(weak) Perfect Bayesian equilibrium

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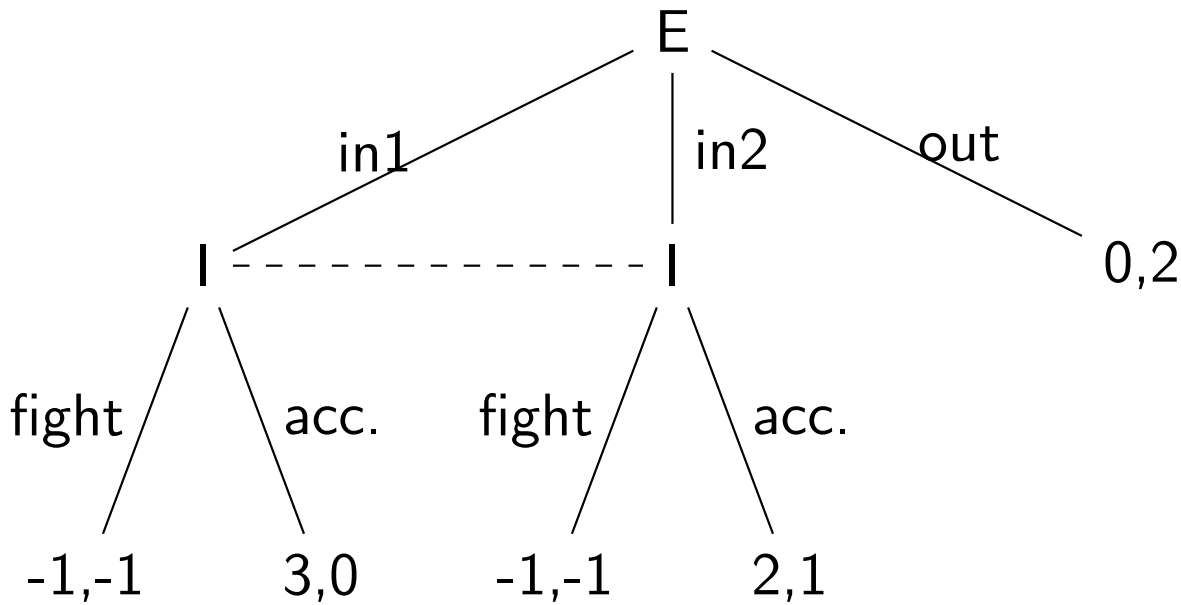
Outline

Introduction

- previous lecture: market can breakdown if only sellers know product attributes
- possible countermeasure: try to demonstrate good quality (independent tests etc.)
  - *dynamic* games of incomplete information
    - dynamic: players take (partially observed) actions sequentially
  - new solution concepts
    - with complete information: dynamics $\rightarrow$ *subgame perfect* Nash equilibrium (SPNE)
    - with incomplete information: dynamics $\rightarrow$ *(weak) perfect* Bayesian Nash equilibrium
- we introduce the equilibrium concept with simpler examples this time and return to the motivation next time

When SPNE is too weak



- entrant can enter with two technologies (in1 and in2)
- what are the (SP)NE?
- which SPNE is a (not so) reasonable prediction?

Beliefs

- at every information set H the acting player has to have a "belief" over the nodes in H
  - the players assigns probabilities to all decision nodes in his information set
  - these probabilities sum to 1

System of beliefs

A system of beliefs μ assigns to every node x in an extensive form game Γ_E a probability $\mu(x) \in [0, 1]$ such that $\sum_{y \in H(x)} \mu(y) = 1$ (where $H(x)$ is the information set in which x lies).

Sequential rationality

- sequentially rational: no player wants to change his behavior in some information set given his beliefs and strategies of the other players
  - each player's strategy maximizes expected utility at every information set
  - expected utility is calculated using
    - own beliefs at the information set and
    - strategies of all players

Sequential rationality

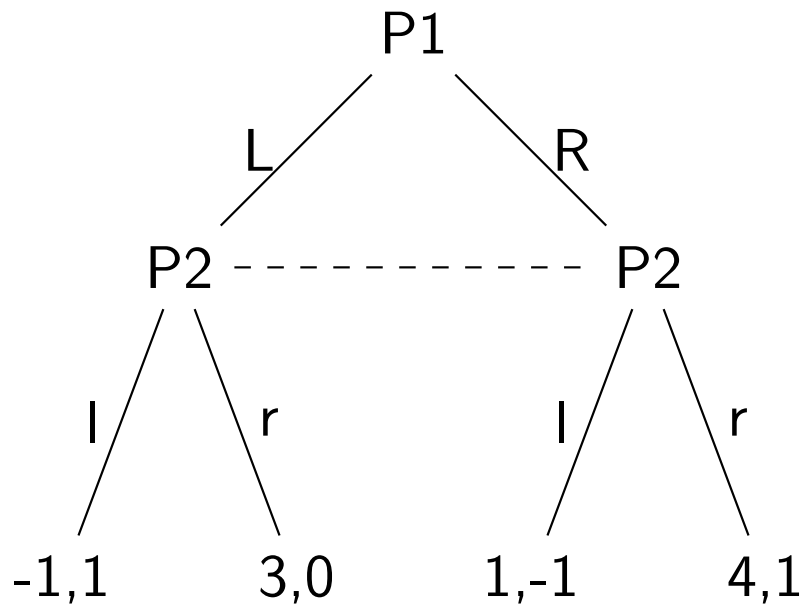
A strategy profile σ is sequentially rational given system of beliefs μ if for every information set H, the player acting at H cannot increase his expected utility by deviating from σ_i at H:

$$\mathbb{E}[u_i(\sigma_i, \sigma_{-i})|H, \mu] \geq \mathbb{E}[u_i(\tilde{\sigma}_i, \sigma_{-i})|H, \mu]$$

for all $\tilde{\sigma}_i$ differing from σ_i only at H.

- which strategy profile is sequentially rational in the previous example game (for which system of beliefs)?

Why sequential rationality may not be enough



- is there a belief system such that (R, l) is sequentially rational?

Weak perfect Bayesian Nash equilibrium

- beliefs should be consistent with the strategies used
  - $prob(x|H,\sigma) = \frac{prob(x|\sigma)}{\sum_{x' \in H} prob(x'|\sigma)}$ if $\sum_{x' \in H} prob(x'|\sigma) > 0$ and $x \in H$
  - arbitrary μ_H if $\sum_{x' \in H} prob(x'|\sigma) = 0$

Weak perfect Bayesian Nash equilibrium (weak PBE)

A profile of strategies σ together with a system of beliefs μ is a weak PBE in an extensive form game Γ_E if

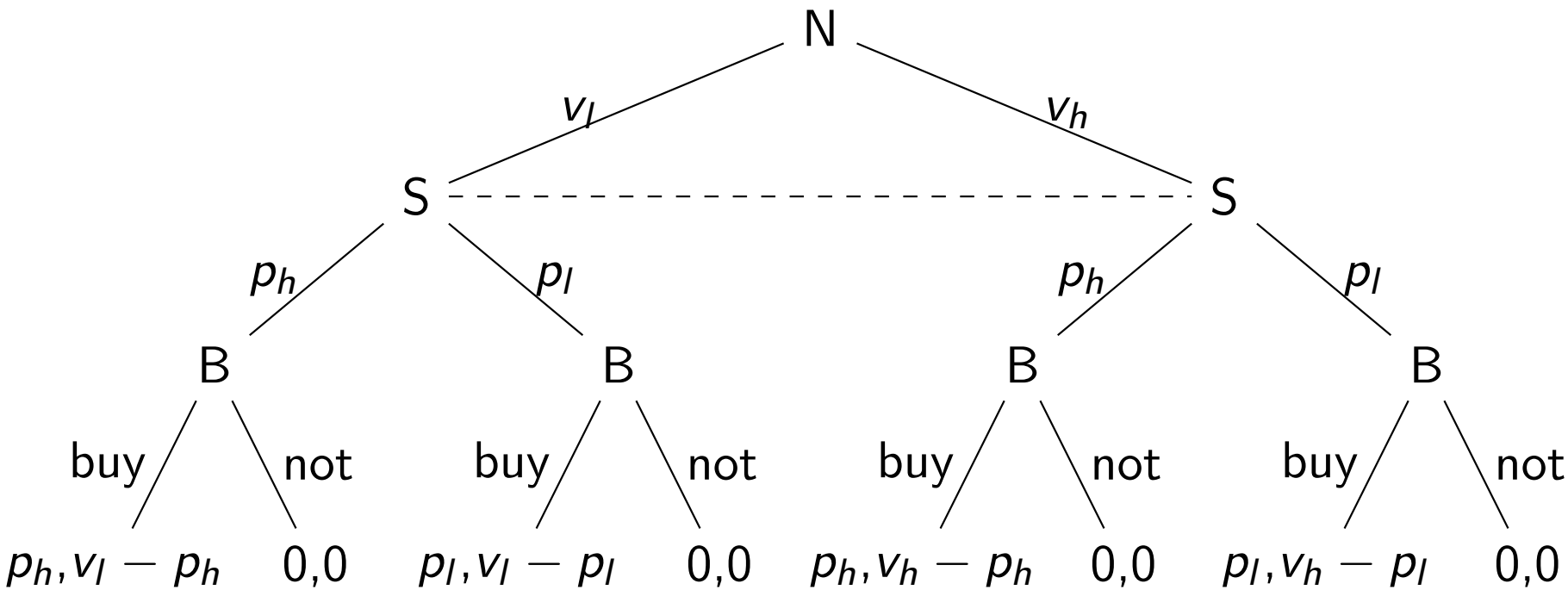
- σ is sequentially rational given μ,
- μ is derived from σ using Bayes' rule at all information sets reached with positive probability under σ.

- weak PBE in previous example?

incomplete information and Harsanyi's trick I

- so far: "imperfect information" (i.e. do not observe another player's prior actions)
- what about "incomplete information" (i.e. do not observe another player's "type")?
- example
  - buyer has private information about his valuation for one indivisible good
    - valuation is either v_h or v_l (both with probability $1/2$)
  - monopoly seller with zero costs sets a price $p \in \{p_h, p_l\}$
  - after observing p buyer decides whether to buy or not
- Harsanyi's trick: introduce artificial player "nature"
  - nature chooses type of buyer (each with probability $1/2$)
  - seller chooses price without observing nature's choice
  - buyer chooses to buy or not observing all prior choices
  - taking nature's strategy as fixed, we have a game as before and use wPBE as before

incomplete information and Harsanyi's trick II



- assuming $v_h = 7$, $v_l = 4$, $p_h = 5$ and $p_l = 3$, determine wPBE

Behavior based price discrimination I

- same buyer/seller example but 2 periods
  - buyer can buy 1 unit each period
  - seller can charge different prices each period
  - discounting: payoffs realized in period 2 are discounted with discount factor 3/4

Behavior based price discrimination II

- timeline
  - period 1:
    - nature chooses buyer's type $v \in \{7, 4\}$ each with probability $1/2$
    - seller chooses $p_1 \in \{5, 3\}$ (not observing v)
    - buyer decides whether to buy at price p_1, i.e. $b_1 \in \{0, 1\}$
  - period 2:
    - seller chooses p_2 (after observing b_1)
    - buyer decides whether to buy at price p_2, i.e. $b_2 \in \{0, 1\}$
- wPBE components
  - seller strategy: p_1 and $p_2(p_1, b_1)$
  - buyer strategy: $b_1(p_1)$ and $b_2(p_1, b_1, p_2)$
  - seller beliefs: μ_1 and $\mu_2(p_1, b_1)$

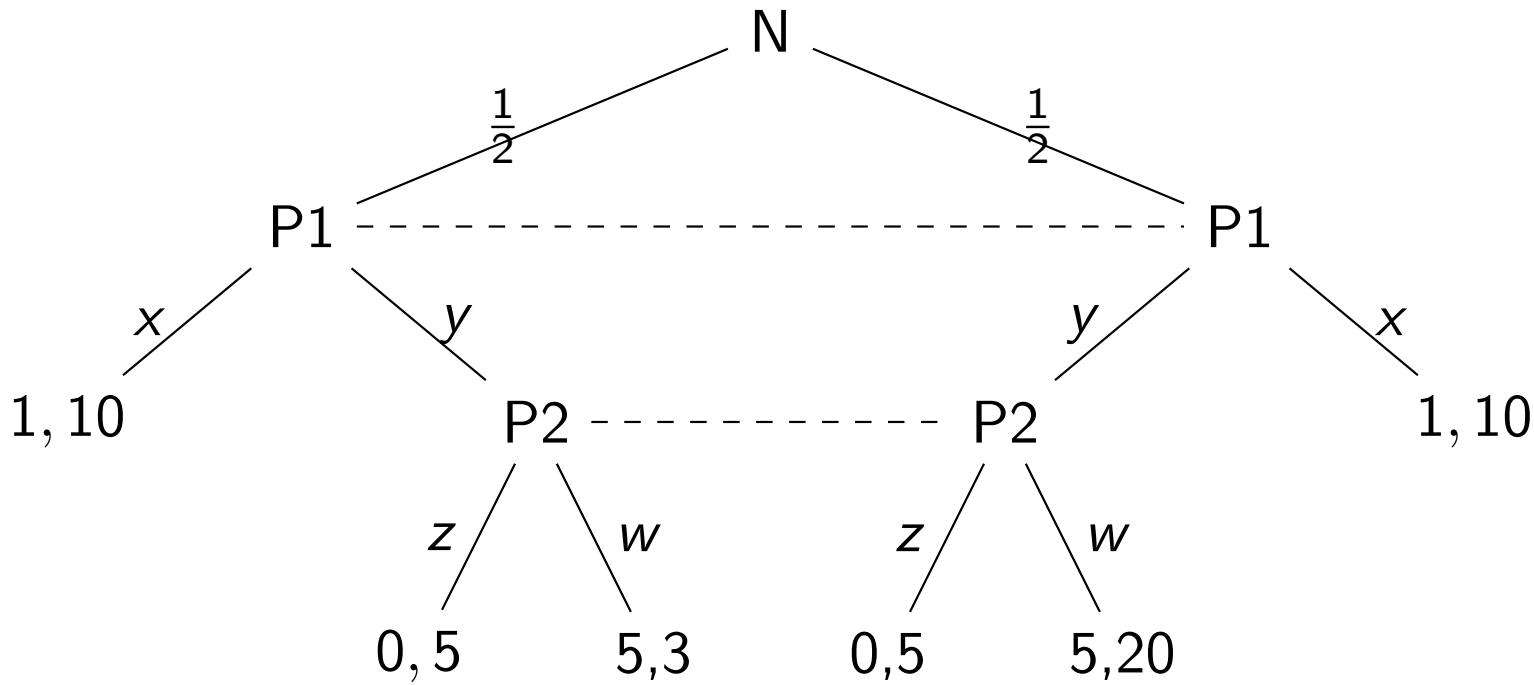
Behavior based price discrimination III

- sequential rationality: buyer buys in period 2 if and only if price is below his valuation
- sequential rationality seller: $p_2 = 5$ if and only if $\mu_2 5 \geq 3 \Leftrightarrow \mu_2 \geq 3/5$
- why is the following not a wPBE:
  - seller: $p_1 = 3$ and $p_2 = 3$ (regardless of p_1 and b_1)
  - buyer: buy in each period if and only if valuation is above price
  - beliefs: $\mu_1 = \mu_2 = 1/2$ (regardless of p_1 and b_1)

Behavior based price discrimination IV

- wPBE
  - seller: $p_1 = 5$, $p_2(5,1) = 5$, $p_2(5,0) = 3$, $p_2(3,1) = p_2(3,0) = 3$
  - buyer: buy in each period if and only if valuation is above price
  - beliefs: $\mu_1 = 1/2$, $\mu_2(5,1) = 1$, $\mu_2(5,0) = 0$, $\mu_2(3,1) = 1/2$, $\mu_2(3,0) = 1/2$
- seller uses period 1 to screen buyer types
- seller benefits in period 2 from conditioning his prices on purchase history

Why "weak" PBE is (sometimes too) weak: unreasonable beliefs



- one wPBE:
  - strategies: x and z
  - beliefs: $\mu_1 = (1/2, 1/2)$, $\mu_2 = (0.9, 0.1)$
- why is P2's belief inconsistent?

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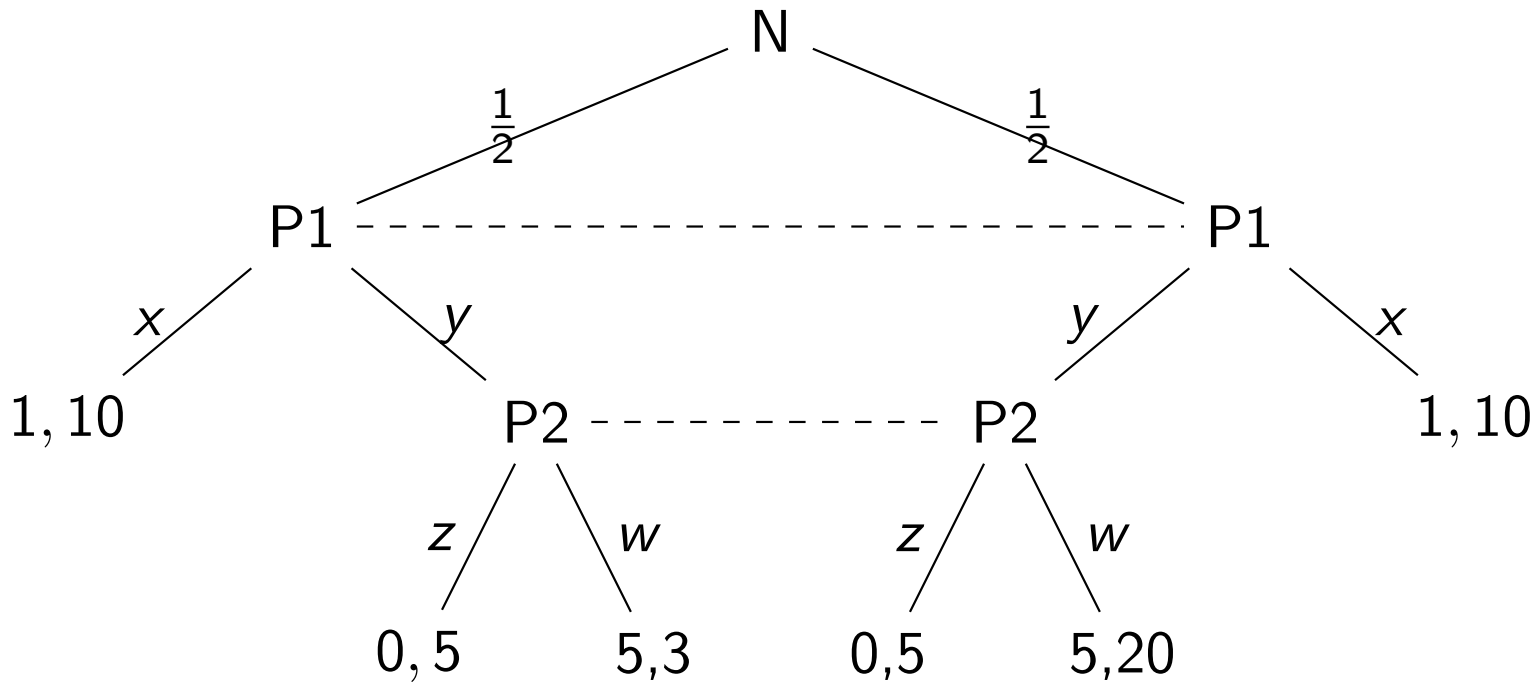
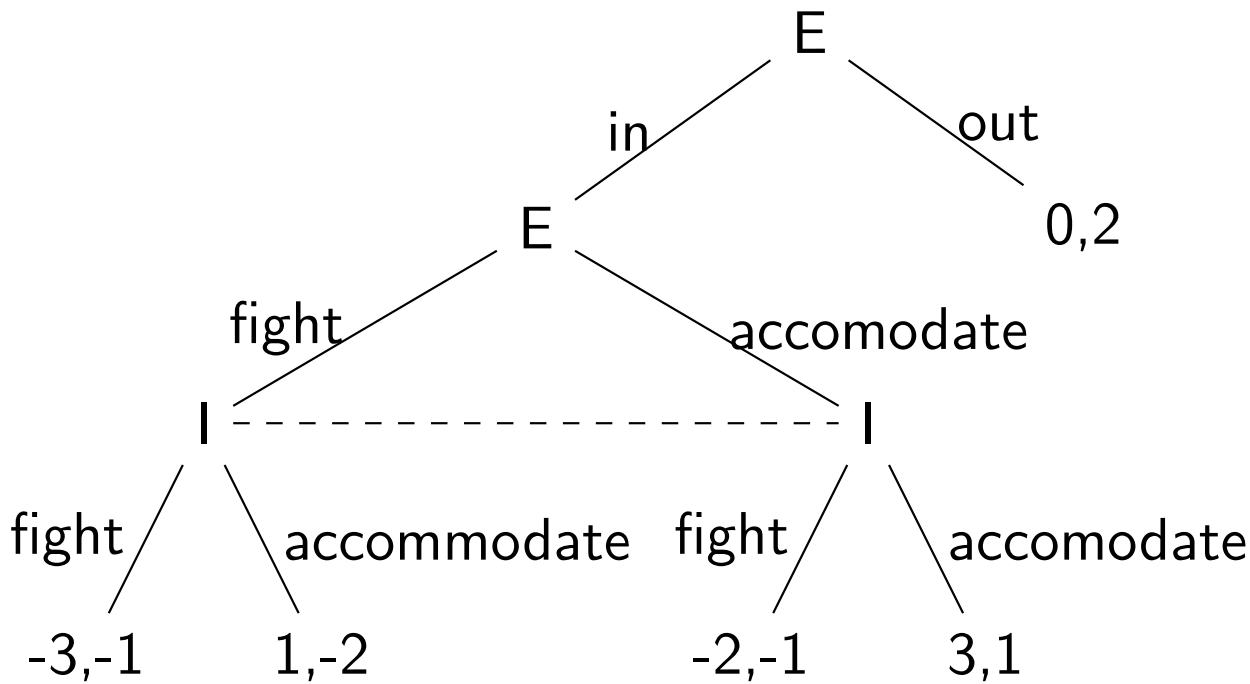


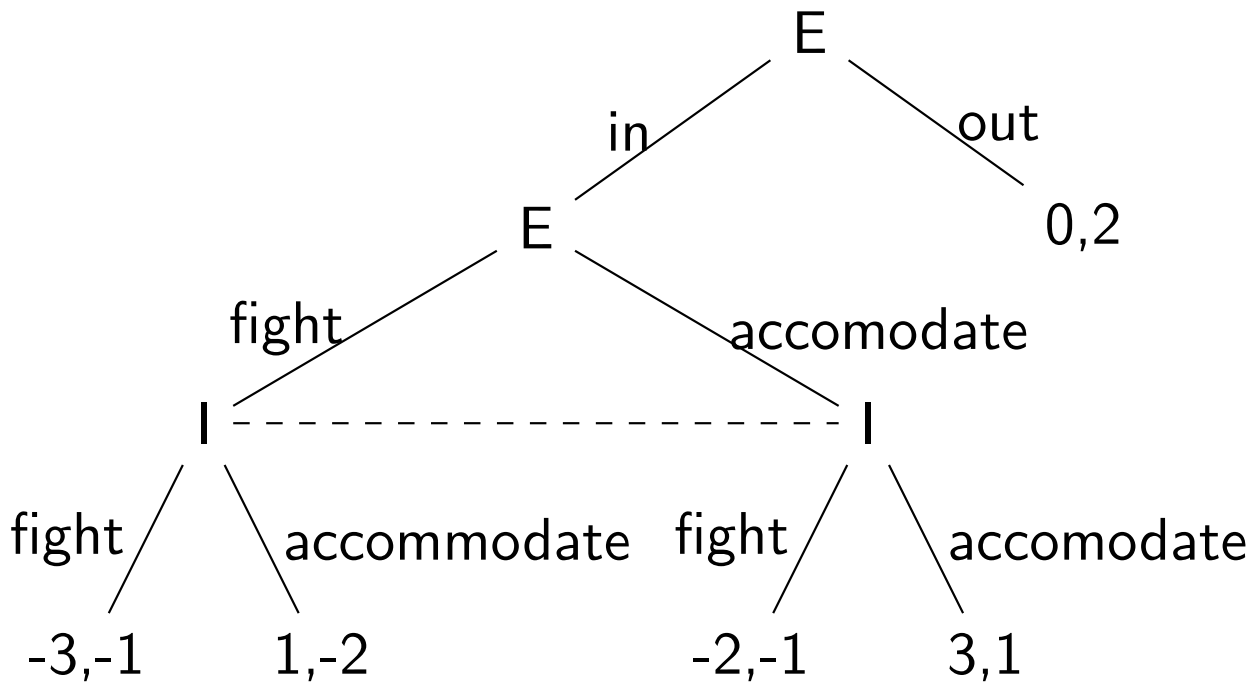
- one wPBE:
  - strategies: x and z
  - beliefs: $\mu_1 = (1/2, 1/2)$, $\mu_2 = (0.9, 0.1)$
- why is P2's belief inconsistent?
- (add requirement: some strategy profile leading to off path beliefs has to exist; "structural consistency")

Why "weak" PBE is (sometimes too) weak: not subgame perfect



- with which belief system would (out+accommodate, fight) be a weak PBE?
- is (out+accommodate, fight) subgame perfect NE?

Why "weak" PBE is (sometimes too) weak: not subgame perfect



- with which belief system would (out+accommodate, fight) be a weak PBE?
- is (out+accommodate, fight) subgame perfect NE?
- (add requirement "weak PBE in every subgame")

Perfect Bayesian equilibrium

- caution: different authors use different ways of defining *perfect Bayesian equilibrium*

perfect Bayesian equilibrium (PBE)

A perfect Bayesian equilibrium is a weak perfect Bayesian equilibrium which
(i) induces a weak perfect Bayesian equilibrium in every subgame
(ii) satisfies structural consistency, i.e. beliefs at every information set are such that a strategy profile consistent with these beliefs exists.

- PBE in previous example?