CANONICAL BIG COHEN-MACAULAY ALGEBRAS AND RATIONAL SINGULARITIES

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Abstract. We give a canonical construction of a balanced big Cohen-Macaulay algebra for a domain of finite type over $\mathbb{C}$ by taking ultraproducts of absolute integral closures in positive characteristic. This yields a new tight closure characterization of rational singularities in characteristic zero.

1. Introduction

In [7], Hochster proves the existence of big Cohen-Macaulay modules for a large class of Noetherian rings containing a field. Recall that a module M over a Noetherian local ring R is called a *big Cohen-Macaulay module*, if there is a system of parameters of R which is M-regular (the adjective *big* is used to emphasize that M need not be finitely generated). He also exhibits in that paper the utility of big Cohen-Macaulay modules in answering various homological questions. Often, one can even obtain a big Cohen-Macaulay module M such that *every* system of parameters is M-regular; these are called *balanced* big Cohen-Macaulay modules. In [8], Hochster and Huneke show that for equicharacteristic excellent local domains, one can even find a balanced big Cohen-Macaulay *algebra*, that is to say, M admits the structure of a (commutative) R-algebra. In fact, for R a local domain of positive characteristic, they show that the absolute integral closure of R, denoted R^+, is a (balanced) big Cohen-Macaulay algebra (it is easy to see that this is false in characteristic zero). In [9], using lifting techniques similar to the ones developed in the original paper of Hochster, they obtain also the existence of big Cohen-Macaulay algebras in characteristic zero. However, the construction is no longer canonical and one looses the additional information one had in positive characteristic. Nonetheless, many useful applications follow, see [11, §9] or [9].

In this paper, I will show that for a local domain R of finite type over $\mathbb{C}$ (henceforth, a *local* $\mathbb{C}$-*affine domain*), a simple construction of a balanced

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big Cohen-Macaulay algebra $\mathcal{B}(R)$ can be made, which restores canonicity, is weakly functorial and preserves many of the good properties of the absolute integral closure. Namely, to the domain R, one associates certain characteristic p domains R_p, called *approximations* of R, and of these one takes the absolute integral closure R_p^+ and then forms the ultraproduct $\mathcal{B}(R) := \operatorname{ulim}_{p\to\infty} R_p^+$. For generalities on ultraproducts, including Łos' Theorem, see [5]; for a short introduction, see [17, §2]. Recall that an ultraproduct of rings C_p is a certain homomorphic image of the direct product of the C_p. This ultraproduct will be denoted by $\operatorname{ulim}_{p\to\infty} C_p$, or simply by C_∞, and similarly, the image of a sequence $(a_p \mid p)$ in C_∞ will be denoted by $\operatorname{ulim}_{p\to\infty} a_p$, or simply by a_∞.

The notion of approximation goes back to the paper [17], where it was introduced to define a closure operation, called *non-standard tight closure*, on $\mathbb{C}$-affine algebras by means of a so-called *non-standard Frobenius*. Let me briefly recall the construction of an approximation (details and proofs can be found in [17, §3]). Suppose R is of the form $\mathbb{C}[X]/I$, or possibly, a localization of such an algebra with respect to a prime ideal $\mathfrak{p}$. There is a fundamental (but non-canonical) isomorphism between the field of complex numbers on the one hand, and the ultraproduct of all the fields $\mathbb{F}_p^{\mathrm{alg}}$ on the other hand, where $\mathbb{F}_p^{\mathrm{alg}}$ denotes the algebraic closure of the p-element field. Therefore, for every element c in $\mathbb{C}$, we can choose a representative in the product, that is to say, a sequence of elements $c_p \in \mathbb{F}_p^{\mathrm{alg}}$, called an *approximation* of c, such that $\operatorname{ulim}_{p\to\infty} c_p = c$. Applying this to each coefficient of a polynomial $f \in \mathbb{C}[X]$ separately, we get a sequence of polynomials $f_p \in \mathbb{F}_p^{\mathrm{alg}}[X]$ (of the same degree as f), called again an *approximation* of f. If we apply this to the generators of I and $\mathfrak{p}$, we generate ideals I_p and $\mathfrak{p}_p$ in $\mathbb{F}_p^{\mathrm{alg}}[X]$, called once more approximations of I and $\mathfrak{p}$ respectively. One shows that $\mathfrak{p}_p$ is prime for almost all p. Finally, we set $R_p := \mathbb{F}_p^{\mathrm{alg}}[X]/I_p$ (or its localization at the prime ideal $\mathfrak{p}_p$) and call the collection of these characteristic p rings an *approximation* of R. Although the choice of an approximation is not unique, almost all its members are the same; this is true for every type of approximation just introduced (here and elsewhere, *almost all* means with respect to a non-specified but fixed non-principal ultrafilter). Moreover, if we depart from a different presentation of R as a $\mathbb{C}$-affine algebra, then the resulting approximation is isomorphic to R_p, for almost all p. In particular, the ultraproduct $R_\infty := \operatorname{ulim}_{p\to\infty} R_p$ of the R_p is uniquely determined up to R-algebra isomorphism and is called the *non-standard hull* of R. There is a natural embedding $R \to R_\infty$, the main property of which was discovered by van den Dries in [27]: $R \to R_\infty$ is faithfully flat (note that in general, R_∞ is no longer Noetherian nor even separated). In case R is a local domain, almost all R_p are local domains. Therefore, the ultraproduct $\mathcal{B}(R)$ of the R_p^+ is well defined and unique up to R-algebra isomorphism and we get our first main result.

Theorem A. *If R is a local $\mathbb{C}$-affine domain, then $\mathcal{B}(R)$ is a balanced big Cohen-Macaulay algebra.*

In fact, due to canonicity, the operation of taking $\mathcal{B}(\cdot)$ is weakly functorial (see Theorem 2.4 for a precise statement). Moreover, $\mathcal{B}(R)$ has the additional property that every monic polynomial over it splits completely in linear factors. In $\mathcal{B}(R)$, any sum of prime ideals is either the unit ideal or else again a prime ideal. This is explained in §3. In §4, we use $\mathcal{B}(\cdot)$ to define a new closure operation given as $I^+ := I\mathcal{B}(R) \cap R$, and relate it to generic tight closure (this is one of the alternative closure operations in characteristic zero introduced in [17]). One immediate corollary of the canonicity of our construction is the following characteristic zero version of the generalized Briançon-Skoda Theorem in [9, Theorem 7.1].

Theorem B. *If R is a local $\mathbb{C}$-affine domain and I an ideal of R generated by n elements, then the integral closure of I^{n+k} is contained in $(I^{k+1})^+$, for every $k \in \mathbb{N}$.*

In [17] the same result is proven if we replace I^+ by the generic tight closure of I. This suggests that the appropriate characteristic zero equivalent of the conjecture that tight closure equals plus closure is the conjecture that I^+ always equals the generic tight closure of I. We show that in any case, the former is contained in the latter. Moreover, we have equality for parameter ideals, that is to say, the characteristic zero equivalent of Smith's result in [25] also holds. Using this, we give in §5 a characterization of rational singularities in terms of these closure operations, extending the results of Hara [6] and Smith [26], at least in the affine case.

Theorem C. *A local $\mathbb{C}$-affine domain has rational singularities if, and only if, there exists an ideal I generated by a system of parameters for which $I = I^+$.*

Note that we need Hara's result for the proof (see Theorem 5.11 for more details), which itself relies on some deep vanishing theorems. In [23], we will give a similar characterization for log-terminal singularities. Using the above results, we recover the Briançon-Skoda Theorem of Lipman-Teissier. Another application is a new proof of Boutot's main result in [3], at least for Gorenstein rational singularities (this also generalizes the main result of [24]; for a further generalization, see [23, Theorem B]).

Theorem (Boutot [3]). *Let $R \to S$ be a (cyclically) pure homomorphism of local $\mathbb{C}$-affine algebras. If S is Gorenstein and has rational singularities, then R has rational singularities.*

In the final section, some results of [18] are extended to the present characteristic zero situation. In particular, we obtain the following regularity criterion (see Theorem 7.1).

Theorem D. *Let R be a local $\mathbb{C}$-affine domain with residue field k. If R has an isolated singularity and $\operatorname{Tor}_1^R(\mathcal{B}(R), k) = 0$, then R is regular.*

In contrast with the prime characteristic case, I do not know whether for an arbitrary local $\mathbb{C}$-affine domain R, the flatness of $R \to \mathcal{B}(R)$ is equivalent with the regularity of R (that it is a necessary condition is proved in Corollary 2.5).

Remark on the base field. To make the exposition more transparent, I have only dealt in the text with the case that the base field is $\mathbb{C}$. However, the results extend to arbitrary uncountable base fields of characteristic zero by the following observations. First, any uncountable algebraically closed field of characteristic zero is the ultraproduct of (algebraically closed) fields of positive characteristic by the Lefschetz Principle (see for instance [17, Remark 2.5]) and this is the only property we used of $\mathbb{C}$. Second, if A is a local K-affine domain with K an arbitrary uncountable field, then A^+ is a K^{alg}-algebra, where K^{alg} is the algebraic closure of K. Therefore, in order to define $\mathcal{B}(A)$ in case K has moreover characteristic zero, we may replace A by $A \otimes_K K^{\mathrm{alg}}$ and assume form the start that K is uncountable and algebraically closed, so that our first observation applies.

In [2], we will show the existence of a big Cohen-Macaulay algebra for an arbitrary equicharacteristic zero Noetherian local domain. In [16, 20], the same techniques as in this paper are used to obtain an asymptotic version of big Cohen-Macaulay algebras in mixed characteristic.

2. Big Cohen-Macaulay algebras

2.1. **Absolute Integral Closure.** Let A be a domain. The *absolute integral closure* A^+ of A is defined as follows. Let Q be the field of fractions of A and let Q^{alg} be its algebraic closure, We let A^+ be the integral closure of A in Q^{alg}. Since algebraic closure is unique up to isomorphism, any two absolute integral closures of A are isomorphic as A-algebras. To not have to deal with exceptional cases separately, we put $A^+ = 0$ if A is not a domain.

In this paper, we will use the term *K-affine algebra* for an algebra of finite type over a field K or a localization of such an algebra with respect to a prime ideal; the latter will also be referred to as a *local K-affine algebra*.

2.2. **Approximations and non-standard hulls.** Let A be a $\mathbb{C}$-affine algebra and choose an approximation A_p of A (see the introduction; for a precise definition and proofs, see [17, §3]). The ultraproduct of the A_p is called the *non-standard hull* of A and is often denoted A_∞. The assignment sending A to A_∞ is functorial. There is a natural homomorphism $A \to A_\infty$, which is faithfully flat by [14, Theorem 1.7] (for an alternative proof, see [21, A.2]). It follows that if I is an ideal in A and I_p an approximation of I, then IA_∞ is the ultraproduct of the I_p and $I = IA_\infty \cap A$. By [17, Theorem 4.4], almost all A_p

are domains (respectively, local) if, and only if, A is a domain (respectively, local) if, and only if, A_∞ is a domain (respectively, local).

2.3. **The quasi-hull $\mathcal{B}(\cdot)$.** Let A be a $\mathbb{C}$-affine domain with approximation A_p. Define $\mathcal{B}(A)$ as the ultraproduct

$$\mathcal{B}(A) := \operatorname*{ulim}_{p\to\infty} A_p^+.$$

In view of the uniqueness of the absolute integral closure, $\mathcal{B}(A)$ is independent of the choice of the A_p and hence is uniquely determined by A up to A-algebra isomorphism. Using Łos' Theorem, one easily shows that the natural map $\operatorname{Spec}\mathcal{B}(A) \to \operatorname{Spec} A$ is surjective. Given a homomorphism $A \to B$ of $\mathbb{C}$-affine algebras, we obtain homomorphisms $A_p \to B_p$, for almost all p, where B_p is an approximation of B (see [17, 3.2.4]). These homomorphisms induce (non-canonically) homomorphisms $A_p^+ \to B_p^+$, which, in the ultraproduct, yield a homomorphism $\mathcal{B}(A) \to \mathcal{B}(B)$.

Note that the natural homomorphism $A \to \mathcal{B}(A)$ factors through the non-standard hull A_∞, and in particular, $A \to \mathcal{B}(A)$ is no longer integral. Using Łos' Theorem and results on the absolute integral closure in [8] (see also [11, Chapter 9]), we get the following more precise version of Theorem A.

2.4. **Theorem.** *For each local $\mathbb{C}$-affine domain R, the R-algebra $\mathcal{B}(R)$ is a* balanced big Cohen-Macaulay algebra *in the sense that any system of parameters of R is a $\mathcal{B}(R)$-regular sequence. Moreover, if $R \to S$ is a local homomorphism of local $\mathbb{C}$-affine domains, then there exists a $\mathbb{C}$-algebra homomorphism $\mathcal{B}(R) \to \mathcal{B}(S)$ giving rise to a commutative diagram*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathcal{B}(R) & \longrightarrow & \mathcal{B}(S). \end{array} \tag{1}$$

If $R \to S$ is finite and injective, then $\mathcal{B}(R) = \mathcal{B}(S)$.

Proof. Let R_p be an approximation of R and R_∞ its non-standard hull. Let $\mathbf{x}$ be a system of parameters in R with approximation $\mathbf{x}_p$. By [17, Theorem 4.5] almost all $\mathbf{x}_p$ are a system of parameters of R_p. Therefore, by [8, Theorem 1.1], the sequence $\mathbf{x}_p$ is R_p^+-regular, for almost all p. Łos' Theorem then yields that $\mathbf{x}$ is a $\mathcal{B}(R)$-regular sequence.

The existence of the homomorphism $\mathcal{B}(R) \to \mathcal{B}(S)$ and the commutativity of diagram (1) follow from the above discussion. Finally, if S is finite overring of R, then by [17, Theorem 4.7], so will almost all S_p be over R_p, where R_p

and S_p are approximations of R and S respectively. In particular, $R_p^+ = S_p^+$, for almost all p, proving that $\mathcal{B}(R) = \mathcal{B}(S)$. $\square$

2.5. **Corollary.** *For each local $\mathbb{C}$-affine regular ring R, the natural homomorphism $R \to \mathcal{B}(R)$ is faithfully flat.*

Proof. It is well-known that a balanced big Cohen-Macaulay module over a regular local ring is flat (see for instance [19, Theorem IV.1] or [9, Lemma 2.1(d)]). Since the maximal ideal of R extends to a proper ideal in $\mathcal{B}(R)$, the homomorphism $R \to \mathcal{B}(R)$ is faithfully flat. $\square$

As in positive characteristic, we can define $\mathcal{B}(A)$ for any reduced $\mathbb{C}$-affine ring A as the product of all $\mathcal{B}(A/\mathfrak{p})$, where $\mathfrak{p}$ runs over all minimal prime ideals of A. It follows easily from Theorem 2.4 that $\mathcal{B}(R)$ is a big Cohen-Macaulay algebra for every reduced local $\mathbb{C}$-affine ring R. As for localization, we have a slightly less pretty result as in positive characteristic: if A is a $\mathbb{C}$-affine domain with non-standard hull A_∞ and if $\mathfrak{p}$ is a prime ideal of A, then

$$\mathcal{B}(A_\mathfrak{p}) \cong \mathcal{B}(A) \otimes_{A_\infty} (A_\infty)_{\mathfrak{p}A_\infty}. \tag{2}$$

Indeed, if A_p and $\mathfrak{p}_p$ are approximations of A and $\mathfrak{p}$ respectively, then by [8, Lemma 6.5], we have an isomorphism

$$((A_p)_{\mathfrak{p}_p})^+ \cong (A_p^+)_{\mathfrak{p}_p} = A_p^+ \otimes_{A_p} (A_p)_{\mathfrak{p}_p}.$$

Taking ultraproducts, we get isomorphism (2). It follows that Corollary 2.5 also holds if we drop the requirement that A is local (use that $\mathcal{B}(A)_\mathfrak{p} \to \mathcal{B}(A_\mathfrak{p})$ is flat, for every prime ideal $\mathfrak{p}$ of A, by (2)). We also obtain the following characteristic zero analogue of [8, Theorem 6.6].

2.6. **Theorem.** *If A is a $\mathbb{C}$-affine domain and I an ideal in A of height h, then $H_I^j(\mathcal{B}(A)) = 0$, for all $j < h$.*

Proof. As in the proof of [8, Theorem 6.6], it suffices to show that for every maximal ideal $\mathfrak{m}$ of A containing I, we have that $H_I^j(\mathcal{B}(A))_\mathfrak{m} = 0$, for $j < h$. Since $A_\mathfrak{m} \to (A_\infty)_{\mathfrak{m}A_\infty}$ is faithfully flat, as explained in §2.2, it suffices to show that

$$H_I^j(\mathcal{B}(A)) \otimes_{A_\infty} (A_\infty)_{\mathfrak{m}A_\infty} = 0.$$

By (2), the left hand side is simply $H_I^j(\mathcal{B}(A_\mathfrak{m}))$ and therefore, the problem reduces to the case that A is local. Let $(x_1, \dots, x_h)$ be part of a system of parameters of A contained in I. Since $(x_1, \dots, x_h)$ is $\mathcal{B}(A)$-regular by Theorem 2.4, the vanishing of $H_I^j(\mathcal{B}(A))$ for $j < h$ is then clear since local cohomology can be viewed as a direct limit of Koszul cohomology. $\square$

3. Properties of $\mathcal{B}(A)$

Let us call a domain S *absolutely integrally closed* if every monic polynomial over S has a root in S.

3.1. **Lemma.** *For a domain S with field of fractions Q, the following are equivalent.*

(3.1.1) *S is absolutely integrally closed.*
(3.1.2) *Every monic polynomial completely splits in S.*
(3.1.3) *S is integrally closed in Q and Q is algebraically closed.*

Proof. The implications (3.1.3) $\implies$ (3.1.2) and (3.1.2) $\implies$ (3.1.1) are straightforward. Hence assume that S is absolutely integrally closed. It is clear that S is then integrally closed in Q. So remains to show that Q is algebraically closed. In other words, we have to show that every non-zero one-variable polynomial $F \in Q[T]$ has a root in Q. Clearing denominators, we may assume that $F \in S[T]$. Let $a \in S$ be the (non-zero) leading coefficient of F and d its degree. We can find a monic polynomial G over S, such that $a^{d-1}F(T) = G(aT)$. By assumption, $G(b) = 0$ for some $b \in S$. Hence $F(b/a) = 0$, as required. $\square$

It follows from [8, Lemma 6.5] that a domain S is the absolute integral closure of a subring A if, and only if, S is absolutely integrally closed and $A \subset S$ is integral.

3.2. **Proposition.** *If A is a $\mathbb{C}$-affine domain, then $\mathcal{B}(A)$ is absolutely integrally closed.*

Proof. Let $F(T) := T^d + a_1T^{d-1} + \cdots + a_d$ be a monic polynomial in the single variable T with $a_i \in \mathcal{B}(A)$. We need to show that F has a root in $\mathcal{B}(A)$. Choose $a_{ip} \in A_p^+$, such that $\operatorname{ulim}_{p\to\infty} a_{ip} = a_i$, for all i, where A_p is some approximation of A. Hence we can find $b_p \in A_p^+$ such that

$$(b_p)^d + a_{1p}(b_p)^{d-1} + \cdots + a_{dp} = 0.$$

Therefore, by Łos' Theorem, $b := \operatorname{ulim}_{p\to\infty} b_p$ is a root of F. $\square$

3.3. **Corollary.** *Let A be a $\mathbb{C}$-affine domain. The sum of any collection of prime ideals in $\mathcal{B}(A)$ is either prime or the unit ideal. If $\mathfrak{g}_i$ are $\mathfrak{p}_i$-primary ideals, for i in some index set I, and if $\mathfrak{p} := \sum_{i\in I}\mathfrak{p}_i$ is not the unit ideal, then $\sum_{i\in I}\mathfrak{g}_i$ is $\mathfrak{p}$-primary.*

Proof. By Proposition 3.2, the ring $\mathcal{B}(A)$ is quadratically closed and therefore has the stated properties by [8, Theorem 9.2]. $\square$

The next result shows that $\mathcal{B}(R)$, viewed as an R_∞-algebra, also behaves very much like a Cohen-Macaulay algebra.

3.4. **Proposition.** *Let $(R,\mathfrak{m})$ be a local $\mathbb{C}$-affine domain. Let $(x_1,\dots,x_d)$ be part of a system of parameters of R and let $\mathfrak{p}_1,\dots,\mathfrak{p}_s$ be the minimal prime ideals of $(x_1,\dots,x_d)R$. If $t_\infty \in \mathfrak{m}R_\infty$ does not lie in any $\mathfrak{p}_iR_\infty$, then $(x_1,\dots,x_d,t_\infty)$ is a $\mathcal{B}(R)$-regular sequence.*

Proof. Suppose $b_\infty \in \mathcal{B}(R)$ is such that

$$t_\infty b_\infty \in (x_1,\dots,x_d)\mathcal{B}(R).$$

Let R_p, x_{ip} and $\mathfrak{p}_{ip}$ be approximations of R, x_i and $\mathfrak{p}_i$ respectively. It follows from [17, Theorem 4.5] that $(x_{1p},\dots,x_{dp})$ is part of a system of parameters in R_p, and from [17, Theoprem 4.4], that $\mathfrak{p}_{1p},\dots,\mathfrak{p}_{sp}$ are the minimal prime ideals of $(x_{1p},\dots,x_{dp})R_p$, for almost all p. Choose t_p and b_p in R_p and R_p^+ respectively such that their ultraproduct is t_∞ and b_∞. By Łos' Theorem, almost all t_p lie outside any $\mathfrak{p}_{ip}$, and $t_pb_p \in (x_{1p},\dots,x_{dp})R_p^+$. Therefore, $(x_{1p},\dots,x_{dp},t_p)$ is part of a system of parameters in R_p and hence, by [8, Theorem 1.1], is an R_p^+-regular sequence, for almost all p. It follows that $b_p \in (x_{1p},\dots,x_{dp})R_p^+$, for almost all p, whence, by Łos' Theorem, that $b_\infty \in (x_1,\dots,x_d)\mathcal{B}(R)$. □

4. $\mathcal{B}$-closure

In analogy with plus closure in positive characteristic, which is defined via absolute integral closures, we use the quasi-hull $\mathcal{B}(\cdot)$ to define a new closure operation on a local $\mathbb{C}$-affine domain R as follows.

4.1. **Definition** ($\mathcal{B}$-closure). We define the *$\mathcal{B}$-closure* of an ideal I in R to be the ideal

$$I^+ := I\mathcal{B}(R) \cap R.$$

Clearly $I \subset I^+$ and $(I^+)^+ = I^+$, so that this yields indeed a closure operation on ideals. Since $\operatorname{Spec}\mathcal{B}(R) \to \operatorname{Spec} R$ is surjective, $\mathfrak{p} = \mathfrak{p}^+$ for every prime ideal $\mathfrak{p}$ of R. It follows that $I^+ \subset \operatorname{rad} I$, for every ideal I. We will show that $\mathcal{B}$-closure satisfies many of the properties of classical tight closure. For instance, Theorem B is the analogue of the tight closure Briançon-Skoda Theorem and will be proved in §6. Let us record some important properties, all of which follow immediately from the results obtained in the previous sections.

4.2. **Theorem.** *Let R be a local $\mathbb{C}$-affine domain.*

- (4.2.1) *If R is regular, then $I = I^+$ for every ideal I of R.*
- (4.2.2) *If $(x_1,\dots,x_d)$ is a system of parameters in R, then $((x_1,\dots,x_i)R : x_{i+1})$ is contained in the $\mathcal{B}$-closure of $(x_1,\dots,x_i)R$, for all $i < d$* (Colon Capturing).
- (4.2.3) *If $R \to S$ is a local $\mathbb{C}$-algebra homomorphism of local $\mathbb{C}$-affine domains, then $I^+S \subset (IS)^+$ for every ideal I of R* (Persistence).

Proof. The first assertion is immediate from the faithful flatness of $R \to \mathcal{B}(R)$ proved in Corollary 2.5. For (4.2.2), let $I := (x_1, \dots, x_i)R$ and suppose $ax_{i+1} \in I$. Since $(x_1, \dots, x_d)$ is $\mathcal{B}(R)$-regular by Theorem 2.4, we get $a \in I\mathcal{B}(R)$, and hence $a \in I^+$. In fact, the argument yields that x_{i+1} is a non-zero divisor modulo I^+. The last assertion is immediate from the weak functoriality of $\mathcal{B}(\cdot)$ proved in Theorem 2.4. □

One of the advantages of plus closure over tight closure is the fact that it commutes with localization. At present, I cannot yet show the analogue of this for $\mathcal{B}$-closure, but we have at least the following special case (which often suffices).

4.3. **Theorem.** *Let R be a local $\mathbb{C}$-affine domain and let $\mathfrak{p}$ be a prime ideal of R. If I is an ideal in R such that $IR_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$-primary, then*

$$I^+R_\mathfrak{p} = (IR_\mathfrak{p})^+.$$

Proof. Put $S := R_\mathfrak{p}$. By (4.2.3), we have an inclusion $I^+S \subset (IS)^+$, so remains to show the opposite inclusion. Let $a \in (IS)^+$ and let R_∞ be the non-standard hull of R. By definition, the non-standard hull S_∞ of S is isomorphic to $(R_\infty)_{\mathfrak{p}R_\infty}$, so that using (2), we get

$$\mathcal{B}(S) \cong \mathcal{B}(R) \otimes_{R_\infty} S_\infty. \tag{3}$$

Since IS is $\mathfrak{p}S$-primary, $S/IS \cong S_\infty/IS_\infty$ by [17, Theorem 4.5]. Tensoring with $\mathcal{B}(R)_\mathfrak{p}$ and using (3) yields

$$\mathcal{B}(R)_\mathfrak{p}/I\mathcal{B}(R)_\mathfrak{p} \cong \mathcal{B}(S)/I\mathcal{B}(S). \tag{4}$$

Hence we showed that $I\mathcal{B}(S) \cap \mathcal{B}(R)_\mathfrak{p} = I\mathcal{B}(R)_\mathfrak{p}$, so that $a \in I\mathcal{B}(R)_\mathfrak{p}$. Therefore, there is an $s \in R \setminus \mathfrak{p}$ such that $sa \in I\mathcal{B}(R)$. This shows that $sa \in I^+$ whence $a \in I^+S$, as required. □

4.4. *Remark.* We would obtain that $\mathcal{B}$-closure commutes with localization if we can show that the $\mathcal{B}$-closure of an ideal I in a local $\mathbb{C}$-affine domain $(R, \mathfrak{m})$ is equal to the intersection of all $(I + \mathfrak{m}^n)^+$. One obstruction in proving this is the fact that the intersection of all $\mathfrak{m}^n\mathcal{B}(R)$ is not zero (as it is neither so in R_∞).

There is a close connection between $\mathcal{B}$-closure and generic tight closure, the definition of which we now recall. Let A be a (local) $\mathbb{C}$-affine algebra, I an ideal of A and z an arbitrary element. We say that z lies in the *generic tight closure* of I, if z_p lies in the tight closure of I_p, for almost all p, where z_p and I_p are approximations of z and I respectively. In [17] it is shown that this yields a closure operation with similar properties as characteristic zero tight closure, and that it is contained in non-standard tight closure (for the definition of non-standard (tight) closure and for further properties of these closure operations, see [17]; variants can be found in [22, 24]).

4.5. **Corollary.** *Let R be a local $\mathbb{C}$-affine domain. If I is an ideal in R, then I^+ is contained in the generic tight closure of I (whence also in the non-standard tight closure and in the integral closure of I).*

Moreover, if I is generated by a system of parameters of R, then I^+ is equal to the generic tight closure of I.

Proof. Let R_p and I_p be approximations of R and I respectively. Let $f \in R$ with approximation f_p. Assume first that $f \in I^+$. It follows that $f_p \in I_p R_p^+$, for almost all p. Since in general, $JB \cap A$ lies in the tight closure of J, for any integral extension $A \to B$ of prime characteristic rings and any ideal $J \subset A$ ([11, Theorem 1.7]), we get that f_p lies in the tight closure of I_p, for almost all p. However, this just means that f lies in the generic tight closure of I.

Conversely, if f lies in the generic tight closure of $\mathbf{x}R$, where $\mathbf{x}$ is a system of parameters with approximation $\mathbf{x}_p$, then f_p lies in the tight closure of $\mathbf{x}_p R_p$ and $\mathbf{x}_p$ is a system of parameters in R_p by [17, Theorem 4.5], for almost all p. By the result of Smith in [25], tight closure equals plus closure for any ideal generated by a system of parameters, so that $f_p \in \mathbf{x}_p R_p^+$. Taking ultraproducts, we get that $f \in \mathbf{x}\mathcal{B}(R)$. □

5. Rational Singularities

Classical tight closure has also applications in singularity theory: certain (rational) singularities of a ring seem to be (or at least, are conjectured to be) determined by the type of ideals in the ring that are tightly closed. A similar, and even better behaved phenomenon, holds true for the non-standard versions of tight closure (see for instance [23, 24]). It turns out that also $\mathcal{B}$-closure can be used to characterize rational singularities, and so we make the following definitions for a local $\mathbb{C}$-affine domain R.

5.1. **Definition.** We say that R is *$\mathcal{B}$-rational* (respectively, *generically F-rational*), if there is an ideal I generated by a system of parameters, such that $I = I^+$ (respectively, such that I is equal to its generic tight closure).

We say that R is *$\mathcal{B}$-regular* (respectively, *weakly generically F-regular*), if $I = I^+$ (respectively, I is equal to its generic tight closure), for every ideal I in R.

In other words, R is $\mathcal{B}$-regular if, and only if, $R \to \mathcal{B}(R)$ is cyclically pure (recall that a homomorphism $A \to B$ is called *cyclically pure* if $I = IB \cap A$ for all ideals I in A). By (4.2.1), regular implies $\mathcal{B}$-regular. Corollary 4.5 shows that weakly generically F-regular implies $\mathcal{B}$-regular and that generically F-rational and $\mathcal{B}$-rational are equivalent. The reader should also compare the notion of $\mathcal{B}$-regularity with the notion of *CM^n-regularity* from [9].

If every localization of R at a prime ideal is weakly generically F-regular, then we call R *generically F-regular*. Conjecturally, weakly generically F-regular and generically F-regular are equivalent, or more generally, one expects that generic tight closure commutes with localization. It is a direct consequence of Theorem 4.3 that no such complication arises for the notion of $\mathcal{B}$-regularity (whence the absence of the modifier *weak* in the definition).

5.2. **Theorem.** *Any localization of a $\mathcal{B}$-regular local $\mathbb{C}$-affine domain is again $\mathcal{B}$-regular.*

Proof. Let R be a $\mathcal{B}$-regular local $\mathbb{C}$-affine domain with non-standard hull R_∞. Let $\mathfrak{p}$ be a prime ideal in R and put $S := R_{\mathfrak{p}}$. We have to show that $J = J^+$ for every ideal J in S. Suppose we have shown this for all $\mathfrak{p}S$-primary ideals and let J be arbitrary. It follows that $J^+ \subset (J + \mathfrak{p}^n S)^+ = J + \mathfrak{p}^n S$, for all n, so that by Krull's Intersection Theorem, J^+ is contained in J, as we needed to show.

So we may assume J is $\mathfrak{p}S$-primary. If we choose I in R such that $J = IS$, then $J^+ = I^+S$ by Theorem 4.3. However, $I = I^+$ by assumption, showing that $J^+ = IS = J$. □

We now turn our attention to $\mathcal{B}$-rational (or equivalently, generically F-rational) rings. The following is just a rephrasing of [24, Theorem 6.2] in our new terminology (its converse also holds and will be proved in Theorem 5.11 below).

5.3. **Theorem.** *If a local $\mathbb{C}$-affine domain is $\mathcal{B}$-rational, then it has rational singularities.*

Proof. Let R a local $\mathbb{C}$-affine domain and assume R is $\mathcal{B}$-rational. This means that there exists a system of parameters $\mathbf{x}$ in R such that $\mathbf{x}R = \mathbf{x}\mathcal{B}(R) \cap R$. Since $\mathbf{x}R$ is then equal to its own generic tight closure by Corollary 4.5, we get from [24, Remark 6.3] that R has rational singularities. For the reader's convenience, let me briefly repeat the argument. Let $\mathbf{x}_p$ be an approximation of $\mathbf{x}$. By Corollary 4.5, almost all $\mathbf{x}_p R_p$ are tightly closed. Since almost all $\mathbf{x}_p$ are systems of parameters by [17, Theorem 4.5], almost all R_p are F-rational. Therefore, by [26], almost all R_p are pseudo-rational. This in turn implies that R is pseudo-rational by [24, Theorem 5.1]. Let me also sketch the argument of this last result. Let $W \to X$ be a desingularization of $X := \operatorname{Spec} R$. Hence $W = \operatorname{Proj} B$ for some blow-up algebra B of R. Put $W_p := \operatorname{Proj} B_p$, for some choice of approximation B_p of B (note that almost all B_p are graded). One shows, using the results from [17], that $W_p \to X_p$ is a desingularization of R_p for almost all p, where $X_p := \operatorname{Spec} R_p$. By definition of pseudo-rationality, we get isomorphisms $H_0(W_p, \omega_{W_p}) \cong H_0(X_p, \omega_{X_p})$ for almost all p, where in general, ω_Y denotes the canonical sheaf of a scheme Y. Using results from [15], we derive from this an isomorphism $H_0(W, \omega_W) \cong H_0(X, \omega_X)$, proving that R has rational singularities. □

5.4. **Proposition.** *For R a local $\mathbb{C}$-affine domain with approximation R_p, almost all R_p are F-rational if, and only if, R is generically F-rational (or, equivalently, $\mathcal{B}$-rational).*

Proof. Let $\mathbf{x}$ be a system of parameters of R and let $\mathbf{x}_p$ be an approximation of $\mathbf{x}$. By [17, Theorem 4.5] almost all $\mathbf{x}_p$ are a system of parameters of R_p. Suppose first that almost all R_p are F-rational. Let y be in the generic tight closure of $\mathbf{x}R$ and let y_p be an approximation of y. Hence almost all y_p lie in the tight closure of $\mathbf{x}_pR_p$, whence in $\mathbf{x}_pR_p$ by F-rationality. Therefore, $y \in \mathbf{x}R_\infty$ by Łos' Theorem, whence $y \in \mathbf{x}R$ by faithful flatness.

Conversely, assume almost all R_p are not F-rational. This means that for almost all p, the tight closure of $\mathbf{x}_pR_p$ is strictly bigger than $\mathbf{x}_pR_p$. Let J_∞ be the ultraproduct of the tight closures of the $\mathbf{x}_pR_p$. By Łos' Theorem, $\mathbf{x}R_\infty \subsetneq J_\infty$. Since $\mathbf{x}R$ is primary to the maximal ideal in R, we have an isomorphism $R/\mathbf{x}R \cong R_\infty/\mathbf{x}R_\infty$. Symbolically, this means that $R_\infty = R + \mathbf{x}R_\infty$ (as sets), and hence that $J_\infty = (J_\infty \cap R) + \mathbf{x}R_\infty$. Therefore, putting $J := J_\infty \cap R$, we showed that $J_\infty = JR_\infty$. Since $\mathbf{x}R_\infty \subsetneq J_\infty$, we get that $\mathbf{x}R \subsetneq J$. However, one easily checks that J is just the generic tight closure of $\mathbf{x}R$. Hence, for no system of parameters $\mathbf{x}$ is $\mathbf{x}R$ equal to its generic tight closure, showing that R is not generically F-rational. □

5.5. *Remark.* In the course of the proof we actually established the following more general result. Let $(R, \mathfrak{m})$ be a local $\mathbb{C}$-affine domain and let I be $\mathfrak{m}$-primary. The ultraproduct of the tight closures of an approximation of I is equal to the extension of the generic tight closure of I to R_∞. It follows that if almost all R_p are weakly F-regular, then R is weakly generically F-regular. Indeed, let $\tilde{I}$ be the generic tight closure of an ideal I and let I_p be an approximation of I. Suppose first that I is $\mathfrak{m}$-primary. Since each I_p is tightly closed, our previous remark yields that $\tilde{I}R_\infty$ is equal to the ultraproduct of the I_p, that is to say, equal to IR_∞. Hence by faithful flatness, $I = \tilde{I}$. For I arbitrary, $\tilde{I}$ is contained in the generic tight closure of $I + \mathfrak{m}^n$, and by the previous argument that is just $I + \mathfrak{m}^n$. Since this holds for all n, Krull's Intersection Theorem yields $I = \tilde{I}$.

However, this argument does not prove the converse (since the ideals that disprove the weak F-regularity of each R_p might be of unbounded degree). Nonetheless, we suspect the converse to be true as well. Proposition 5.12 below gives the converse under the additional Gorenstein assumption.

5.6. **Proposition.** *For a local $\mathbb{C}$-affine domain R, the following are true.*

(5.6.1) *If I is generated by a regular sequence $(x_1, \dots, x_d)$ and if $I = I^+$, then*
$$(x_1^t, \dots, x_d^t)R = ((x_1^t, \dots, x_d^t)R)^+,$$
for all $t \geq 1$.

(5.6.2) *If I is an ideal of R for which $I = I^+$ and if J is an arbitrary ideal of R, then $(I : J) = (I : J)^+$.*

(5.6.3) *If R is $\mathcal{B}$-rational, then $I = I^+$, for every ideal I generated by part of a system of parameters.*

Proof. We translate the usual tight closure proofs from [11] to the present situation. For (5.6.1), induct on t, where $t = 1$ is just the hypothesis. Put $J := (x_1^t, \dots, x_d^t)R$ and let z be an element in J^+. If $x_i z \notin J$, then we may replace z by $x_i z$. Therefore, we may assume without loss of generality that $zI \subset J$. Since $(x_1, \dots, x_d)$ is R-regular, $(J : I) = J + x^{t-1}R$, where x is the product of all x_i. Hence we may assume that $z = wx^{t-1}$, for some $w \in R$. By assumption, $z = wx^{t-1} \in J\mathcal{B}(R)$. Since $(x_1, \dots, x_d)$ is $\mathcal{B}(R)$-regular by Theorem 2.4, we get that $w \in I\mathcal{B}(R)$, whence $w \in I^+ = I$. However, this shows that $z = wx^{t-1} \in J$, as required.

Assertion (5.6.2) is clear, since $z \in (I : J)\mathcal{B}(R) \cap R$ implies that $zJ \subset I\mathcal{B}(R) \cap R = I$. To prove the last assertion, assume that R is $\mathcal{B}$-rational, say, $\mathbf{x}\mathcal{B}(R) \cap R = \mathbf{x}R$ for some system of parameters $\mathbf{x} := (x_1, \dots, x_d)$. Let I be an ideal generated by an arbitrary system of parameters $(y_1, \dots, y_d)$. Since we can calculate the top local cohomology group $\mathrm{H}^d_{\mathfrak{m}}(R)$ as the direct limit of the system $R/(x_1^t, \dots, x_d^t)R$ or, alternatively, as the direct limit of the system $R/(y_1^t, \dots, y_d^t)R$, we must have an embedding $R/I \to R/(x_1^t, \dots, x_d^t)R$ for sufficiently large t. Put differently, for large enough t, we have that $I = ((x_1^t, \dots, x_d^t)R : a_t)$, for some $a_t \in R$ (see for instance [11, Exercise 4.4]). It follows therefore from (5.6.1) and (5.6.2) that $I = I^+$.

So remains to prove the result in case I is generated by part of a system of parameters. Let $\mathbf{y} := (y_1, \dots, y_d)$ be an arbitrary system of parameters and put $I_i := (y_1, \dots, y_i)R$. We need to show that $I_i = I_i^+$, for all i, and we will do this by a downward induction on i. The case $i = d$ holds by the previous argument. Suppose we already know that $I_{i+1} = I_{i+1}^+$. Let $z \in I_i^+$. In particular, $z \in I_{i+1}^+ = I_{i+1}$, so that we can write $z = a + ry_{i+1}$, for some $a \in I_i$ and some $r \in R$. Hence $z - a = ry_{i+1} \in I_i\mathcal{B}(R)$. Since y_{i+1} is a non-zero divisor modulo $I_i\mathcal{B}(R)$ by Theorem 2.4, we get that $r \in I_i\mathcal{B}(R)$, whence $r \in I_i^+$. In conclusion, we showed that $I_i^+ = I_i + y_{i+1}I_i^+$. Nakayama's Lemma therefore yields $I_i = I_i^+$, as required. □

5.7. **Models.** Let K be a field and R a K-affine algebra. With a *model of* R (called *descent data* in [10]) we mean a pair (Z, R_Z) consisting of a subring Z of K which is finitely generated over $\mathbb{Z}$ and a Z-algebra R_Z essentially of finite type, such that $R \cong R_Z \otimes_Z K$. Oftentimes, we will think of R_Z as being the model. Clearly, the collection of models R_Z of R forms a direct system whose union is R. We say that R has *F-rational type* (respectively, has *weakly F-regular type*), if there exists a model (Z, R_Z), such that $R_Z/\mathfrak{p}R_Z$ is F-rational (respectively, weakly F-regular) for all maximal ideals $\mathfrak{p}$ of Z (note that we may always localize Z at a suitably chosen element so that the property holds for all maximal ideals). See [10] or [11] for more details.

In order to compare the notions of F-rational type and generic F-rationality, we need to better understand the relation between reduction modulo p and approximations. We will see that approximations are base changes to the algebraic closure of the residue field of reductions modulo p, where the choice of the embedding of the residue field in its algebraic closure is determined by the ultrafilter.

5.8**. Lemma.** *Let Z be a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$. For almost all p, there exists a homomorphism $\gamma_p\colon Z \to \mathbb{F}_p^{alg}$, such that the sequence $\gamma_p(z)$ is an approximation of z, for each $z \in Z$.*

Proof. Write $Z \cong \mathbb{Z}[Y]/(g_1,\dots,g_m)\mathbb{Z}[Y]$, with Y a finite tuple of variables. Let $\mathbf{y}$ be the image of the tuple Y in $\mathbb{C}$ under the embedding $Z \subset \mathbb{C}$ and take an approximation $\mathbf{y}_p$ of $\mathbf{y}$ in $\mathbb{F}_p^{\mathrm{alg}}$. By Łos' Theorem, $(g_1,\dots,g_m)\mathbb{F}_p[Y]$ is contained in the kernel of the algebra homomorphism $\mathbb{F}_p[Y] \to \mathbb{F}_p^{\mathrm{alg}}$ given by $Y \mapsto \mathbf{y}_p$, for almost all p. This induces a homomorphism $\gamma_p\colon Z \to \mathbb{F}_p^{\mathrm{alg}}$ as asserted. Remains to verify the approximation property. To this end, let $z \in Z$ be represented by the image of $G \in \mathbb{Z}[Y]$, that is to say, $z = G(\mathbf{y})$. By construction, $\gamma_p(z) = G(\mathbf{y}_p)$. Since in the ultraproduct

$$\operatorname*{ulim}_{p\to\infty} G(\mathbf{y}_p) = G(\operatorname*{ulim}_{p\to\infty} \mathbf{y}_p) = G(\mathbf{y}) = z,$$

we showed that $\gamma_p(z)$ is an approximation of z. □

Note that almost all $\gamma_p(Z) \subset \mathbb{F}_p^{\mathrm{alg}}$ are in fact separable field extensions.

5.9**. Corollary.** *Let R be a local $\mathbb{C}$-affine domain with approximation R_p. For each finite subset of R, we can find a model (Z, R_Z) of R containing this subset, and, for almost all p, a homomorphism $\gamma_p\colon Z \to \mathbb{F}_p^{alg}$ inducing a separable field extension $\gamma_p(Z) \subset \mathbb{F}_p^{alg}$, such that*

$$R_p := R_Z \otimes_Z \mathbb{F}_p^{alg} \tag{5}$$

is an approximation of R.

Moreover, for each $r \in R_Z$, we get an approximation of r by taking its image in R_p via the canonical homomorphism $R_Z \to R_p$.

Proof. Suppose R is the localization of $\mathbb{C}[X]/I$ at the prime ideal $\mathfrak{m}$. Take any model (Z, R_Z) of R containing the prescribed subset. After possibly enlarging this model, we may moreover assume that there exist ideals I_Z and $\mathfrak{m}_Z$ in $Z[X]$ such that

$$R_Z = (Z[X]/I_Z)_{\mathfrak{m}_Z}$$

(whence $I = I_Z\mathbb{C}[X]$ and $\mathfrak{m} = \mathfrak{m}_Z\mathbb{C}[X]$). Let $\gamma_p\colon Z \to \mathbb{F}_p^{\mathrm{alg}}$ be a homomorphism as in Lemma 5.8 such that $\gamma_p(z)$ is an approximation of z, for each $z \in Z$. Let I_p (respectively, $\mathfrak{m}_p$) be the ideal in $\mathbb{F}_p^{\mathrm{alg}}[X]$ generated by all f^{γ_p} with $f \in I_Z$ (respectively, $f \in \mathfrak{m}_Z$), where we write f^{γ_p} for the polynomial obtained from f by applying γ_p to each of its coefficients. It follows that I_p and $\mathfrak{m}_p$ are approximations of I and $\mathfrak{m}$ respectively. Therefore

$$(\mathbb{F}_p^{\mathrm{alg}}[X]/I_p)_{\mathfrak{m}_p} \cong R_Z \otimes_Z \mathbb{F}_p^{\mathrm{alg}}$$

is an approximation of R, proving the first assertion. The last assertion is now also clear. $\square$

5.10**. Proposition.** *Let R be a local $\mathbb{C}$-affine domain. If R has F-rational type (weakly F-regular type), then R is generically F-rational (respectively, weakly generically F-regular).*

Proof. Suppose first that R has F-rational type. By definition, we can find a model (Z, R_Z) of R such that $R_Z/\mathfrak{p}R_Z$ is F-rational for all maximal ideals $\mathfrak{p}$ of Z. Let γ_p and R_p be as in (5) of Corollary 5.9. Note that $\gamma_p(Z)$ is the residue field of Z at the maximal ideal given by the kernel of γ_p. Hence each $R_Z \otimes_Z \gamma_p(Z)$ is F-rational. Since R_p is obtained from this by base change over the field extension $\gamma_p(Z) \to \mathbb{F}_p^{\mathrm{alg}}$, we get that almost all R_p are F-rational. Hence R is generically F-rational by Proposition 5.4.

The argument for weak generic F-regularity is the same, using Remark 5.5. $\square$

5.11**. Theorem.** *For a local $\mathbb{C}$-affine domain R, the following four statements are equivalent.*

(5.11.1) *R has F-rational type.*
(5.11.2) *R is generically F-rational.*
(5.11.3) *R is $\mathcal{B}$-rational.*
(5.11.4) *R has rational singularities.*

Proof. The implication (5.11.1) $\Longrightarrow$ (5.11.2) is given by Proposition 5.10 and the implication (5.11.2) $\Longrightarrow$ (5.11.3) by Corollary 4.5. Theorem 5.3 gives (5.11.3) $\Longrightarrow$ (5.11.4) and the implication (5.11.4) $\Longrightarrow$ (5.11.1) is proven by Hara in [6]. $\square$

In particular, this proves Theorem C from the introduction. Note that Smith has already proven (5.11.1) $\Longrightarrow$ (5.11.4) in [26]. Recall that we showed in [24, Theorem 6.2] that non-standard difference rational implies rational singularities. It is natural to ask whether the converse is also true. There is another related notion which is expected to be equivalent with rational singularities, to wit, *F-rationality*, that is to say, the property that some ideal generated by a system of parameters is equal to its (classical) characteristic zero tight closure. Since characteristic zero tight closure (more precisely, equational tight closure) is the smallest of all closure operations (see [17, Theorem 10.4]), F-rationality is implied by $\mathcal{B}$-rationality. Of all implications, (5.11.4) $\Longrightarrow$ (5.11.1) is the least elementary, since Hara's proof rests on some deep vanishing theorems.

5.12**. Proposition.** *If a local $\mathbb{C}$-affine domain R is Gorenstein and generically F-rational, then it is generically F-regular whence $\mathcal{B}$-regular.*

Proof. Since generic F-rationality is preserved under localization, it suffices to show that R is weakly generically F-regular. Let R_p be an approximation of R. By [17, Theorem 4.6], almost all R_p are Gorenstein. By Proposition 5.4, almost all R_p are F-rational. Therefore, almost all R_p are F-regular, by [11, Theorem 1.5]. Hence R is weakly generically F-regular by Remark 5.5, whence $\mathcal{B}$-regular by Corollary 4.5. □

Recall that a homomorphism $A \to B$ is called *cyclically pure*, if $IB \cap A = I$, for every ideal I of A.

5.13. **Proposition.** *If $R \to S$ is a cyclically pure homomorphism of local $\mathbb{C}$-affine domains and if S is weakly generically F-regular, then so is R. The same is true upon replacing weakly generically F-regular by $\mathcal{B}$-regular.*

Proof. Let I be an ideal in R and z an element in its generic tight closure. Let $R_p \to S_p$ be an approximation of $R \to S$ (that is to say, choose approximations R_p and S_p for R and S as well as approximations for the polynomials that induce the homomorphism $R \to S$; these then induce the homomorphism $R_p \to S_p$, for almost all p; see [17, 3.2.4] for more details). Let z_p and I_p be approximations of z and I. For almost all p, we have that z_p lies in the tight closure of I_p. By persistence ([11, Theorem 2.3]), z_p lies in the tight closure of I_pS_p, for almost all p, showing that z lies in the generic tight closure of IS. In fact, the preceding argument shows that generic tight closure is persistent (we have not yet used the purity of $R \to S$ nor even its injectivity). Now, by assumption, S is weakly generically F-regular, so that $z \in IS$ and hence, by cyclic purity, $z \in IS \cap R = I$.

To prove the last statement, (4.2.3) yields $I^+S \subset (IS)^+$ and the latter ideal is by assumption just IS. Cyclical purity therefore yields $I^+ \subset I$. □

5.14. **Proof of Boutot's Theorem under the additional Gorenstein hypothesis.** Let $R \to S$ be a cyclically pure homomorphism of local $\mathbb{C}$-affine domains and assume S is Gorenstein and has rational singularities. It follows that S is $\mathcal{B}$-rational, by Theorem 5.11, whence $\mathcal{B}$-regular, by Proposition 5.12. Therefore, R is $\mathcal{B}$-regular by Proposition 5.13 and hence has rational singularities by Theorem 5.11 again. □

Note that Boutot proves the same result without the Gorenstein hypothesis. It follows from his result that being generically F-rational (or, equivalently, being of F-rational type) descends under pure maps. However, it is not clear how to prove this from the definitions alone.

6. Briançon-Skoda Theorems

6.1. **Proof of Theorem** B. Let R and I be as in the statement and let z be an element in the integral closure of I^{n+k}, for some $k \in \mathbb{N}$. Take approximations

R_p, I_p and z_p of R, I and z respectively. Since z satisfies an integral equation

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

with $a_i \in I^{(n+k)i}$, we have for almost all p an equation

$$(z_p)^n + a_{1p}(z_p)^{n-1} + \cdots + a_{np} = 0$$

with $a_{ip} \in (I_p)^{(n+k)i}$ an approximation of a_i. In other words, z_p lies in the integral closure of $(I_p)^{n+k}$, for almost all p. By [9, Theorem 7.1], almost all z_p lie in $(I_p)^{k+1}R_p^+$. Taking ultraproducts, we get that $z \in I^{k+1}\mathcal{B}(R)$, as we needed to show. □

In fact, the ideas in the proof of [9, Theorem 7.1] can be used to carry out the argument directly in $\mathcal{B}(R)$. Using Theorem B, we also get a new proof of a result of Lipman and Teissier in [12]. We need a result on powers of parameter ideals.

6.2. **Proposition.** *Let R be a local $\mathbb{C}$-affine domain with rational singularities. If I is an ideal generated by a regular sequence, then $I^n = (I^n)^+$, for each n.*

Proof. Let $\mathbf{x}$ be a regular sequence generating I. We induct on n. If $n = 1$, the assertion follows from (5.6.3) in Proposition 5.6 since R is $\mathcal{B}$-rational by Theorem 5.11. Hence assume $n > 1$ and let $a \in (I^n)^+$. By induction, $a \in I^{n-1}$, so that $a = F(\mathbf{x})$ with F a homogeneous polynomial over R of degree $n-1$. Since $\mathbf{x}$ is a $\mathcal{B}(R)$-regular sequence by Theorem 2.4, it is $\mathcal{B}(R)$-quasi-regular ([13, Theorem 16.2]). In particular $a = F(\mathbf{x}) \in I^n\mathcal{B}(R)$ implies that all coefficients of F lie in $I\mathcal{B}(R)$, whence in $I = I^+$. Therefore, $a = F(\mathbf{x}) \in I^n$. □

6.3. *Remark.* More generally, we have that $J = J^+$ for any ideal J generated by monomials in some regular sequence $(x_1, \dots, x_d)$ such that J contains a power of every x_i. Indeed, by [4], any such ideal is the intersection of ideals of the form $(x_1^{t_1}, \dots, x_d^{t_d})R$ for some choice of $t_i \in \mathbb{N}$. Hence it suffices to prove the claim for J of the latter form, and this is just (5.6.3).

6.4. **Theorem** (Lipman-Teissier). *If a d-dimensional local $\mathbb{C}$-affine domain R has rational singularities, then for any ideal I of R and any $k \geq 0$, the integral closure of I^{d+k} is contained in I^{k+1}.*

Proof. Assume first that I is generated by a system of parameters. By Theorem B, the integral closure of I^{d+k} lies in $(I^{k+1})^+$ and the latter ideal is just I^{k+1} by Proposition 6.2. Next assume that I is $\mathfrak{m}$-primary, where $\mathfrak{m}$ denotes the maximal ideal of R. By [13, Theorem 14.14], we can find a system of parameters $\mathbf{x}$ of R such that $J := \mathbf{x}R$ is a reduction of I. Since I^{d+k} and J^{d+k} have then the same integral closure, our previous argument shows that this integral closure lies inside J^{k+1} whence inside I^{k+1}. Finally, let I be arbitrary and put $J_n := I + \mathfrak{m}^n$. If a lies in the integral closure of I^{d+k}, then for each n, it lies also in the integral closure of J_n^{d+k}, whence in J_n^{k+1} by our previous argument. Since

$$J_n^{k+1} \subset I^{k+1} + \mathfrak{m}^n,$$

we get that a lies in right hand side ideal for each n, and hence by Krull's Intersection Theorem, in I^{k+1}, as required. $\square$

7. Regularity and Betti numbers

In this section, we extend the main results of [18] to $\mathbb{C}$-affine domains. We start with proving Theorem D from the introduction.

7.1. **Theorem.** *Let $(R, \mathfrak{m})$ be a local $\mathbb{C}$-affine domain with residue field k. If R has at most an isolated singularity or has dimension at most two and if $\operatorname{Tor}_1^R(\mathcal{B}(R), k) = 0$, then R is regular.*

Proof. Let $(R_p, \mathfrak{m}_p)$ be an approximation of $(R, \mathfrak{m})$ and let k_p be the corresponding residue fields. It follows from [17, Theorems 4.5 and 4.6] that R_p has at most an isolated singularity or has dimension at most two, for almost all p. I claim that $\operatorname{Tor}_1^{R_p}(R_p^+, k_p) = 0$, for almost all p. Assuming the claim, we get by [18, Theorem 1.1] that almost all R_p are regular. By another application of [17, Theorem 4.6], we get that R is regular, as required.

To prove the claim, we argue as follows. Write each R_p^+ as $R_p[X]/\mathfrak{n}_p$, where X is an infinite tuple of variables and $\mathfrak{n}_p$ some ideal. Put $A_p := R_p[X]$ and let A_∞ and $\mathfrak{n}_\infty$ be the ultraproduct of the A_p and the $\mathfrak{n}_p$ respectively. Therefore, $\mathcal{B}(R) = A_\infty/\mathfrak{n}_\infty$. The vanishing of $\operatorname{Tor}_1^R(\mathcal{B}(R), k)$ means that $\mathfrak{m}A_\infty \cap \mathfrak{n}_\infty = \mathfrak{m}\mathfrak{n}_\infty$. The vanishing of $\operatorname{Tor}_1^{R_p}(R_p^+, k_p)$ is then equivalent with the equality $\mathfrak{m}_p A_p \cap \mathfrak{n}_p = \mathfrak{m}_p\mathfrak{n}_p$. Therefore, assume that this equality does not hold for almost all p, so that there exists f_p which lies in $\mathfrak{m}_p A_p \cap \mathfrak{n}_p$, but, for almost p does not lie in $\mathfrak{m}_p\mathfrak{n}_p$. Let f_∞ be the ultraproduct of the f_p. It follows from Łos' Theorem that f_∞ lies in $\mathfrak{m}A_\infty \cap \mathfrak{n}_\infty$ whence in $\mathfrak{m}\mathfrak{n}_\infty$. Let $\mathfrak{m} := (y_1, \dots, y_s)R$ and let y_{ip} be an approximation of y_i, so that $\mathfrak{m}_p = (y_{1p}, \dots, y_{sp})R_p$, for almost all p. Since $f_\infty \in \mathfrak{m}\mathfrak{n}_\infty$, there exist $g_{i\infty} \in \mathfrak{n}_\infty$, such that $f_\infty = g_{1\infty}y_1 + \dots + g_{s\infty}y_s$. Hence, if we choose $g_{ip} \in \mathfrak{n}_p$ such that their ultraproduct is $g_{i\infty}$, then by Łos' Theorem, $f_p = g_{1p}y_{1p} + \dots + g_{sp}y_{sp}$ for almost all p, contradicting our assumption on f_p. $\square$

In general, we can prove at least the following.

7.2. **Corollary.** *Let R be a local $\mathbb{C}$-affine domain with residue field k. If $\operatorname{Tor}_1^R(\mathcal{B}(R), k)$ vanishes, then R has rational singularities.*

Proof. By [18, Theorem 2.2], the vanishing of $\operatorname{Tor}_1^R(\mathcal{B}(R), k)$ implies that $R \to \mathcal{B}(R)$ is cyclically pure, that is to say, R is $\mathcal{B}$-regular. Hence R has rational singularities by Theorem 5.3. $\square$

We actually showed that R as above is $\mathcal{B}$-regular.

7.3. *Remark.* Let R be a local $\mathbb{C}$-affine domain with residue field k and approximation R_p. Let $\mathcal{F}(R)$ be the subring of $\mathcal{B}(R)$ defined as the ultraproduct of the $R_p^{1/p}$. The following are equivalent:

(7.3.1) R is regular;
(7.3.2) $R \to \mathcal{F}(R)$ is flat;
(7.3.3) $\mathrm{Tor}_1^R(\mathcal{F}(R), k) = 0$.

Indeed, let k_p the residue field of R_p. By Kunz's Theorem, the regularity of R_p is equivalent to the flatness of $R_p \to R_p^{1/p}$, and by the Local Flatness Criterion, this in turn is equivalent to the vanishing of $\mathrm{Tor}_1^{R_p}(R_p^{1/p}, k_p)$. Moreover, R is regular if, and only if, almost all R_p are regular ([17, Theorem 4.6]) whereas the same argument as in the proof of Theorem D shows that the vanishing of $\mathrm{Tor}_1^R(\mathcal{F}(R), k)$ is equivalent with the vanishing of almost all $\mathrm{Tor}_1^{R_p}(R_p^{1/p}, k_p)$. This proves that all assertions are equivalent.

7.4. *Remark.* Recently, Aberbach has announced in [1] a proof of the Main Theorem of [18] without the isolated singularity assumption. From this it would follow immediately that we can also omit the isolated singularity condition in Theorem 7.1.

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