STOCHASTIC MODELS FOR FINANCIAL APPLICATIONS

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1 Introduction

Black-Scholes Model (BSM) is one of the most famous models in the world of finance. During the 1960s, pricing options was something of a black art, which meant neither banks nor investors could use these with confidence. So once the BSM introduced a scientific way to price options (1973), the Chicago Board of Trade opened an options exchange and the market for derivatives took off.
In this report we will look at the Geometric Brownian Motion (GBM) as a model for stock price movement and Black-Scholes Model used to price European Call Options and study the results.

2 Modelling of Stock Price Trajectory

A Stochastic Differential Equation (SDE) is similar to a regular Differential Equation. The dissimilarity arises due to the fact that an SDE contains terms which are stochastic in nature, typically white noise, where white noise is defined as the derivative of a Wiener Process.[1] The most widely known and used model for stock price prediction is the Geometric Brownian Motion (GBM) model, where GBM is a process defined as the solution of an SDE.

2.1 GBM for Stock Price Prediction

The general form of the SDE used to model fluctuating stock prices is given as follows

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t \tag{1}$$

where B_t represents the regular Wiener Process, μ is the drift coefficient, and σ is the diffusion coefficent, which is also referred to as volatility by the finance community. Note that, in general, μ and σ are variables. However, the most widely used models for stock prices treat these as constants, for ease of estimation from historical data. The process S_t in (1) is called Geometric Brownian Motion (GBM).

[1] We know that the Wiener Process leads to a discontinuous trajectory, making the definition of the derivative of the process to be ambiguous. This topic has been extensively discussed in the literature, and is beyond the scope of this report. For our study, we use Ito's representation of Calculus in the stochastic framework.

One of the most important features of this SDE is that it is possible to obtain a closed-form solution for the resulting process in terms of the initial conditions. We now proceed to solve the SDE :-

Let us make the substitution $Z_t = \ln(S_t)$. Using Ito's Lemma (Appendix), we may write the differential $\mathrm{d}(Z_t)$ as

$$\mathrm{d}(Z_t) = \frac{1}{S_t}\mathrm{d}S_t + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)(\mathrm{d}S_t)^2 \tag{2}$$

$$= \frac{1}{S_t}(\mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t) + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)(\mathrm{d}S_t)^2 \tag{3}$$

To simplify the second term in (3), we use

$$\begin{aligned}(\mathrm{d}S_t)^2 &= (\mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t)^2 \\ &= (\mu S_t \mathrm{d}t)^2 + (\sigma S_t \mathrm{d}B_t)^2 + 2(\mu S_t \mathrm{d}t)(\sigma S_t \mathrm{d}B_t) \\ &\approx (\sigma S_t \mathrm{d}B_t)^2 = \sigma^2 S_t^2 \mathrm{d}t\end{aligned}$$

after ignoring super-linear powers of dt and using the known result that $\mathrm{d}B_t = \sigma\sqrt{\mathrm{d}t}$. We now have for $\mathrm{d}(Z_t)$

$$\mathrm{d}(Z_t) = \left(\mu - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma \mathrm{d}B_t \tag{4}$$

Integrating both sides, and writing the equation in terms of S_t we get a closed-form expression for S_t

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\} \tag{5}$$

The above equation can be used to simulate a stock price trajectory. Since $B_t \sim \mathcal{N}(0, t)$, $\ln\left(\frac{S_t}{S_0}\right)$ is a normally distributed random variable with mean $\left(\mu - \frac{\sigma^2}{2}\right)$. This leads to

$$E[S_t] = S_0 e^{\mu t} \tag{6}$$

This tells us that according to the GBM model of stock price trajectory evolution, the price of the stock is expected to grow at the same rate as a principal amount S_0 in a bank with interest rate μ.

2.2 Parameter Estimation

In this subsection, we briefly address the issue of parameter estimation for the GBM model. The two parameters to be estimated are μ and σ which are the drift and diffusion parameters. The drift parameter controls the deterministic part of the trajectory and the diffusion parameter influences the fluctuations. The diffusion parameter is also called 'Volatility'.

Let us define the log-returns variables u as below

$$u = \ln \frac{S_t}{S_{t-1}} \tag{7}$$

where S_{t-1} represents the stock prices at the previous time step. From (5), we see that

$$u = \left(\mu - \frac{\sigma^2}{2}\right) \mathrm{d}t + \sigma B_t \tag{8}$$

The unbiased estimator for the log-returns is

$$\hat{u} = \frac{\sum_{i=1}^{n} u_i}{n} \tag{9}$$

where n is the total number of time steps. From (8), we can see that the standard deviation of u is $\sigma\sqrt{\mathrm{d}t}$. Therefore, and estimate of volatility $\hat{\sigma}$ is obtained by scaling the estimate of standard deviation of u by a factor of $\sqrt{\mathrm{d}t}$. The standard deviation ν of u is estimated using the estimator

$$\hat{\nu} = \sqrt{\frac{\sum_{i=1}^{n}(u - \hat{u})^2}{n-1}} \tag{10}$$

From (8), we can also clearly see that the expected value of u is $\left(\mu - \frac{\sigma^2}{2}\right) \mathrm{d}t$. An estimator for the drift parameter is thus given by

$$\hat{\mu} = \frac{\hat{u}}{\mathrm{d}t} + \frac{\sigma^2}{2\mathrm{d}t} \tag{11}$$

In real life computations, dt would be approximated by a finite time interval.

2.3 Validity of the GBM Model

Using GBM to model stock price movement has some advantages as well as disadvantages. Some arguments in favour of the model are :-

- The movement of the stock price is independent of the value of the stock price.
- GBM exhibits the same type of random fluctuations are seen in observed stock price data.
- GBM does not allow negative values in the trajectory unlike a regular Wiener Process, which makes it a more intuitive choice.
- Using GBM as the model allows a closed-form solution for the stock price at any time t to be analytically computed.

While the advantages given above are important, there is a drawback that often makes it fail as a reliable model for stock price prediction. The main disadvantage is that the parameters μ and σ are assumed constant. In reality, this is not always the case, as will be illustrated through simulation in the next section.

3 Simulation

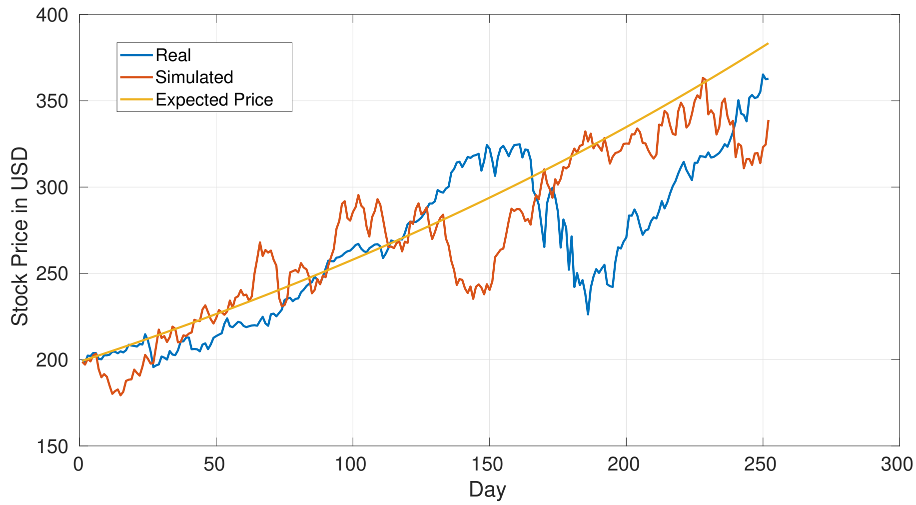

Figure 1: Comparison of real and stimulated stock price trajectories of Apple from 25 June 2019 to 25 June 2020.

From Fig. 1, we observe that the actual stock price conforms reasonably well to the expected trajectory, and shows a roughly exponential growth over the given time period. The simulated trajectory also roughly emulates the trends of the actual trajectory in the considered time frame. We observe similar random fluctuations in price in the simulated as well as actual stock prices.

In Section 2.2, we remarked that the GBM model for stock price prediction often fails. This is, in part, due to the fact that the parameters μ and σ are assumed to be constant. This means that, if there is a world event, or something else that causes a shift of confidence in traders at large, it will not be reflected in the GBM model. This is clearly showcased in Fig. 2, where we estimate the parameters μ and σ using historical data until the coronavirus was first detected in the US. Using that estimate, we simulate the trajectory for the entire time duration of 2017 - present for the Boeing stock. We observe that the model dramatically fails after the coronavirus caused a huge drop of confidence in investors of the company, but it performed reasonably well until then.

The problem of choosing the drift and diffusion parameters for the GBM model has been a long-standing topic of research in the quantitative finance community. Some questions that need to be answered are :-

- How much memory should we give the model in order for it to accurately predict future stock prices? In other words, how much historical data must one use for estimation of parameters?
- Does the answer to the above question change depending on whether we are trying to predict stock prices in the short-term or long-term? If yes, how do we factor this into the model?
- How do we estimate the variability of the parameters, if choosing constant parameters does not work well in the given case?

Each of these questions have been the focus of many researchers, but the common consensus within the finance community is that no model or method of estimation can ever be used completely reliably for

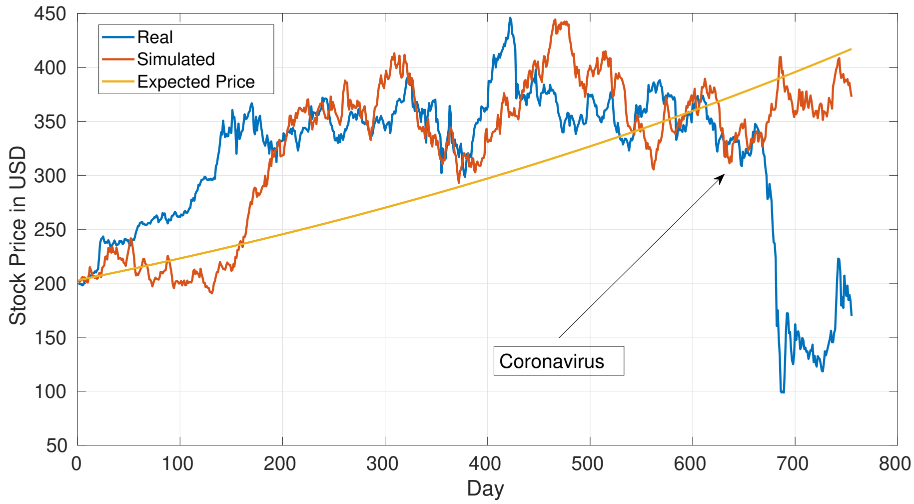

Figure 2: Comparison of real and stimulated stock price trajectories of Apple from 25 June 2019 to 25 June 2020.

stock price prediction, as we would expect intuitively. This is why there is always some risk involved while trading stocks.

4 Black Scholes Model

Borrowing the idea of underlying stock price movement from the Geometric Brownian Motion, and assuming stock price movement is analogous to diffusion of particles in Brownian motion.
Now this model assumes that there are no transaction costs, no dividend payment, option is of the European type, no restriction on short selling and it is possible to short sell too.

4.1 Black Scholes PDE

Notation used:

- V(S,t): Price of Call/Put Option
- S: Stock Price of underlying Stock
- E: Strike price of the Option
- T: Time to expiry of the Option contract
- r: Risk free rate(Yield of Government Bonds)

Let's consider a portfolio of one option and ϕ number of stocks. Hence, the value of portfolio is,

$$P = V + \phi S \tag{12}$$

Now for a no arbitrage condition to hold, the money made from this portfolio should be the same if the amount P was invested in a government bond giving r percent continuous interest, i.e.

$$dP = rPdt \tag{13}$$

Now, from GBM formalism we get,

$$dS = \mu S dt + \sigma S dB \tag{14}$$

Applying Ito's Lemma, we get the differential of the option value as,

$$dV = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})dt + \sigma S\frac{\partial V}{\partial S}dB \tag{15}$$

Taking the differential of (12) we get,

$$dP = dV + \phi dS \tag{16}$$

Now, we want a portfolio that is independent of risk characteristic and hence risk neutral. In order to do that, we want to remove the stochastic component of the portfolio, or equate its coefficient to zero. Substituting (14) and (15) in the R.H.S of (16) we get,

$$dP = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \phi\mu S)dt + (\sigma S\frac{\partial V}{\partial S} + \phi\sigma S)dB \tag{17}$$

In order to get rid of the stochastic component we choose $\phi = -\frac{\partial V}{\partial S}$
Using (13) we get the final Black Scholes PDE,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{18}$$

Having developed the Black-Scholes PDE, we make one slight change of variable so as to assist us in our numerical simulation of this equation.
Let's take $\tau = T - t$,and so Black Scholes PDE on change of variable becomes,

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \tag{19}$$

4.2 Explicit Numerical Method for pricing a European Call option

The initial condition for $V(S,\tau)$ is,

$$V(S,0) = max(S - E, 0) \tag{20}$$

Please note that there has been a change of variable from V(S,t) to $V(S,\tau)$ and so the boundary condition changed from V(S,T) to the initial condition $V(S,\tau = 0)$.

The other two boundary conditions are

$$V(0, \tau) = 0 \tag{21a}$$

$$V(S_{max}, \tau) = S_{max} - E\exp(-r\tau) \tag{21b}$$

4.2.1 Space Discretization

The interval $[0, \tau]$ is divided in M blocks, so there are M+1 points in the time axis.
At the same time, $[0, S_{max}]$ is divided into N blocks giving us N+1 points in the S axis. So we finally have a grid space of [M+1,N+1] points where we need to evaluate the option value.

4.2.2 Explicit Method

We use a forward difference approximation for the time derivative, a central difference approximation for the first order derivative in S, and a symmetric central difference approximation for the second order derivative in S, as given by,

$$\frac{\partial V}{\partial \tau} = \frac{v_n^{m+1} - v_n^m}{\Delta\tau} + O(\Delta\tau) \tag{22a}$$

$$\frac{\partial V}{\partial S} = \frac{v_{n+1}^m - v_{n-1}^m}{2\Delta S} + O((\Delta S)^2) \tag{22b}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{v_{n+1}^m - 2v_n^m + v_{n-1}^m}{(\Delta S)^2} + O((\Delta S)^2) \tag{22c}$$

It should be noted that the superscript gives the points in time axis while the subscript denotes the points in the stock price axis.
Substituting these values in equation (19) we get the following explicit equation which needs to be solved iteratively to get the pathway followed by the option pricing model and find the final call option value,

$$v_n^{m+1} = \frac{1}{2}(\sigma^2 n^2 \Delta\tau - rn\Delta\tau)v_{n-1}^m + (1 - \sigma^2 n^2 \Delta\tau - r\Delta\tau)v_n^m + \frac{1}{2}(\sigma^2 n^2 \Delta\tau + rn\Delta\tau)v_{n+1}^m, \tag{23}$$

for n = 1, 2, ..., N-1 and m = 0, 1, 2, ..., M-1
Refer Dura, Gina "Numerical Approximation of Blac Scholes Equation" for the vectorised implementation of Explicit method as used in the Matlab files.
Using equation (23) with the given initial condition and boundary conditions enables us to calculate the stock prices at every time point.

4.3 Analysis of Asian Paints Option Price

According to the analytical solution of the Black-Scholes model, price of call option is,

$$C(S, t) = SN(d_1) - E\exp\{(-r(T-t))\}N(d_2) \tag{24}$$

where N(.) is the cumulative distribution function of a standard normal random variable and,

$$d_1 = \frac{log(S/E) + (r + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \tag{25a}$$

$$d_2 = \frac{log(S/E) + (r - 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \tag{25b}$$

We take the case study of Asian Paints Option whose closing stock price was **S** $=$ **1500.45** on 20th May, 2020. Kindly refer to Table 1 below for the stock level data. Let's consider for the strike price **E** $=$ **1560** for the option expiring on 25th june, 2020. Then according to analytical equation, exact price of call option for the parameters calculated in the matlab file is, C_{exact} $=$ **42.5672** . The C_{calc} using the explicit method for the stock price **S** $=$ **1500.00** is taken since building a grid that will be more granular is extremely computationally expensive. Using the explicit method, we find C_{calc} $=$ **42.3647**. Thus, we see close agreement between the analytical and numerical solution and find our validation for executing a successful numerical procedure for evaluting the Black Scholes model for pricing European Call option.
However, the twist in the story comes as the closing price of the option(premium) on 20th May,2020 for this Asian Paints option expiring 25th June, 2020 is $C_{premium}$ $=$ **51.40**, sufficiently overpriced compared to Black Scholes Model.
Now, looking at the data on 25th June, we find that the closing stock price S_T $=$ **1689**.
Interestingly, the pay-off is **payoff** $=$ **1689 - 1560** $=$ **129**.
So, even though the Black Scholes Model tells us to not buy the option when it is trading at **51.40**, we see that we would have made a profit on this deal nevertheless.
It is even more interesting to note that had we looked at the payoff on 19th June for which it was simulated the payoff would have been 1622.50 - 1560 $=$ **62.50** which is still higher than the premium of **51.40**. The most interesting part is the stock price jumped to **1617.90** in 2 days, i.e. May 22nd, 2020 and stayed above that price level except for one day on June 18th.
Following is a table of Asian Paints stock price for the corresponding period,

The stock price is taken to be the close price of each day as shown in **Table 1**. **Table 2** notes that the error between the Call price calculated analytically(**AnalyticalCall**) and the Call price simulated using explicit method(**SimulatedCall**) is of the order of 10^{-2} when strike price is low at 1500 and the error reduces as strike price increases. At a strike price of 1800, the absolute error between analytical and numerical solution reduces to $2.65 * 10^{-3}$, enabling better convergence between the two.
One possible reason why this is happening might be that at a low strike price, the payoff is non-zero for more number of stock prices compared to when the strike price is higher and the payoff is zero for most stock prices. Referring to (23) we suggest that if in your initial condition most v_n^0 are zero, the value at the next time step v_n^1 is less prone to error propagation.

Looking at **Table 2** we notice that the vanilla Black Scholes Option pricing model that we used always undervalues the call option. As noted earlier in Section (3), this could be due to the fact that the underlying GBM used to simulate stock prices considers μ and σ as constants, and hence the model is unable to explain

Table 1: Stock price movement of Asian Paints stock

Date A	*Open* V	High	Low	Close	Adj Close	Volume
$2020-05-20$	1500.500000	1,514.2	1,494	1,500.45	1,500.45	50,292
$2020-05-21$	1501.000000	1,582	1,501	1,575	1,575	$1.16 \cdot 10^5$
$2020-05-22$	1571.000000	1,624.75	1,556	1,617.85	1,617.85	$1.21 \cdot 10^5$
$2020-05-26$	1625.000000	1,656	1,621.45	1,630.5	1,630.5	77,530
$2020-05-27$	1628.150024	1,637.55	1,596.85	1,619.75	1,619.75	44,186
$2020-05-28$	1621.000000	1,657	1,616.6	1,643.75	1,643.75	62,689
$2020-05-29$	1643.500000	1,697.95	1,634.75	1,684.4	1,684.4	67,099
$2020-06-01$	1693.449951	1,710	1,682	1,693.5	1,693.5	64,045
$2020-06-02$	1693.500000	1,732.35	1,681.45	1,709.85	1,709.85	49,661
$2020-06-03$	1723.000000	1,738.1	1,705.9	1,716.3	1,716.3	45,143
$2020-06-04$	1727.000000	1,727	1,626.6	1,633	1,633	$1.32 \cdot 10^5$
$2020-06-05$	1654.000000	1,658.8	1,621.35	1,638.4	1,638.4	72,995
$2020-06-08$	1654.000000	1,665.75	1,629	1,634.45	1,634.45	$1.35 \cdot 10^5$
$2020-06-09$	1653.000000	1,694.3	1,638	1,641.55	1,641.55	94,930
$2020-06-10$	1642.000000	1,680.8	1,628.3	1,636.2	1,636.2	54,345
$2020-06-11$	1636.000000	1,645.25	1,606.65	1,609.55	1,609.55	42,854
$2020-06-12$	1586.500000	1,642.6	1,575	1,637.95	1,637.95	55,787
$2020-06-15$	1635.000000	1,650.7	1,602.3	1,618.05	1,618.05	41,365
$2020-06-16$	1640.000000	1,640	1,608.75	1,621.05	1,621.05	74,648
$2020-06-17$	1619.800049	1,622.55	1,591	1,596.1	1,596.1	40,030
$2020-06-18$	1601.800049	1,612	1,588.05	1,598.9	1,598.9	58,275
$2020-06-19$	1595.000000	1,630	1,592.55	1,622.25	1,622.25	77,035
$2020-06-22$	1632.000000	1,673	1,624.55	1,653.45	1,653.45	53,386
$2020-06-23$	1655.550049	1,688	1,641.15	1,683.65	1,683.65	93,279
$2020-06-24$	1689.000000	1,813.1	1,686.2	1,746.4	1,746.4	$2.98 \cdot 10^5$
$2020-06-25$	1730.500000	1,746.3	1,683.85	1,688.85	1,688.85	90,372

Table 2: Asian Paints Stock Option data for various Strike Prices

StrikePrices	*Premium*	*AnalyticalCall*	*SimulatedCall*	*AbsoluteError*	payoff
1500	74.85	67.90311	67.87699	0.02612	188.85
1520	65	58.43568	58.40997	0.02571	168.85
1540	60	49.94199	49.91722	0.02477	148.85
1560	51.4	42.38809	42.3647	0.02339	128.85
1580	57.2	35.72826	35.7066	0.02166	108.85
1600	36.65	29.90735	29.88768	0.01967	88.85
1620	35	24.86324	24.8457	0.01754	68.85
1640	37.4	20.52926	20.51392	0.01535	48.85
1700	19.8	11.10051	11.0913	0.00922	-11.15
1760	15	5.66145	5.65679	0.00466	-71.15
1800	11.1	3.50353	3.50088	0.00265	-111.15

unexpected rises or falls in the stock price.

Finally we note that up until a strike price of 1640, as referenced in **Table 2** ,the payoff is always greater

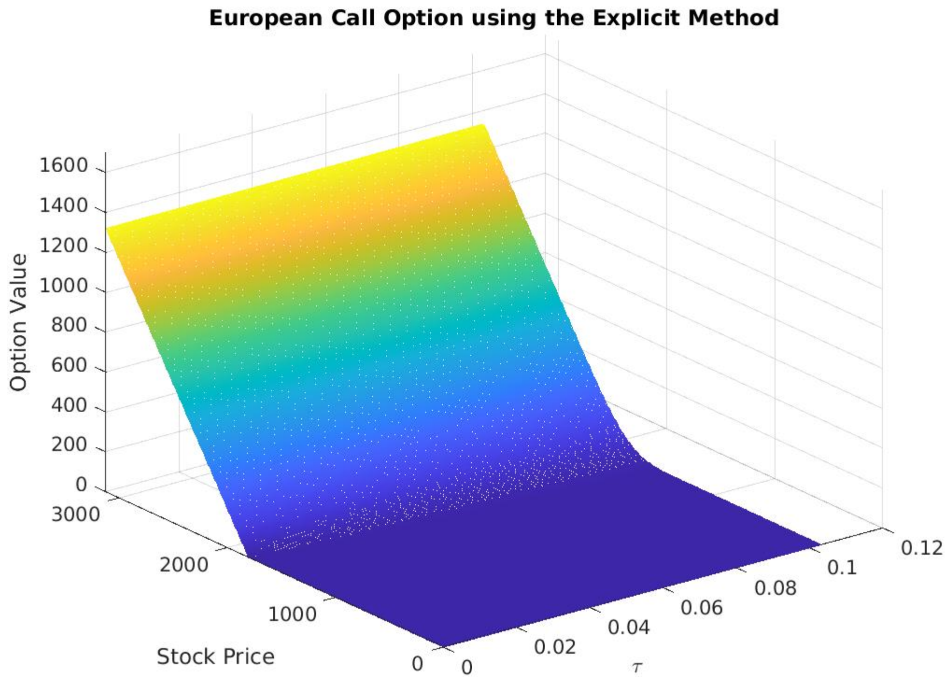

Figure 3: Option pricing surface as a function of S and τ for strike=1800.

than the premium, making them profitable investments.
However, according to the calculated call option prices, the premium is always overvalued and hence the option at that strike should not be invested in which is the same as losing out on a profitable venture.

5 Conclusion

The Black Scholes Model(BSM) is one of the most celebrated models in finance that enabled numerous researchers and academicians to build upon this model to better mimic reality. However, there are some assumptions taken for the model that do not always hold true in reality, and these drawbacks are topics of research.
The BSM is based on the basic GBM model for stock price prediction. The GBM model in turn, is based on the assumption that asset log-returns are normally distributed. However, this assumption is often not entirely correct, and the distribution sometimes showcases leptokurtosis. One attempt at a solution to this problem is given by Kou(2002), where he tries to incorporate this in his jump-diffusion model by modelling asset returns as a double exponential distribution to capture the leptokurtic feature of asset prices.
Black and Scholes in their seminal paper note that in the emperical tests, the BSM model consistently undervalues the option price at which they are bought. There are also heavy transaction costs in the market, which are not taken into account by the model. So, while the GBM model for stock price prediction and the BSM for option pricing perform reasonably well, they are not entirely accurate, and there are constant improvements being suggested by the research community.

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Appendix

Ito's Lemma

Assuming X(t) is an Ito process that satisfies the stochastic differential equation(SDE),

$$dX_t = \mu_t dt + \sigma_t dB_t \tag{26}$$

where μ_t is the drift parameter, σ_t is the volatility parameter and B_t is the Weiner process.
Assuming $f(X,t)$ is twice differentiable, we expand its taylor series,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + ... \tag{27}$$

Substituting dX_t for X we get,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu_t dt + \sigma_t dB_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu_t dt + \sigma_t dB_t)^2 + ... \tag{28}$$

As dt -> 0, we remove all higher order terms of dt, higher than the order O(dt). So the dt_2 and $dtdB_t$ terms are removed as they are of the order greater than O(dt) while the dB_t^2 term is kept since it is of the order of O(dt).
This is an intuitive derivation of the Ito's lemma for finding the differential of a stochastic process. Hence, we obtain the final equation:

$$df = (\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2})dt + \sigma_t \frac{\partial f}{\partial x}dB_t \tag{29}$$