What is the right auction format?

- It depends!
- Second-price (and ascending) auctions:
  - appealing properties and are easy to understand (only require dominant strategies!)
  - more robust to distributional assumptions, concerns about secrecy
  - susceptible to collusion, small asymmetries...
- First-price auctions:
  - need more sophisticated bidders, sensitive to distributional assumptions etc (require Bayesian NE!)
  - often more robust to collusion and small asymmetries
- Competition between enough bidders can be more important than any theoretical details!
- Klemperer (2002): "Good auction design is really good undergraduate industrial organization; the two issues that really matter are attracting entry and preventing collusion."

Reserve price I

- independent private value setting
- can the seller obtain more expected revenue than in SPA/FPA?
- consider reserve price $\bar{b}$ in a SPA
  - if winning bid is below $\bar{b}$ seller keeps the good
  - if second highest bid but not winning bid is below $\bar{b}$, the winner has to pay $\bar{b}$
- effectively as if the seller was additional bidder bidding $\bar{b}$
- still weakly dominant to bid true value
- can expected revenue be increased by a properly chosen reserve price?

Reserve price II (intuition)

- start from $\bar{b} = 0$, i.e. no reserve price
- what is effect of increasing reserve price slightly to $\varepsilon > 0$?
  - less revenue if all bidders have value below ε
    - probability: $F(\varepsilon)^I$
  - more revenue if all but one bidder have value below ε
    - probability: $IF(\varepsilon)^{I-1}(1 - F(\varepsilon))^1$
- gain is more likely than loss (for ε small)
- (of course size of gain/loss matters as well but the probability effect can be shown to dominate for small ε)

Reserve price III (example)

- two bidders with valuation uniformly and independently distributed on $[0, 1]$
- expected revenue without reserve price: $1/3$
- reserve price of $\bar{b}$
- expected revenue from bidder i:

$$Pr(v_j < \bar{b}) * Pr(\bar{b} < v_i) * \bar{b} + \int_{\bar{b}}^{1} Pr(\bar{b} \leq v_j < v_i) \mathbb{E}[v_j | \bar{b} \leq v_j < v_i] \, dv_i$$

$$= \bar{b} * (1 - \bar{b}) * \bar{b} + \int_{\bar{b}}^{1} (v_i - \bar{b}) \frac{v_i + \bar{b}}{2} \, dv_i$$

$$= (1 - \bar{b})\bar{b}^2 + \int_{\bar{b}}^{1} \frac{v_i^2 - \bar{b}^2}{2} \, dv_i$$

$$= (1 - \bar{b})\bar{b}^2 + \frac{1}{6} - \frac{\bar{b}^3}{6} - \frac{(1 - \bar{b})\bar{b}^2}{2} = \frac{(1 - \bar{b})\bar{b}^2}{2} + \frac{1 - \bar{b}^3}{6}$$

$$= \frac{-4\bar{b}^3 + 3\bar{b}^2 + 1}{6}$$

Reserve price III (example)

- expected revenue from bidder i:

$$\frac{-4\bar{b}^3 + 3\bar{b}^2 + 1}{6}$$

- maximal for $\bar{b} = 1/2$ with maximum $5/24$
- maximal revenue: $2 * 5/24 = 5/12 > 1/3$
- note:
  - inefficiency: seller does not value good but good is not sold with probability $1/4$
  - similar to monopoly distortion: reduce probability/quantity below efficient level to increase prices and expected profits
  - market incompleteness: monopoly power!

Reserve prices and RET

- RET assumption "the object is won by the bidder with the highest valuation and" no longer holds with reserve price
- however, can be substituted by "the probability of getting a good as a function of type $P_i(v_i)$ is the same in the compared (auction format, equilibrium) pairs"
  (proof works similarly)
- hence, a version of RET also holds with reserve prices (though the reserve prices may have to differ between, e.g., SPA and FPA to get the same $P_i(v_i)$)

Position auction I

- ad position auction (as for Google ads)
- S ad slots on a website
- slot s has a click through rate of $x_s > 0$ where $x_1 \geq x_2 \geq \cdots \geq x_S$ (and define $x_{S+1} = 0$ for notational convenience)
- $I > S$ bidders with independent private values for an ad slot for a given search term (say "hotel in Cologne")
- value of slot s for bidder i is $v_i * x_s$
  - value per click is v_i

- new: several slots/prizes of differing prominence

Position auction II

- consider the following (Vickrey style) auction
  - s-highest bidder gets slot s
  - and pays $\sum_{t=s+1}^{S+1}(x_{t-1} - x_t)b_t$ where b_t is the t highest bid
  - interpretation: the price of increasing the click through rate from x_t to x_{t-1} is the bid of the t-highest bidder
- claim: bidding v_i is weakly dominant strategy
  - bidding above v_i: risk of overpaying, i.e. paying more than v_i for the last few clicks if someone bids between v_i and your bid
  - bidding below v_i: risk of missing out, i.e. if the next highest bid is below v_i, a deviation to v_i would yield additional clicks at a price less than v_i

Position auction III

- in practice:
  - click through rates depend on ads (or ad "relevance" to the search term)
  - assign to each advertiser a quality factor q_i
  - rank advertisers according to $score_i = b_i * q_i$, i.e. the advertiser with the s highest score $b_i * q_i$ gets slot s
  - prices are directly per actual click
  - used to be:
    price per click of s-highest bidder equals $score_{s+1}/q_s$ (where subscript s refers to s-highest bidder etc.)

Position auction IV

- price per click of s-highest bidder equals $score_{s+1}/q_s$
- suppose click through rate of ad i in position s is $q_i * x_s$
- bidding v_i is not a dominant strategy:
  - say this leads to slot s and payoff $(v_i - b_{s+1} * q_{s+1}/q_i)x_s q_i = x_s(v_i q_i - b_{s+1} q_{s+1})$
  - deviating to higher bids and getting a better position $t < s$ is not profitable: deviation payoff $x_t(v_i q_i - b_t q_t)$ where b_t is the t-highest bid under equilibrium bidding; deviation is negative as $b_t q_t$ is a higher score than $b_s = v_i q_i$
  - deviating to lower bids and getting a worse position $t > s$:
    - deviation payoff $x_t(v_i q_i - b_{t+1} q_{t+1})$ may be higher than $x_s(v_i q_i - b_{s+1} q_{s+1})$ (think, for example, of $t = s + 1$ and $x_s \approx x_t$)
  - $\Rightarrow$ bid shading
- $\Rightarrow$ industry has moved to first price bidding, i.e. price per click is own bid

Markets as auctions I

- how is a market price determined?
- stock market:
  - buyers state a bid price
  - sellers state an ask price
  - assume unit demand/supply (otherwise split a buyer who demands q units up into q unit demand buyers etc.)
  - stock exchange executes as many trades as possible such that bid price of buyer is above ask price of seller
    - how do you determine which trades to execute?

Markets as auctions II

- how do you determine which trades to execute?
  - order buyers according to bid price:

$$b_1 \geq b_2 \geq \cdots \geq b_I$$

  - order sellers according to ask price:

$$a_1 \leq a_2 \leq \cdots \leq a_J$$

  - execute trades 1 to k where k is the last index where $b_k \geq a_k$
  - reminds you of anything?

Markets as auctions III

- what is the trading price?
  - most natural: $(a_k + b_k)/2$
  - bidding game like first price auction
  - bid shading:
    - bidding true valuation ensures that a buyer (seller) trades if and only if the trading price is below (above) valuation
    - but: positive probability that own bid determines trading price
    - $\Rightarrow$ buyers (sellers) bid below (above) their valuation to affect trading price favorably in this case
    - if number of buyers and sellers is large, the chance of being buyer/seller k is small $\Rightarrow$ hardly any bid shading
    - with bid shading: possible inefficiency as, e.g., trade $k+1$ may be efficient
  - equilibrium strategies can be determined similar to first price auction but somewhat involved

Markets as auctions IV

can we do something like a second price auction?

- hopefully simpler bidding strategies etc.
- first try:
  - buyers pay b_{k+1} and sellers receive a_{k+1}
  - upside: bids of trading players do not affect price
  - $\Rightarrow$ no incentive for bid shading
  - problem: as $a_{k+1} > b_{k+1}$, there is a deficit!

Markets as auctions V

can we do something like a second price auction?

- second try:
  - execute only trades 1 to $k-1$
  - buyers pay b_k
  - sellers receive a_k
  - now: possibly surplus (say, profit of the stock exchange)
  - weakly dominant strategy to bid true valuation
    for buyer:
    - if trading price (after truthful bid) above valuation no deviation can bring price down $\Rightarrow$ no profitable deviation
    - if trading price (after truthful bid) below valuation, any deviation would either not affect outcome or lead to not trading $\Rightarrow$ no profitable deviation
- inefficiency: trade k not executed
  - if many buyers and sellers a_k and b_k are close $\Rightarrow$ inefficiency small
  - for large number of buyers and sellers: no market power $\Rightarrow$ complete market