Chapter 1

Characteristic p methods in characteristic zero via ultraproducts

Hans Schoutens

1.1 Introduction

In the last three decades, all the so-called Homological Conjectures have been settled completely for Noetherian local rings containing a field by work of Peskine-Szpiro, Hochster-Roberts, Hochster, Evans-Griffith, et. al. (some of the main papers are [20, 27, 29, 42, 57]; for an overview, see [10, §9] or [84]). More recently, Hochster and Huneke have given more simplified proofs of most of these results by means of their elegant tight closure theory, including a more canonical construction of big Cohen-Macaulay algebras (see [35, 39, 40]; for a survey, see [44, 76, 82]). However, tight closure theory also turned out to have applications to other fields, including the study of rational singularities; see for instance [33, 34, 36, 37, 46, 79, 80, 81].

Most of these results have in common that they are based on characteristic p methods, where results in characteristic zero are then obtained by reduction to characteristic p. To control the behavior under this reduction in its greatest generality, strong forms of Artin Approximation [58, 83, 85] are required, rendering the theory highly non-elementary. Moreover, there are plenty technical difficulties, which offset the elegance of the characteristic p method. It is the aim of this survey paper to show that when using ultraproducts as a means of transfer from positive to zero characteristic, the resulting theory is, in comparison, (i) easier and more elementary (at worst, we need Rotthaus's version of Artin Approximation [62]); (ii) more elegant; and (iii) more powerful. In Section 1.4, I will substantiate the former two claims, and in Section 1.5, the latter. In a final section, I discuss briefly the status in mixed characteristic.

1.2 Characteristic p methods

Let A be a ring of prime characteristic p. One feature that distinguishes it immediately from any ring in characteristic zero is the presence of the *Frobenius* morphism $x \mapsto x^p$. We will denote this ring homomorphism by Frob_A, or, when there is little room for confusion, by Frob.[1] In case A is a domain, with field of fractions K, we fix an algebraic closure $\bar{K}$ of K, and let A^+ be the integral closure of A in $\bar{K}$. We call A^+ the *absolute integral closure* of A; it is uniquely defined up to isomorphism. Although no longer Noetherian, it has many good properties. We start with a result, the proof of which we will discuss below (the reader be warned that we are presenting the results in a reversed logical, as well as historical, order).

Theorem 1.2.1. *If A is an excellent regular local ring of characteristic p, then A^+ is a flat A-algebra.*

[1] The key fact about this map, of course, is its additivity: $(a+b)^p = a^p + b^p$, since in the expansion of the former, all non-trivial binomials $\binom{p}{i}$ are divisible by p, whence zero in A.

From this fact, many deep theorems can be deduced. To discuss these, we need a definition.

Big Cohen-Macaulay algebras

Given a Noetherian local ring R, we call an R-module M a *big Cohen-Macaulay module*,[2] if there exists a system of parameters[3] which is an M-regular sequence. If every system of parameters is M-regular, then we say that M is a *balanced big Cohen-Macaulay module*. There are big Cohen-Macaulay modules which are not balanced, but this can be overcome by taking their completion ([10, Corollary 8.5.3]). We will make use of the following criterion (whose proof is rather straightforward).

Lemma 1.2.2 ([2, Lemma 4.8]). *If M is a big Cohen-Macaulay module in which every permutation of an M-regular sequence is again M-regular, then M is a balanced big Cohen-Macaulay module.* □

If M has moreover the structure of an R-algebra, we call it a *(balanced) big Cohen-Macaulay algebra.* Hochster [27] proved the existence of big Cohen-Macaulay modules in equal characteristic, and showed how they imply several homological conjectures (we will give an example below). These ideas went back to the characteristic p methods introduced by Peskine and Szpiro [57], which together with Kunz's theorem [49] and the Hochster-Roberts theorem [42] form the precursors of tight closure theory (see [44, Chapter 0]).

Theorem 1.2.3. *Every Noetherian local ring of characteristic p admits a balanced big Cohen-Macaulay algebra.*

Proof. Since completion preserves systems of parameters, it suffices to prove this for R a complete Noetherian local ring. Killing a minimal prime of maximal dimension, we may moreover assume that R is a domain. Let $(x_1, \dots, x_d)$ be a system of parameters. By Cohen's structure theorem, there exists a regular subring $S \subseteq R$ with maximal ideal $(x_1, \dots, x_d)S$ such that R is finite as an S-module. In particular, $R^+ = S^+$. By Theorem 1.2.1, the map $S \to S^+$ is flat, and hence $(x_1, \dots, x_d)$ is a regular sequence in $S^+ = R^+$. □

Remark 1.2.4. We cheated by deriving the existence of big Cohen-Macaulay algebras from Theorem 1.2.1, since currently the only known proof of the latter theorem is via big Cohen-Macaulay algebras. Here is the correct logical order: Hochster and Huneke show in [35], by different, and rather technical means, that R^+ is a balanced big Cohen-Macaulay R-algebra whenever R is an excellent local domain (see [44, Chapter 7] or [45]). This result in turn

---

[2] The nomenclature is meant to emphasize that the module need not be finitely generated.

[3] A tuple of the same length as the dimension of R is called a *system of parameters* if it generates an $\mathfrak{m}$-primary ideal; such an ideal is then called a *parameter ideal.*

implies the flatness of R^+ if R is regular, by the following flatness criterion ([44, Theorem 9.1] or [70, Theorem IV.1]).

Proposition 1.2.5. *A module over a regular local ring is a balanced big Cohen-Macaulay module if and only if it is flat.*

Proof. One direction is immediate since flat maps preserve regular sequences. So let M be a balanced big Cohen-Macaulay module over the d-dimensional regular local ring R. Since all modules have finite projective dimension, the functors $\mathrm{Tor}_i^R(M,\cdot)$ vanish for $i \gg 0$. Let e be maximal such that $\mathrm{Tor}_e^R(M,N) \neq 0$ for some finitely generated R-module N. We need to show that $e = 0$, so, by way of contradiction, assume $e \geq 1$. Using that N admits a filtration $0 = N_0 \subseteq N_1 \subseteq \dots N_s = N$ in which each subsequent quotient has the form $R/\mathfrak{p}_i$ with $\mathfrak{p}_i$ a prime ideal ([16, Proposition 3.7]), we may assume that $N = R/\mathfrak{p}$ for some prime $\mathfrak{p}$. Let h be the height of $\mathfrak{p}$, and choose a system of parameters $(x_1,\dots,x_d)$ in R such that $\mathfrak{p}$ is a minimal prime of $I := (x_1,\dots,x_h)R$. Since $\mathfrak{p}$ is then an associated prime of R/I, we can find a short exact sequence

$$0 \to R/\mathfrak{p} \to R/I \to C \to 0$$

for some finitely generated R-module C. Tensoring the above exact sequence with M, yields part of a long exact sequence

$$\mathrm{Tor}_{e+1}^R(M,C) \to \mathrm{Tor}_e^R(M,R/\mathfrak{p}) \to \mathrm{Tor}_e^R(M,R/I). \tag{1.1}$$

The first module in (1.1) is zero by the maximality of e. The last module is isomorphic to $\mathrm{Tor}_e^{R/I}(M/IM,R/I) = 0$ since $(x_1,\dots,x_h)$ is both R-regular and M-regular. Hence the middle module is zero too, contradiction. $\square$

Since the Frobenius preserves regular sequences, we immediately get one half of Kunz's theorem [49]:

Corollary 1.2.6 (Kunz). *The Frobenius is flat on a regular ring.* $\square$

To illustrate the power of the existence of big Cohen-Macaulay modules, let me derive from it one of the so-called Homological Conjectures:

Theorem 1.2.7 (Monomial Conjecture). *In a Noetherian local ring R of characteristic p, every system of parameters $(x_1,\dots,x_d)$ is* monomial*, in the sense that $(x_1x_2\cdots x_d)^t \notin (x_1^{t+1},\dots,x_d^{t+1})R$, for all t.*

Proof. Assume that the statement is false for some system of parameters $(x_1,\dots,x_d)$ and some t. Let B be a balanced big Cohen-Macaulay R-algebra. Hence $(x_1x_2\cdots x_d)^t$ belongs to $(x_1^{t+1},\dots,x_d^{t+1})B$. However, $(x_1,\dots,x_d)$ is B-regular, and it is not hard to prove that a regular sequence is always monomial, leading to the desired contradiction. $\square$

Tight closure

Let I be an ideal in a Noetherian ring A of characteristic p. We denote the ideal generated by the image of I under Frob_A by $\mathrm{Frob}_A(I)A$ (one also writes $I^{[p]}$). If $I = (f_1, \dots, f_s)A$, then $\mathrm{Frob}(I)A = (f_1^p, \dots, f_s^p)A$. In particular, $\mathrm{Frob}(I)A \subseteq I^p$, but most of the time, this is a strict inclusion. Hochster and Huneke defined the tight closure of an ideal as follows. Let A° be the multiplicative set in A of all elements not contained in any minimal prime ideal of A. An element z belongs to the *tight closure* $\mathrm{cl}_A(I)$ of I (in the literature, the tight closure is more commonly denoted I^*), if there exists some $c \in A^\circ$ such that

$$c\,\mathrm{Frob}_A^n(z) \in \mathrm{Frob}_A^n(I)A \tag{1.2}$$

for all $n \gg 0$. Using that A° is multiplicative, one easily verifies that $\mathrm{cl}(I)$ is an ideal, containing I, which itself is *tightly closed*, meaning that it is equal to its own tight closure. Equally easy to see is that, if $I \subseteq J$, then $\mathrm{cl}(I) \subseteq \mathrm{cl}(J)$.

Remark 1.2.8. The following observations all follow very easily from the definitions (see ([44, §1] for details).

1. If A is a domain, then the only restriction on c is that it be non-zero. We may always reduce to the domain case since z belongs to the tight closure of I if and only if the image of z belongs to the tight closure of $I(A/\mathfrak{p})$, where $\mathfrak{p}$ runs over all minimal prime ideals of A.
2. If A is reduced—a situation we may always reduce to by Remark 1.2.8(1)—or if I has positive height, then we may require (1.2) to hold for all n.
3. It is crucial to note that c is independent from n. If A is a domain, then we may take p^n-th roots in (1.2), to get $c^{1/p^n} z \in IA^{1/p^\infty}$, for all n, where A^{1/p^∞} is the subring of A^+ consisting of all p^n-th roots of elements of A. If we think of c^{1/p^n} approaching 1 as n goes to infinity, then the condition says, loosely speaking, that in the limit z belongs to IA^{1/p^∞}. The strictly weaker condition that $z \in IA^{1/p^\infty}$ is equivalent with requiring c to be 1 in (1.2), and leads to the notion of *Frobenius closure*. This latter closure does not have properties as good as tight closure.
4. A priori, c does depend on z as well as I. However, in many instances there is a single c which works for all z, all I, and all n; such an element is called a *test element*. Unlike most properties of tight closure, the existence of test elements is a more delicate issue (see, for instance, [36] or [44, §2]). Fortunately, for most of applications, it is not needed.

Example 1.2.9. It is instructive to look at an example. Let K be a field of characteristic $p > 3$, and let $A := K[x, y, z]/(x^3 - y^3 - z^3)K[x, y, z]$ be the projective coordinate ring of the cubic Fermat curve. Let us show that x^2 is in the tight closure of $I := (y, z)A$. For a fixed e, write $2p^e = 3h + r$ for some $h \in \mathbb{N}$ and some remainder $r \in \{1, 2\}$, and let $c := x^3$. Hence

$$cx^{2p^e} = x^{3(h+1)+r} = x^r(y^3 + z^3)^{h+1}.$$

A quick calculation shows that any monomial in the expansion of $(y^3+z^3)^{h+1}$ is a multiple of either y^{p^e} or z^{p^e}, showing that (1.2) holds for all e, and hence that $(x^2, y, z)A \subseteq \mathrm{cl}(I)$.

It is often much harder to show that an element does not belong to the tight closure of an ideal. By Theorem 1.2.10 below, any element outside the integral closure is also outside the tight closure. Since $(x^2, y, z)A$ is integrally closed, we conclude that it is equal to $\mathrm{cl}(I)$.

The following five properties all have fairly simple proofs, yet are powerful enough to deduce many deeper theorems.

Theorem 1.2.10. *Let A and B be Noetherian rings of prime characteristic p, and let I be an ideal in A.*

(weak persistence) *In an extension of domains $A \subseteq B$, tight closure is preserved in the sense that $\mathrm{cl}_A(I)B \subseteq \mathrm{cl}_B(IB)$.*
(regular closure) *If A is a regular local ring, then I is tightly closed.*
(plus closure) *If A is a of domain, then $IA^+ \cap A \subseteq \mathrm{cl}_A(I)$.*
(colon capturing) *If A is a homomorphic image of a local Cohen-Macaulay ring then $((x_1, \dots, x_i)A : x_{i+1}) \subseteq \mathrm{cl}((x_1, \dots, x_i)A)$, for each i and each system of parameters $(x_1, \dots, x_d)$.*
(integral closure) *Tight closure, $\mathrm{cl}(I)$, is contained in integral closure, $\bar{I}$; if I is principal, then $\mathrm{cl}(I) = \bar{I}$.*

Proof. Weak persistence is immediate from the fact that (1.2) also holds, by functoriality of the Frobenius, in B, and c remains non-zero in B. In fact, the much stronger property, *persistence*, where the homomorphism does not need to be injective, holds in many cases. However, to prove this, one needs test elements (see Remark 1.2.8(4)).

To prove the regularity property, suppose A is regular but I is not tightly closed. Hence there exists $z \in \mathrm{cl}(I)$ not in I. In particular, $(I : z)$ is contained in the maximal ideal $\mathfrak{m}$ of A. By definition, there is some non-zero c such that $c\,\mathrm{Frob}^n(z) \in \mathrm{Frob}^n(I)A$ for all $n \gg 0$. Since the Frobenius is flat on a regular ring by Corollary 1.2.6, and since flat maps commute with colons (see for instance [76]), we get

$$c \in (\mathrm{Frob}^n(I)A : \mathrm{Frob}^n(z)) = \mathrm{Frob}^n(I : z)A$$

for all n. Since $(I : z) \subseteq \mathfrak{m}$, we get $c \in \mathrm{Frob}^n(\mathfrak{m})A \subseteq \mathfrak{m}^n$, for all n, yielding the contradiction that $c = 0$ by Krull's intersection theorem.

To prove the plus closure property, let $z \in IA^+ \cap A$. Hence, there exists a finite extension $A \subseteq B \subseteq A^+$ such that already $z \in IB$. Choose an A-linear (module) morphism $g\colon B \to A$ sending 1 to a non-zero element $c \in A$.[4]

[4] Let K be the field of fractions of A. Embed B in a finite dimensional vector space K^n and choose a projection $K^n \to K$ so that the image of 1 under the composition is non-zero. The required map $B \to A$ is obtained from this composition by clearing denominators.

Applying the Frobenius to $z \in IB$, yields $\mathrm{Frob}^n_A(z) \in \mathrm{Frob}^n_B(IB)B$ for all n. Applying g to the latter shows $c\,\mathrm{Frob}^n_A(z) \in \mathrm{Frob}^n_A(I)A$, for all n, that is to say, $z \in \mathrm{cl}_A(I)$.

Colon capturing knows many variants. Let me only discuss the special, but important case that A is moreover complete. By Cohen's structure theorem, we can find a regular local subring $(S, \mathfrak{n})$ of A such that A is finite as an S-module and $\mathfrak{n}A = (x_1, \dots, x_d)A$. Suppose $zx_{i+1} \in (x_1, \dots, x_i)A$. Applying powers of Frobenius, we get

$$\mathrm{Frob}^n(zx_{i+1}) \in (\mathrm{Frob}^n(x_1), \dots, \mathrm{Frob}^n(x_i))A \tag{1.3}$$

for all n. Let R be the S-subalgebra of A generated by z, and as above, choose an R-linear morphism $g\colon A \to R$ with $c := g(1) \neq 0$. Applying g to (1.3) yields a relation

$$c\,\mathrm{Frob}^n(z)\,\mathrm{Frob}^n(x_{i+1}) \in (\mathrm{Frob}^n(x_1), \dots, \mathrm{Frob}^n(x_i))R \tag{1.4}$$

for all n. Since R is a hypersurface ring, it is Cohen-Macaulay. In particular, $(x_1, \dots, x_d)$, being a system of parameters in R, is R-regular, and so is therefore the sequence $(\mathrm{Frob}^n(x_1), \dots, \mathrm{Frob}^n(x_d))$. This allows us to cancel $\mathrm{Frob}^n(x_{i+1})$ in (1.4), getting the tight closure relations $c\,\mathrm{Frob}^n(z) \in (\mathrm{Frob}^n(x_1), \dots, \mathrm{Frob}^n(x_i))R$. Weak persistence then shows that z also belongs to the tight closure of $(x_1, \dots, x_i)A$, as we needed to show.

Finally, the containment $\mathrm{cl}(I) \subseteq \bar{I}$ is immediate from the integrality criterion that $z \in \bar{I}$ if and only if $cz^n \in I^n$ for some $c \in A^\circ$ and infinitely many n (note that $\mathrm{Frob}^n(I)A \subseteq I^{p^n}$). If I is principal, then $\mathrm{Frob}^n(I)A = I^{p^n}$. ☐

Remark 1.2.11. It follows from the last property that the tight closure of an ideal is contained in its radical. In particular, radical ideals are tightly closed. It had been conjectured that $IA^+ \cap A$ is equal to the tight closure of I, but this has now been disproved by the counterexample in [8]. Nonetheless, for parameter ideals they are the same by [78]; see also Remark 1.2.14(3).

To convince the reader of the strength of these properties, I provide a short tight closure proof of the following celebrated theorem of Hochster and Roberts (see also Theorem 1.4.4 below):

Theorem 1.2.12 (Hochster-Roberts [42]). *If $R \to S$ is a* cyclically pure *extension of Noetherian local rings (that is to say, if $IS \cap R = I$ for all ideals $I \subseteq R$), and if S is regular, then R is Cohen-Macaulay.*

Proof. We leave it to the reader to verify that all properties pass to the completion, and so we may assume that R and S are moreover complete. Let $(x_1, \dots, x_d)$ be a system of parameters in R. We have to show that it is R-regular, that is to say, that $(J_i : x_{i+1})$ is equal to $J_i := (x_1, \dots, x_i)R$, for all i. By colon capturing, the former ideal is contained in the tight closure of J_i, whence by weak persistence (note that R is a domain since $R \to S$ is in particular injective) in the tight closure of J_iS. Since S is regular, the latter ideal is tightly closed, showing that $(J_i : x_{i+1})$ is contained in $J_iS \cap R = J_i$, where the last equality follows from cyclic purity. □

Tight closure and singularities

The regularity property suggests the following paradigm: the larger the collection of tightly closed ideals in a Noetherian ring A of prime characteristic p, the closer it is to being regular.

Definition 1.2.13. If every ideal is tightly closed, A is called *weakly F-regular*; if every localization is weakly F-regular, then A is called *F-regular*. If A is local and some parameter ideal is tightly closed, the ring is called *F-rational*. If A is reduced and the Frobenius is pure on A, that is to say, each base change of the Frobenius is injective, then A is called *F-pure*.

Table 1.1 Correspondence (partly conjectural)

F-singularity	classical singularity
F-rational	rational singularities
F-pure	log-canonical
F-regular	log-terminal

Remark 1.2.14. In §1.5, I will discuss the connection with the singularities in the above table. For now, let me just make some remarks on the tight closure versions.

1. Regular rings are F-regular by Theorem 1.2.10, but these are not the only ones. In fact, the proof of Theorem 1.2.12 shows that any cyclically pure subring of a regular local ring is F-regular. It is a major open question whether in general F-regular and weakly F-regular are the same. This is tied in with the problem of the behavior of tight closure under localization (only recently, [8], tight closure has been shown to not always commute with localization).
2. Notwithstanding, the property of being F-rational is preserved under localization. Therefore, any weakly F-regular ring is F-rational. Moreover, in an F-rational local domain, every parameter ideal, and more generally, every ideal generated by part of a system of parameters is tightly closed ([44, Theorem 4.2]). In particular, every principal ideal is tightly closed, since it is generated by a parameter. Hence every principal ideal is integrally closed by the last property in Theorem 1.2.10, from which it follows that an F-rational ring is normal. Moreover, by colon capturing, any system of

parameters is regular (same argument as in the proof of Theorem 1.2.12), proving that an F-rational ring is Cohen-Macaulay.
3. Smith has shown in [78] that for a local domain A, the tight closure of a parameter ideal I is equal to $IA^+ \cap A$, and hence such an A is F-rational if and only if some parameter ideal is contracted from A^+.
4. A weakly F-regular ring is F-pure: given an ideal $I \subseteq A$, we have

$$IA^{1/p} \cap A \subseteq IA^+ \cap A \subseteq \mathrm{cl}(I),$$

by Remark 1.2.11. Hence, by weak F-regularity, the latter ideal is just I. This shows that the Frobenius is cyclically pure. Since A is normal by remark (2) above, this in turn implies the purity of the Frobenius by [28, Theorem 2.6].

1.3 Difference closure and the ultra-Frobenius

From the proofs of the five basic properties of tight closure listed in Theorem 1.2.10, we extract the following three key properties of the Frobenius: its functoriality, its contractive nature (sending a power of an ideal into a higher power of the ideal),[5] and its preservation of regular sequences. Moreover, it is not necessary that the Frobenius acts on the ring itself; it suffices that it does this on some faithfully flat overring. So I propose the following formalization of tight closure.

Definition 1.3.1 (Difference hull). Let $\mathfrak{C}$ be a category of Noetherian rings (at this point we do not need to make any characteristic assumption). A *difference hull* on $\mathfrak{C}$ is a functor $\mathsf{D}(\cdot)$ from $\mathfrak{C}$ to the category of difference rings,[6] and a natural transformation η from the identity functor to $\mathsf{D}(\cdot)$, with the following three additional properties:

1. each $\eta_A \colon A \to \mathsf{D}(A)$ is faithfully flat;
2. the endomorphism σ_A of $\mathsf{D}(A)$ preserves $\mathsf{D}(A)$-regular sequences;
3. for any ideal $I \subseteq A$, we have $\sigma_A(I) \subseteq I^2\mathsf{D}(A)$.

Functoriality here means that, for each A in $\mathfrak{C}$, we have a ring $\mathsf{D}(A)$ together with an endomorphism σ_A, and a ring homomorphism $\eta_A \colon A \to \mathsf{D}(A)$, such that for each morphism $A \to B$ in $\mathfrak{C}$, we get an induced morphism of difference rings $\mathsf{D}(A) \to \mathsf{D}(B)$ for which the diagrams

---

[5] The Frobenius is a contractive homeomorphism on the metric space given by the maximal adic topology; this is not to be confused with tight closure coming from contraction in finite extensions.

[6] A *difference ring* is a ring with an endomorphism; a *morphism of difference rings* is a ring homomorphism between difference rings that commutes with the respective endomorphisms.

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d] & D(A) \arrow[r, "\sigma_A"] \arrow[d] & D(A) \arrow[d] \\
B \arrow[r, "\eta_B"] & D(B) \arrow[r, "\sigma_B"] & D(B).
\end{tikzcd}$$

commute.

Since η_A is in particular injective, we will henceforth view A as a subring of $D(A)$ and omit η_A from our notation. Given a difference hull $D(\cdot)$ on some category $\mathfrak{C}$, we define the *difference closure* $\mathrm{cl}^D(I)$ of an ideal $I \subseteq A$ of a member A of $\mathfrak{C}$ as follows: an element $z \in A$ belongs to $\mathrm{cl}^D(I)$ if there exists $c \in A^\circ$ such that

$$c\sigma^n(z) \in \sigma^n(I)D(A) \tag{1.5}$$

for all $n \gg 0$. Here, $\sigma^n(I)D(A)$ denotes the ideal in $D(A)$ generated by all $\sigma^n(y)$ with $y \in I$, where σ is the endomorphism of the difference ring $D(A)$. It is crucial here that c belongs to the original ring A, although the membership relations in (1.5) are inside the bigger ring $D(A)$. An ideal that is equal to its difference closure will be called *difference closed*. One easily checks that $\mathrm{cl}^D(I)$ is a difference closed ideal containing I.

Example 1.3.2 (Frobenius hull). It is clear that our definition is inspired by the membership test (1.2) for tight closure, and indeed, this is just a special case. Namely, for a fixed prime number p, let $\mathfrak{C}_p$ be the category of all Noetherian rings of characteristic p and let $D(\cdot)$ be the functor assigning to a ring A the difference ring (A, Frob_A). It is easy to see that this makes $D(\cdot)$ a difference hull in the above sense, and the difference closure with respect to this hull is just the tight closure of the ideal.

Remark 1.3.3. Let $\mathfrak{C}$ be a category with a difference hull $D(\cdot)$, consisting of local rings. We can extend this difference hull to any Noetherian ring A all of whose localizations belong to $\mathfrak{C}$. Indeed, let $D(A)$ be the Cartesian product of all $D(A_\mathfrak{m})$ where $\mathfrak{m}$ runs over all maximal ideals of A. The product of all $\sigma_{A_\mathfrak{m}}$ is then an endomorphism satisfying the conditions of difference hull in Definition 1.3.1. In particular, this yields a difference closure on A as well.

Before we discuss how we will view tight closure in characteristic zero as a difference closure, we discuss the difference analogue of three of the five key properties of tight closure (for the remaining two, one needs some additional assumptions, which I will not discuss here).

Theorem 1.3.4. *Let $\mathfrak{C}$ be a category endowed with a difference hull $D(\cdot)$, and let $A \to B$ be a morphism in $\mathfrak{C}$ with B a domain.*

(weak persistence) *If $A \subseteq B$ is injective, then $\mathrm{cl}^D(I)B \subseteq \mathrm{cl}^D(IB)$, for each ideal $I \subseteq A$.*

(regular closure) *If A is a regular local ring, then every ideal is difference closed.*

(colon capturing) *If A is a homomorphic image of a local Cohen-Macaulay ring, then $((x_1,\dots,x_i)A : x_{i+1}) \subseteq \mathrm{cl}^{\mathsf{D}}((x_1,\dots,x_i)A)$, for each i and each system of parameters $(x_1,\dots,x_d)$.*

Proof. The arguments in the proof of Theorem 1.2.10 carry over easily, once we have shown that $\sigma\colon A \to \mathsf{D}(A)$ is flat whenever A is regular. This follows from the fact that σ preserves regular sequences, so that $\mathsf{D}(A)$ is a balanced big Cohen-Macaulay A-algebra, whence flat by Proposition 1.2.5. □

Remark 1.3.5. Note that exactly these three properties were required to deduce Theorem 1.2.12.

Lefschetz rings

By a *Lefschetz ring*,[7] we mean a ring of characteristic zero which is realized as the ultraproduct of rings of prime characteristic. More precisely, let W be an infinite index set and A_w a ring, for each $w \in W$. Let A_∞ be the Cartesian product of the A_w. We may view A_∞ as a $\mathbb{Z}_\infty$-algebra, where $\mathbb{Z}_\infty$ is the corresponding Cartesian power of $\mathbb{Z}$. We call a prime ideal $\mathfrak{p}$ in $\mathbb{Z}_\infty$ *non-standard*, if it is a minimal prime ideal of the direct sum ideal $\oplus_w \mathbb{Z} \subseteq \mathbb{Z}_\infty$. An *ultraproduct* of the rings A_w is any residue ring of the form $A_\natural := A_\infty/\mathfrak{p}A_\infty$, where $\mathfrak{p} \subseteq \mathbb{Z}_\infty$ is some non-standard prime ideal. Given elements $a_w \in A_w$, we call the image of the sequence $(a_w)_w$ in $A_\natural$ the *ultraproduct* of the a_w. If each A_w is a Z-algebra, for some ring Z, then so is $A_\natural$. The structure map $Z \to A_\natural$ is given as follows: if we write z_w for z viewed as an element of A_w, then we send z to the ultraproduct $z_\natural$ of the z_w.

Although a simple and elegant algebraic definition, the aforesaid is not the usual definition of an ultraproduct, and to formulate the main properties and prove them, we need to turn to its classical definition from logic. Namely, let $\mathcal{U}$ be an ultrafilter on W—that is to say, a collection of infinite subsets of W closed under finite intersections and supersets, and such that any subset of W or its complement belongs to $\mathcal{U}$. Let $\mathfrak{a}_{\mathcal{U}}$ be the ideal in A_∞ of all sequences almost all of whose entries are zero (a property is said to hold *for almost all w* if the subset of all indices w for which it holds belongs to the ultrafilter). We call the residue ring $A_\natural := A_\infty/\mathfrak{a}_{\mathcal{U}}$ the *ultraproduct*[8] of the A_w. To connect this to our previous definition, one then shows that $\mathfrak{a}_{\mathcal{U}}$ is of the form $\mathfrak{p}A_\infty$ for

---

[7] The designation alludes to an old heuristic principle in algebraic geometry regarding transfer between positive and zero characteristic, which Weil [87] attributes to Lefschetz.

[8] More generally, if A_w are certain algebraic, or more precisely, first-order structures, then their ultraproduct is defined in a similar way, by taking the quotient of the Cartesian product A_∞ modulo the equivalence relation that two sequences are equivalent if and only if almost all their entries are the same.

some non-standard prime $\mathfrak{p}$. More precisely, $\mathfrak{p}$ is generated by all characteristic functions 1_W (viewed as elements in $\mathbb{Z}_\infty$) with $W \notin \mathcal{U}$. The main property of an ultraproduct is the following version of what logicians call Łos' Theorem (its proof is a straightforward verification of the definitions [76, Theorem 2.3.1]).

Proposition 1.3.6 (Equational Łos' Theorem). *Let $A_\natural$ be the ultraproduct of rings A_w, let $\mathbf{a}_w$ be a tuple of length n in A_w and let $\mathbf{a}_\natural$ be their ultraproduct in $A_\natural$. Given a finite set of polynomials $f_1, \dots, f_s \in \mathbb{Z}[x]$ in n indeterminates x, we have that $f_1(\mathbf{a}_w) = \dots = f_s(\mathbf{a}_w) = 0$ in A_w for almost all w if and only if $f_1(\mathbf{a}_\natural) = \dots = f_s(\mathbf{a}_\natural) = 0$ in $A_\natural$.*

Remark 1.3.7. Instead of just equations, we may also include inequations. If all A_w are Z-algebras, over some ring Z, then so is $A_\natural$, and we may take the polynomials f_i with coefficients over Z. The full, model-theoretic, version, Łos' Theorem, allows for arbitrary first-order sentences, which are obtained from equational formulae by taking finite Boolean combinations and quantification (for in-depth discussions of ultraproducts, see [11, 17, 43]; for a brief review see [68, §2] or [76, §2]). However, the above version is often sufficient to prove transfer results between an ultraproduct and its components. For instance, one easily deduces from it that $A_\natural$ is reduced (respectively a domain, or a field), if and only if almost all components A_w are. Indeed, reducedness follows from the equation $x^2 = 0$ only having the zero solution. Unfortunately, one of the most fundamental properties used in commutative algebra, Noetherianity, is rarely preserved under ultraproducts (the case of fields mentioned above is a providential exception). One of our main tasks, therefore, will be to circumvent this major obstacle.

Let $A_\natural$ be an ultraproduct of rings A_w of positive characteristic p_w. Using the above theorem, one can show that $A_\natural$ has equal characteristic zero if and only if the p_w are *unbounded*, meaning that for every N, almost all $p_w > N$. A ring with this property will be called a *Lefschetz* ring. Our key example is:

Proposition 1.3.8. *The field of complex numbers is a Lefschetz ring.*

Proof. For each prime number p, let $\bar{\mathbb{F}}_p$ be the algebraic closure of the p-element field, and let $F_\natural$ be their ultraproduct (with respect to some ultrafilter on the set of prime numbers). By our above discussions $F_\natural$ is again a field of characteristic zero. Since one can express in terms of equations that a field is algebraically closed, Proposition 1.3.6 proves that $F_\natural$ is algebraically closed. One checks that its cardinality is that of the continuum. So we may invoke Steinitz's theorem to conclude that it must be the unique algebraically closed field of characteristic zero of that cardinality, to wit, $\mathbb{C}$. $\square$

Remark 1.3.9. It is clear from the above proof that the isomorphism $F_\natural \cong \mathbb{C}$ is far from explicit. This is the curse when working with ultraproducts: they are highly non-constructive; after all, the very existence of ultrafilters hinges on the Axiom of Choice. Steinitz's theorem holds of course also in higher cardinalities, and we may therefore extend the above result to: every algebraically closed field of characteristic zero of cardinality 2^κ for some infinite cardinal κ is Lefschetz.[9] In particular, any field of characteristic zero is contained in some Lefschetz field. No countable field can be Lefschetz because of cardinality reasons. In particular, the algebraic closure of $\mathbb{Q}$ is an example of an algebraically closed field of characteristic zero which is not Lefschetz.

Our main interest in Lefschetz rings comes from the following observation. Let $\mathrm{Frob}_\infty := \prod_w \mathrm{Frob}_{A_w}$ be the product of the Frobenii on the components. It is not hard to show that any non-standard prime ideal $\mathfrak{p}$ is generated by idempotents, and hence $\mathrm{Frob}_\infty(\mathfrak{p}A_\infty) = \mathfrak{p}A_\infty$. In particular, we get an induced homomorphism on $A_\natural$, which we call the *ultra-Frobenius* of $A_\natural$ and which we continue to denote by Frob_∞. In other words, Lefschetz rings are difference rings in a natural way, and this is our point of departure to define tight closure in characteristic zero.

1.4 Tight closure in characteristic zero

As mentioned above, we will use Lefschetz rings as difference hulls to define tight closure in characteristic zero. However, before we describe the theory, let us see how Hochster and Huneke arrive at a tight closure operation in characteristic zero, which for emphasis, we will denote $\mathrm{cl}^{HH}(\cdot)$.[10] Their method goes back, once more, to the seminal work [57] of Peskine and Szpiro: use generic flatness and Artin Approximation to lift results in characteristic p to characteristic zero. The method was elaborated upon further by Hochster [27], and can be summarized briefly as follows (see also [10, Chapter 8] or [84]). Given a Noetherian ring A of equal characteristic zero we first construct a suitable finitely generated subalgebra $A_0 \subseteq A$ and then reduce modulo p, to obtain the rings A_0/pA_0 of characteristic p. Of course, we must do this in such way that properties of A are reflected by properties of A_0 and these in turn should be reflected by properties of the closed fibers A_0/pA_0. The former requires Artin Approximation (see below) and the latter generic flatness. Moreover, due to these two techniques, only properties expressible by systems of equations stand a chance of being transferred. To carry this out, quite some machinery is needed, which unfortunately drowns tight closure's elegance in technical prerequisites; see [44, Appendix] or [40].

So, let us describe the ultraproduct method, in which transfer will be achieved mainly through Łos' Theorem. Given a Noetherian ring of equal

[9] If one assumes the (generalized) Continuum Hypothesis, then this just means any uncountable algebraically closed field.

[10] In fact, there are several candidates for tight closure in characteristic zero, depending on the choice of a base field, which are only conjecturally equivalent; $\mathrm{cl}^{HH}(\cdot)$ is the smallest of these variants.

characteristic zero, we must construct a Lefschetz ring $\mathsf{L}(A)$ containing A in such a way that the functor $\mathsf{L}(\cdot)$ constitutes a difference hull; we will call $\mathsf{L}(\cdot)$ a *Lefschetz hull*. Condition 1.3.1.(3) is clear, whereas a simple application of Łos' Theorem can be used to prove that any ultraproduct of regular sequences is again a regular sequence, showing that also condition 1.3.1.(2) is automatically satisfied. So, apart from functoriality, remains to construct $\mathsf{L}(A)$ so that the embedding $A \subseteq \mathsf{L}(A)$ is faithfully flat. Part of functoriality is easily obtained: if we have a Lefschetz hull $\mathsf{L}(A)$ for A, and $I \subseteq A$ is an ideal, then we can take $\mathsf{L}(A/I) := \mathsf{L}(A)/I\mathsf{L}(A)$ as a Lefschetz hull for A/I. Indeed, $\mathsf{L}(A)/I\mathsf{L}(A)$ is again Lefschetz by Lemma 1.4.1 below; all three properties in Definition 1.3.1 now follow by base change.

Lemma 1.4.1. *Any residue ring of a Lefschetz ring modulo a finitely generated ideal is again Lefschetz.*

Proof. Let $B_\natural$ be an arbitrary Lefschetz ring and $J \subseteq B_\natural$ a finitely generated ideal. By assumption, $B_\natural$ is the ultraproduct of rings B_w of positive characteristic. Let $f_1, \dots, f_s \in B_\natural$ generate J, and choose $f_{iw} \in B_w$ so that for each i, the ultraproduct of the elements f_{iw} is equal to f_i. Let $J_w := (f_{1w}, \dots, f_{sw})B_w$. It is an easy but instructive exercise on Łos' Theorem to show that the image of $\prod J_w \subseteq B_\infty$ in $B_\natural$ is equal to J, and that the ultraproduct of the B_w/J_w is equal to $B_\natural/J$, proving in particular that the latter is again Lefschetz. □

With notation as in the proof, we call J the *ultraproduct* of the J_w. A note of caution: infinitely generated ideals in $B_\natural$ need not be realizable as an ultraproduct of ideals, and so their residue rings need not be Lefschetz. An example is the ideal of infinitesimals, introduced in Proposition 1.4.9 below.

We will prove the existence of Lefschetz hulls for two classes of rings: affine algebras over a field K of characteristic zero,[11] and equicharacteristic zero Noetherian local rings. Since the only issue is the flatness of the hull, we may always pass to a faithfully flat extension of the ring A. Hence, in either case, we can find, by Remark 1.3.9, a sufficient large algebraically closed Lefschetz field K, such that in the affine case, A is finitely generated over K, and in the local case, A is complete, with residue field K. By Noether Normalization and Cohen's structure theorem, A is a residue ring of respectively the polynomial ring $K[x]$ or the power series ring $K[[x]]$, where x is a finite tuple of indeterminates. By our above discussion on residue rings, this then reduces the problem to finding a Lefschetz hull of the polynomial ring and the power series ring respectively. We start with the easiest case, the polynomial case.

[11] We call an algebra A *affine* if it is finitely generated over a field.

Tight closure for affine algebras

Let K be an (algebraically closed) Lefschetz field, realized as the ultraproduct of (algebraically closed) fields K_w of positive characteristic, and let x be a finite tuple of indeterminates. Put $A_w := K_w[x]$ and let $A_\natural$ be their ultraproduct. Clearly, K is a subring of $A_\natural$. For each i, let us write also x_i for the ultraproduct of the constant sequence x_i. By Łos' Theorem, each x_i, is transcendental over K, and hence the polynomial ring $A = K[x]$ embeds in $A_\natural$. So remains to show that this embedding is faithfully flat. This fact was first observed by van den Dries in [14], and used by him and Schmidt in [63] to deduce several uniform bounds in polynomial rings; further extensions based on this method can be found in [64, 65, 74]; for an overview, see [76].

Proposition 1.4.2. *The embedding $A \subseteq A_\natural$ is faithfully flat.*

Proof. Since K is algebraically closed, every maximal ideal $\mathfrak{m} \subseteq A$ is of the form $(x_1 - a_1, \dots, x_n - a_n)A$ by Hilbert's Nullstellensatz. After a change of coordinates, we may assume that $\mathfrak{m} = (x_1, \dots, x_d)A$. Since $\mathfrak{m}A_\natural$ is the ultraproduct of the maximal ideals $\mathfrak{m}_w := (x_1, \dots, x_d)A_w$, it too is maximal, and $(A_\natural)_{\mathfrak{m}A_\natural}$ is the ultraproduct of the localizations $(A_w)_{\mathfrak{m}_w}$. It suffices therefore to show that the homomorphism $A_\mathfrak{m} \to (A_\natural)_{\mathfrak{m}A_\natural}$ is flat. We already remarked that ultraproducts preserve regular sequences, showing that $(x_1, \dots, x_d)$ is $(A_\natural)_{\mathfrak{m}A_\natural}$-regular. Hence $(A_\natural)_{\mathfrak{m}A_\natural}$ is a big Cohen-Macaulay $A_\mathfrak{m}$-algebra. Moreover, every permutation of an $(A_\natural)_{\mathfrak{m}A_\natural}$-regular sequence is again regular, since this is true in each $(A_w)_{\mathfrak{m}_w}$. So $(A_\natural)_{\mathfrak{m}A_\natural}$ is a balanced big Cohen-Macaulay algebra by Lemma 1.2.2. Since $A_\mathfrak{m}$ is regular, $A_\mathfrak{m} \to (A_\natural)_{\mathfrak{m}A_\natural}$ is flat by Proposition 1.2.5. □

By our previous discussion, we have now constructed a Lefschetz hull $\mathsf{L}(C)$ for any affine algebra C over a field k of characteristic zero. Namely, with notation as above, if $C \otimes_k K \cong A/I$, then $\mathsf{L}(C) := A_\natural / IA_\natural$. The positive characteristic affine algebras C_w whose ultraproduct equals $\mathsf{L}(C)$ will be called *approximations* of C. It is justified to call $\mathsf{L}(C)$ a hull, since any K-algebra homomorphism $C \to B_\natural$ into an ultraproduct $B_\natural$ of finitely generated K_w-algebras B_w, induces a unique homomorphism of Lefschetz rings $\mathsf{L}(C) \to B_\natural$ (that is to say, an ultraproduct of K_w-algebras $C_w \to B_w$). It follows that $\mathsf{L}(C)$ only depends on the choice of algebraically closed Lefschetz field K, and on the way we represent the latter as an ultraproduct of algebraically closed fields of positive characteristic, that is to say, on the choice of ultrafilter. An example due to Brenner and Katzman [7] indicates that different choices of ultrafilter may lead to different tight closure notions: this is true for ultraclosure as defined below in §1.4.10, and most likely also for tight closure to be defined shortly; see [7, Remark 4.10]. Nonetheless, this dependence on the ultrafilter is in all what we will do with Lefschetz hulls of no consequence, and we will henceforth pretend that, once K has been fixed, the Lefschetz hull is unique. Nonetheless, even when fixing the ultrafilter, the approximations

C_w are not uniquely defined by C: given a second approximation C'_w, we can at best conclude that $C_w \cong C'_w$ for almost all w. Again, this seems not to matter. The approximations do carry a lot of the structure of the affine algebra (to a much larger extent than the characteristic p reductions of C used in the Hochster-Huneke tight closure $\mathrm{cl}^{HH}(\cdot)$ in characteristic zero; see [72, §2.17]), and we summarize this in the following theorem, stated without proof.

Theorem 1.4.3 ([68, Theorem 4.18]). *Let K be an algebraically closed Lefschetz field, let C be a K-affine algebra, and let C_w be approximations of C. Then C has the same dimension and depth as almost all C_w. Moreover, C is a domain, normal, regular, Cohen-Macaulay, or Gorenstein, if and only if almost all C_w are.* $\square$

In particular, we can extend the Lefschetz hull to any localization of a K-affine algebra: if $\mathfrak{p}$ is a prime ideal in C, then by construction, $L(C)/\mathfrak{p}L(C)$ is a Lefschetz hull of $C/\mathfrak{p}$. By Theorem 1.4.3, the approximations of $C/\mathfrak{p}$ are domains, and hence, by Łos' Theorem, so is their ultraproduct, proving that $\mathfrak{p}L(C)$ is a prime ideal. Hence, we can take $L(C)_{\mathfrak{p}L(C)}$ as a Lefschetz hull of $C_\mathfrak{p}$. Since we will treat the local case below, I skip the details.

In any case, we may apply the difference closure theory from §1.3, to define the *tight closure*[12] $\mathrm{cl}_C(I)$ of an ideal $I \subseteq C$ as the collection of all $z \in C$ for which there exists $c \in C^\circ$ such that $c\,\mathrm{Frob}^n_\infty(z) \in \mathrm{Frob}^n_\infty(I)L(C)$ for all $n \gg 0$, where Frob_∞ is the ultra-Frobenius on the Lefschetz hull $L(C)$. We remind the reader that the analogues of all five properties in Theorem 1.2.10 hold in characteristic zero (we did not give details for two of these; they can be found in [68]).[13] In particular, any ideal in a polynomial ring, and more generally, in a regular K-algebra, is tightly closed.

As in positive characteristic, we immediately get the characteristic zero version of the Hochster-Roberts theorem for affine algebras (see Remark 1.3.5). To state the original version, let us call an affine scheme X a *quotient singularity*, if there exists a smooth scheme $Y = \mathrm{Spec}(A)$ over $\mathbb{C}$, and a linearly reductive algebraic group G (meaning, that G is the complexification of a compact real Lie group), acting $\mathbb{C}$-rationally on Y by $\mathbb{C}$-algebra automorphisms, so that X is the quotient Y/G given as the affine scheme $\mathrm{Spec}(A^G)$, where A^G is the subring of A of G-invariant elements.

Theorem 1.4.4 (Hochster-Roberts, [42]). *A quotient singularity is Cohen-Macaulay.*

Proof. With notation as above, from Lie theory or a general argument about linearly reductive groups, we get the so-called *Reynolds operator*

---

[12] I previously referred to it as *non-standard* tight closure.

[13] In fact, the plus closure property is almost trivial in characteristic zero, since for A normal, we always have $IA^+ \cap A = I$ (see [10, Remark 9.2.4]).

$\rho_G\colon A \to A^G$, that is to say, a homomorphism of A^G-modules. In particular, the inclusion $A^G \subseteq A$ is split whence cyclically pure, and the result now follows, after localization, from the analogue of Theorem 1.2.12.

For another application in characteristic zero, first proven using deep methods from birational geometry [15], but subsequently reproved and generalized by a simple tight closure argument in characteristic p in [41], see [66]. Yet another problem requiring sophisticated methods for its solution, but now admitting a very simple tight closure proof, is the Briançon-Skoda Theorem.[14]

Theorem 1.4.5. *The following Briançon-Skoda type properties hold:*

(Briançon-Skoda, [9]) *If f is a power series without constant term in s variables x over $\mathbb{C}$, then f^s lies in the* Jacobian ideal *of f, that is to say, $f^s \in (\partial f/\partial x_1, \dots, \partial f/\partial x_s)\mathbb{C}[[x]]$;*

(Lipman-Sathaye, [51]) *If R is a regular local ring and $\mathfrak{a} \subseteq R$ an ideal generated by s elements, then the integral closure $\overline{\mathfrak{a}^s}$ of $\mathfrak{a}^s$ is contained in $\mathfrak{a}$;*

(Hochster-Huneke, [34]) *If A is a Noetherian ring containing a field and if $\mathfrak{a} \subseteq A$ is an ideal generated by s elements, then $\overline{\mathfrak{a}^s} \subseteq \operatorname{cl}(\mathfrak{a})$.*

Proof. We start with the last assertion. Assume first that A has characteristic p. Let $\mathfrak{a} = (a_1, \dots, a_s)A$ and assume z lies in the integral closure of $\mathfrak{a}^s$. Hence,

$$cz^N \in \mathfrak{a}^{sN}, \tag{1.6}$$

for all N and some $c \in A^\circ$. One easily verifies that $\mathfrak{a}^{sN}$ is contained in $(a_1^N, \dots, a_s^N)A$. In particular, (1.6) with $N = p^n$ yields the tight closure relation (1.2), showing that $z \in \operatorname{cl}(I)$. Suppose next that A is an affine algebra over a field of characteristic zero with approximations A_w and Lefschetz hull $\mathsf{L}(A)$. Choose s-generated ideals $\mathfrak{a}_w \subseteq A_w$ whose ultraproduct equals $\mathfrak{a}\mathsf{L}(A)$ (see the discussion following Lemma 1.4.1), and choose $z_w \in A_w$ with ultraproduct equal to z viewed as an element in $\mathsf{L}(A)$. By the previous argument, we have a tight closure relation

$$c_w \operatorname{Frob}^n_{A_w}(z_w) \in \operatorname{Frob}^n_{A_w}(\mathfrak{a}_w)A_w$$

in each A_w, for some $c_w \in A_w^\circ$. Taking ultraproducts, we get a relation

$$c_\natural \operatorname{Frob}^n_\infty(z) \in \operatorname{Frob}^n_\infty(I)\mathsf{L}(A) \tag{1.7}$$

in $\mathsf{L}(A)$, where $c_\natural$ is the ultraproduct of the c_w. A priori, $c_\natural$ does not belong to the subring A of $\mathsf{L}(A)$, so that (1.7) is not a true tight closure relation. Nevertheless, in [68, Proposition 8.4], I show that there always exist test elements $c_w \in A_w$ such that their ultraproduct lies in A°. By a more careful bookkeeping, we can circumvent this complication altogether, at least when

[14] According to Wall [86], the question was originally posed by Mather.

$\mathfrak{a}$ has positive height, the only case of interest. Namely, an easy calculation (see [2, Theorem 6.13] or [76, Theorem 5.4.1]) yields the following variant of (1.6): for all N, we have an inclusion $\mathfrak{a}^{sd}z^N \subseteq \mathfrak{a}^{sN}$, where d is the degree of an integral equation exhibiting $z \in \overline{\mathfrak{a}^s}$. By Łos' Theorem, we may choose the z_w to satisfy an integral equation of the same degree, and hence in the ultraproduct we get a relation $\mathfrak{a}^{sd}\operatorname{Frob}^n_\infty(z) \subseteq \operatorname{Frob}^n_\infty(I)\mathsf{L}(A)$. Taking therefore any c in $\mathfrak{a}^{sd} \cap A^\circ$ yields a true tight closure relation, proving that $z \in \operatorname{cl}(\mathfrak{a})$. We will shortly define tight closure for Noetherian local rings containing $\mathbb{Q}$, and by Remark 1.3.3, we may then extend this to any Noetherian ring containing the rationals (see also [2, §6.17]); the previous argument is still applicable, thus completing the proof of the last assertion.

From this the validity of the second property in equal characteristic follows immediately,[15] since any ideal is tightly closed in a regular ring. To obtain the first property, a nice little exercise on the chain rule—which, incidentally, requires us to be in characteristic zero—and using that an element lies in the integral closure of an ideal if and only if it lies in the extension of the ideal under any $\mathbb{C}$-algebra homomorphism $\mathbb{C}[[x]] \to \mathbb{C}[[t]]$ for t a single variable (see [69, Fact 5.1]), shows that f lies in the integral closure of its Jacobian ideal. The original Briançon-Skoda theorem then follows from the second property applied to the regular local ring $\mathbb{C}[[x]]$ with $\mathfrak{a}$ equal to the Jacobian ideal of f (see [69] for a different argument deducing the characteristic zero case from the characteristic p case using ultraproducts). $\square$

Local case

To extend tight closure to an arbitrary Noetherian local ring containing $\mathbb{Q}$, we need to construct a Lefschetz hull for $A := K[[x]]$, where K is an algebraically closed Lefschetz field, given as the ultraproduct of algebraically closed fields K_w of prime characteristic. In analogy with the affine case, we expect the Lefschetz hull to be the ultraproduct $A_\natural$ of the power series rings $A_w := K_w[[x]]$. However, there is no immediate $K[x]$-algebra embedding of A into $A_\natural$. The very existence of such an embedding, in fact, has non-trivial ramifications, as we shall see.

Proposition 1.4.6. *There exists an ultraproduct $\mathsf{L}(A)$ of the power series rings A_w and a faithfully flat $K[x]$-algebra embedding of A into $\mathsf{L}(A)$.*

Proof. Once we have defined a $K[x]$-algebra homomorphism $A \to \mathsf{L}(A)$, its flatness follows by the same argument as in the proof of Proposition 1.4.2. To prove the existence of $A \to \mathsf{L}(A)$, we use Proposition 1.4.7 below, with $Z := K[x]$ and $B = A_\natural$. To apply this result, let $f_1(y) = \cdots = f_s(y) = 0$ be a system of equations in the unknowns y with coefficients in Z. Given a solution $\mathbf{y}$ in $A = K[[x]]$, we need to construct a solution $\mathbf{z}$ in $A_\natural$. By Artin

[15] So far, no tight closure proof in mixed characteristic exists.

Approximation ([1, Theorem 1.10]), there exists already a solution $\tilde{\mathbf{y}}$ in the Henselization $Z^\sim$ of Z. Recall that the Henselization of Z (at the maximal ideal $(x_1, \dots, x_n)Z$) is the smallest Henselian subring $Z^\sim$ of A containing Z (see for instance [56]), and is equal to the ring of algebraic power series over K. Since the A_w are complete, they are in particular Henselian, and hence, by Łos' Theorem, so is their ultraproduct $A_\natural$. By the universal property of Henselization, we have a (unique) Z-algebra homomorphism $Z^\sim \to A_\natural$. The image of $\tilde{\mathbf{y}}$ under this homomorphism is then the desired solution in $A_\natural$. By Proposition 1.4.7, there is therefore a Z-algebra homomorphism from A to some ultrapower $\mathsf{L}(A)$ of $A_\natural$. Since an ultraproduct of ultraproducts is itself an ultraproduct, $\mathsf{L}(A)$ is a Lefschetz ring. □

In the above proof, we used the following result, which originates with Henkin [24], and has proven to be useful in other situations related to Artin Approximation; for instance, see [5, Lemma 1.4] and [84, Lemma 12.1.3].

Proposition 1.4.7 ([2, Corollary 2.5] or [76, Theorem 7.1.1]). *For a Noetherian ring Z, and Z-algebras A and B, the following are equivalent:*

1. *every system of polynomial equations with coefficients from Z which is solvable in A, is solvable in B;*
2. *there exists a Z-algebra homomorphism $A \to B_\natural$, where $B_\natural$ is some ultrapower of B.* □

Proposition 1.4.6, which was proven using Artin Approximation, in turn implies the following stronger form of Artin Approximation.

Theorem 1.4.8 (Uniform strong Artin Approximation, [4, Theorem 4.3]). *There exists a function $N\colon \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let K be a field, put $Z := K[x]$ with x an n-tuple of indeterminates, and let $\mathfrak{m}$ be the ideal generated by these indeterminates. Let $f_1(y) = \dots = f_s(y) = 0$ be a polynomial system of equations in the n unknowns y with coefficients from Z, such that each f_i has total degree at most d (in x and y). If there exists some $\mathbf{y}$ in Z such that $f_i(\mathbf{y}) \equiv 0 \mod \mathfrak{m}^{N(n,d)}Z$ for all i, then there exists $\mathbf{z}$ in $K[[x]]$ such that $f_i(\mathbf{z}) = 0$ for all i.*

Proof. Towards a contradiction, assume such a bound does not exist for the pair (d, n), so that for each $w \in \mathbb{N}$ we can find a counterexample consisting of a field K_w, and polynomial equations $f_{1w}(y) = \dots = f_{sw}(y) = 0$ in the unknowns y over $Z_w := K_w[x]$ of total degree at most d, admitting an approximate solution $\mathbf{x}_w$ in Z_w modulo $\mathfrak{m}^w Z_w$ but no actual solution in $A_w := K_w[[x]]$. Note that the size, s, of these systems can be bounded in terms of d and n only (see for instance [76, Lemma 4.4.2]), and hence, in particular, can be taken independent from w. Let K and $A_\natural$ be the ultraproduct of the K_w and A_w respectively, and let f_i and $\mathbf{x}$ be the ultraproduct of the f_{iw} and $\mathbf{x}_w$ respectively. Since ultraproducts commute with finite sums, each f_i is again a polynomial over K of total degree at most d. Moreover, by Łos'

Theorem, $f_i(\mathbf{x}) \equiv 0 \mod \mathfrak{m}^N R_\natural$ for all N. By Proposition 1.4.9 below, we have an epimorphism $A_\natural \to A := K[[x]]$ having kernel equal to the intersection of all $\mathfrak{m}^N A_\natural$. In particular, the image of $\mathbf{x}$ under this surjection is a solution in A of the system $f_1 = \dots = f_s = 0$.

Since there exists a Z-algebra homomorphism $A \to L(A)$ by Proposition 1.4.6, where $L(A)$ is some ultrapower of $A_\natural$ (note that nowhere in the proof we used that the fields were algebraically closed nor that they had a certain characteristic), the image of $\mathbf{x}$ in $L(A)$ remains a solution of this system, and hence by Łos' Theorem, we can find, contrary to our assumptions, for almost each w (with respect to the larger ultrafilter defining $L(A)$), a solution of $f_{1w}(y) = \dots = f_{sw}(y) = 0$ in A_w. □

Proposition 1.4.9. *There is a canonical epimorphism $A_\natural \to A$ whose kernel is the* ideal of infinitesimals $\mathfrak{I}_{A_\natural} := \bigcap_N \mathfrak{m}^N A_\natural$.

Proof. Given $f \in A_\natural$, choose $f_w \in A_w$ whose ultraproduct is equal to f, and expand as a power series

$$f_w = \sum_{\nu \in \mathbb{N}^n} a_{\nu,w} x^\nu$$

for some $a_{\nu,w} \in K_w$. For each ν, let $a_\nu \in K$ be the ultraproduct of the $a_{\nu,w}$ and define

$$\tilde{f} := \sum_{\nu \in \mathbb{N}^n} a_\nu x^\nu \in A.$$

One checks that the map $f \mapsto \tilde{f}$ is well-defined (that is to say, independent of the choice of the f_w), and is a ring homomorphism, which is surjective, with kernel equal to the ideal of infinitesimals (see [76, Proposition 7.1.7]). □

In §1.6.2 below, we will rephrase this as *A is the cataproduct $A_\sharp$ of the A_w*. For some other uniform versions of Artin Approximation proven using ultraproducts, see [12, 13]. Returning to the issue of defining a Lefschetz hull, whence a tight closure operation, on the category of Noetherian local rings containing $\mathbb{Q}$, there is, however, a catch. Let $\tilde{x}$ be a subtuple of x. Put $\tilde{A} := K[[\tilde{x}]]$, and let $\tilde{A}_\natural$ be the ultraproduct of the $\tilde{A}_w := K_w[[\tilde{x}]]$. In the polynomial case, the inclusion $K[\tilde{x}] \subseteq K[x]$ extends to a homomorphism of Lefschetz rings $\tilde{B}_\natural \to B_\natural$, where $\tilde{B}_\natural$ and $B_\natural$ are the respective ultraproducts of the $K_w[\tilde{x}]$ and $K_w[x]$, making the whole construction functorial. However, it is no longer true that the inclusion $\tilde{A} \subseteq A$ leads to a similar homomorphism $\tilde{A}_\natural \to A_\natural$. In fact, in [2, §4.33] we give a counterexample based on an observation of Roberts in [60]—which itself was intended as a counterexample to an attempt of Hochster [30] to generalize tight closure via the notion of solid closure; see footnote 21. Nonetheless, such a homomorphism does exist if $\tilde{x}$ is an initial tuple of x. To prove this, and thus salvage the functoriality of the construction, we have to prove a filtered version of Proposition 1.4.7. To apply this filtered version, however, a deeper Artin Approximation result, due to Rotthaus [62], is needed. In turn, we derive a filtered version of Theorem 1.4.8.

All this needs some work and is explained in full detail in [2]; for a weaker form of functoriality, still sufficient for applications, see [76, §7.3].

In sum, we have now a tight closure operation on any equicharacteristic Noetherian local ring, and by Remark 1.3.3, even on any Noetherian ring containing a field. It has the five properties listed in Theorem 1.2.10. So, in equal characteristic zero, there are many, potentially different notions: the tight closure $\mathrm{cl}^{\mathsf{HH}}(\cdot)$ (and its variants) introduced by Hochster and Huneke, our notion $\mathrm{cl}(\cdot)$ (which a priori depends on the choice of ultrafilter), and some variants that I will now discuss briefly. Let A be either an affine algebra or a local ring, with Lefschetz hull $\mathsf{L}(A)$, realized as the ultraproduct of positive characteristic rings A_w, called *approximations* of A. Given an ideal $I \subseteq A$, choose $I_w \subseteq A_w$ with ultraproduct equal to $I\mathsf{L}(A)$.

Definition 1.4.10 (Ultra-tight closure). We define the *ultra-tight closure*[16] of I as the ideal $\text{ultra-cl}(I) := J_\natural \cap A$, where $J_\natural \subseteq \mathsf{L}(A)$ is the ultraproduct of the $J_w := \mathrm{cl}_{A_w}(I_w)$.

In other words, $z \in \text{ultra-cl}(I)$, if almost each z_w belongs to the tight closure of I_w, for some choice of $z_w \in A_w$ with ultraproduct equal to z. We have the following comparison between these three notions

$$\mathrm{cl}^{\mathsf{HH}}(I) \subseteq \text{ultra-cl}(I) \subseteq \mathrm{cl}(I) \tag{1.8}$$

where the latter inclusion holds under some mild conditions; see [68, Theorems 8.5 and 10.4] for the affine, and [2, Corollaries 6.23 and 6.26] for the local case.

Another variant is derived from the observation that a single power of the ultra-Frobenius has all the contractive power, in the sense of Definition 1.3.1(3), needed to prove that regular rings are F-regular. So we may define the *simple tight closure* of an ideal I as the collection of all elements $z \in A$ such that there exists $c \in A^\circ$ for which $c\,\mathrm{Frob}_\infty(z) \in \mathrm{Frob}_\infty(I)\mathsf{L}(A)$. Clearly, simple tight closure contains tight closure. All these variants satisfy the five main properties listed in Theorem 1.2.10, except that I do not know whether simple tight closure admits strong persistence. To define a last variant, we turn again to big Cohen-Macaulay algebras.

Big Cohen-Macaulay algebras

Hochster and Huneke [38] also construct (balanced) big Cohen-Macaulay algebras in equal characteristic zero, but their construction is highly non-canonical.[17] Using ultraproducts, we can restore canonicity modulo the choice of ultrafilter (for the affine case, see [71]).

[16] Elsewhere, I called this *generic* tight closure.

[17] The naive guess that absolute integral closure also yields big Cohen-Macaulay algebras in characteristic zero is manifestly false; see footnote 13.

Theorem 1.4.11. *Any equal characteristic Noetherian local ring A admits a balanced big Cohen-Macaulay algebra $\mathsf{B}(A)$.*

Proof. By the same argument as in Theorem 1.2.3, we may reduce to the case that A is a complete d-dimensional Noetherian local domain with algebraically closed residue field of characteristic zero. The local analogue of Theorem 1.4.3 gives the same transfer results between the ring A and its approximations A_w (see [2, §5]). In particular, by the same argument as in the discussion after Theorem 1.4.3, almost each approximation A_w is a domain. Hence, almost each absolute integral closure A_w^+ is a balanced big Cohen-Macaulay A_w-algebra (see Remark 1.2.4). Let $\mathsf{B}(A)$ be the ultraproduct of the A_w^+, so that we have a canonical homomorphism $\mathsf{L}(A) \to \mathsf{B}(A)$. Hence $\mathsf{B}(A)$ is an A-algebra via the composition $A \to \mathsf{L}(A) \to \mathsf{B}(A)$. Let $\mathbf{x}$ be a system of parameters in A, and choose tuples $\mathbf{x}_w$ over A_w with ultraproduct equal to $\mathbf{x}$ (as a tuple in $\mathsf{L}(A)$). Since $\mathsf{L}(A)/\mathbf{x}\mathsf{L}(A)$ is the Lefschetz hull of $A/\mathbf{x}A$, almost all $A_w/\mathbf{x}_wA_w$ have, like $A/\mathbf{x}A$, dimension zero. Since almost each A_w has dimension d, almost each $\mathbf{x}_w$ is therefore a system of parameters in A_w, whence A_w^+-regular by the proof of Theorem 1.2.3 (see Remark 1.2.4). By Łos' Theorem, their ultraproduct $\mathbf{x}$ is then $\mathsf{B}(A)$-regular. □

This proves the characteristic zero version of Theorem 1.2.7, as well as all the other Homological Conjectures that follow from the existence of big Cohen-Macaulay modules (see [27] for an exhaustive list). The big Cohen-Macaulay algebra construction is even weakly functorial (for details see [2, §7]). Unlike the Hochster-Huneke construction, we can preserve some of the good properties of the absolute integral closure from positive characteristic: $\mathsf{B}(A)$ is absolutely integrally closed (though not integral over A!). In particular, the sum of prime ideals is either the unit ideal or again prime (see [71, Proposition 3.2 and Corollary 3.3]).

One can also define a closure operation using the big Cohen-Macaulay algebra $\mathsf{B}(A)$ by taking for closure of the ideal $I \subseteq A$, the ideal $\mathrm{cl}^{\mathsf{B}}(I) := I\mathsf{B}(A) \cap A$. It satisfies the five main properties of tight closure ([71, Theorem 4.2] and [2, Theorem 7.14]), and it even commutes with localization in certain cases ([71, Theorems 4.3 and 5.2]). If A is a complete local domain, then $\mathrm{cl}^{\mathsf{B}}(I) \subseteq \text{ultra-cl}(I)$, with equality, by [78], if I is a parameter ideal.

1.5 Rational Singularities

So far in our discussion, the tight closure theory in characteristic zero via ultraproducts has not brought anything new to the table: it merely gives an alternative, more streamlined, theory than the Hochster-Huneke constructions, but anything proven in our theory can also be proven with theirs. However, this is no longer true when it comes to F-singularities (the definitions in 1.2.13 extend verbatim to characteristic zero). Since the Hochster-Huneke tight closure is contained in ours, to be F-regular or F-rational in their theory is a priori weaker than in ours. For instance, it is not known whether F-rational in their sense implies rational singularities, but it does for our notion. They also introduced the notions of *F-regular type* and *F-rational type*, which do characterize the corresponding singularity notions given in Table (1.1), but these notions do not (a priori) behave well enough with respect to quotients.[18]

Definition 1.5.1 (Rational Singularities). An equicharacteristic zero excellent local domain R is said to have *rational singularities* if it is normal, analytically unramified, and Cohen-Macaulay, and the canonical embedding

$$H^0(W,\omega_W) \to H^0(X,\omega_X) \tag{1.9}$$

is surjective (it is always injective), where $W \to X := \operatorname{Spec} R$ is a resolution of singularities, and where in general, ω_Y denotes the canonical sheaf on a scheme Y. To make the definition in the absence of a resolution of singularities, one calls $(R,\mathfrak{m})$ (in either characteristic) *pseudo-rational* if the canonical map

$$\delta_W\colon H^d_{\mathfrak{m}}(R) \to H^d_E(\mathcal{O}_W) \tag{1.10}$$

is injective (it always is surjective), for all proper birational maps $\pi\colon W \to X$ with W normal, where d is the dimension of R and $E = \pi^{-1}(\mathfrak{m})$ the closed fiber of π.

Note that in (1.9) we take sheaf cohomology, whereas in (1.10) we take cohomology with support, which in the local case amounts to local cohomology. By [52, §2, Remark (a) and Example (b)], if δ_W in (1.10) is injective for some non-singular W, then R is pseudo-rational, and, in fact, has rational singularities. By Matlis duality, therefore, if R is essentially of finite type over a field of characteristic zero then R has rational singularities if and only if it is pseudo-rational.

The key to study rational singularities using tight closure theory is the following result due to Smith:

Theorem 1.5.2 ([79]). *A d-dimensional excellent local ring $(R,\mathfrak{m})$ of characteristic p is F-rational if and only if $H^d_{\mathfrak{m}}(R)$ admits no non-trivial submodule closed under the action of Frobenius.* □

Note that the top local cohomology $H^d_{\mathfrak{m}}(R)$ is the cokernel of the final map in the Čech complex $R_{y_1} \oplus \cdots \oplus R_{y_d} \to R_y$ where $y = x_1x_2\cdots x_d$ and $y_i = y/x_i$, for $(x_1,\dots,x_d)$ a system of parameters in R. In particular, the Frobenius acts on these localizations, whence on the top local cohomology. To formulate the analogue in characteristic zero, we define the *ultra-local cohomology* $\mathrm{UH}^\bullet_{\mathfrak{m}}(R)$ of R as the ultraproduct of the local cohomology of its

[18] In [72], I prove that in the affine case, they are actually equivalent with the notions in this paper, and hence admit the desired properties; this, however, relies on a deeper result due to Hara [21].

approximations. To describe this as the cohomology of a complex over R, we need the notion of a *relative Lefschetz hull* $\mathsf{L}_R(S)$, for S a finitely generated R-algebra (we do not need this in case R itself is essentially of finite type over a field). The construction is a relative version of the affine Lefschetz hull. More precisely, it suffices to make the construction for a polynomial ring $R[x]$, as follows: let R_w be approximations of R, and define $\mathsf{L}_R(R[x])$ as the ultraproduct of the $R_w[x]$. For S arbitrary, say, of the form $R[x]/I$, we put $\mathsf{L}_R(S) := \mathsf{L}_R(R[x])/I\mathsf{L}_R(R[x])$. The natural maps $R[x] \to \mathsf{L}(R)[x] \to \mathsf{L}_R(R[x])$, induce by base change a homomorphism $S \to \mathsf{L}_R(S)$. By the same argument as in the affine case, $\mathsf{L}_R(\cdot)$ is a difference hull on the category of finitely generated R-algebras ([76, Proposition 7.4.3]). We can now also realize $\mathrm{UH}^d_{\mathfrak{m}}(R)$ as the cokernel of

$$\mathsf{L}_R(R_{y_1}) \oplus \cdots \oplus \mathsf{L}_R(R_{y_d}) \to \mathsf{L}_R(R_y).$$

In particular, there is a natural morphism $H^d_{\mathfrak{m}}(R) \to \mathrm{UH}^d_{\mathfrak{m}}(R)$.

Since the ultra-Frobenius acts on relative hulls, it also acts on $\mathrm{UH}^d_{\mathfrak{m}}(R)$. The analogue of Theorem 1.5.2 in characteristic zero is then that R is *ultra-F-rational*, meaning that some parameter ideal is equal to its ultra-closure, if and only if $\mathrm{UH}^d_{\mathfrak{m}}(R)$ has no non-trivial submodule closed under the action of the ultra-Frobenius. From this characterization, we get:

Theorem 1.5.3. *If an equicharacteristic excellent local ring is F-rational, then it is pseudo-rational.*

Proof. Let R be an equicharacteristic excellent F-rational local ring. By Remark 1.2.14(2), and its characteristic zero analogue, R is Cohen-Macaulay and normal. Since R is excellent, it is therefore also analytically unramified. Moreover, R is ultra-F-rational by (1.8). Let $\pi\colon W \to \operatorname{Spec} R$ be a proper birational map with W normal. By functoriality, the kernel of the surjection δ_W in (1.10) is invariant under the action of Frobenius, whence has to be trivial by Theorem 1.5.2 and our previous discussion. ◻

Smith[19] [79] proved Theorem 1.5.3 in characteristic p and Hara [21] has proven its converse; in the affine case, I proved that having rational singularities is equivalent with being ultra-F-rational ([71, Theorem 5.11]); I do not know whether this is also true in general, nor do I know whether ultra-F-rational and F-rational are equivalent. From the discussion at the end of the previous section it follows that R is ultra-F-rational if and only if $I = \mathrm{cl}^{\mathsf{B}}(I)$, for some parameter ideal $I \subseteq R$.

Soon after Hochster and Roberts proved Theorem 1.4.4, Boutot, using some deep vanishing theorems, improved this by showing that quotient singularities have rational singularities. This is just a special case of the following more general result:

---

[19] Much of the work discussed in this section is based on Karen Smith's ideas, and grew out from some stimulating conversations I had with her.

Theorem 1.5.4 ([75, Main Theorem A]). *Let $A \to B$ be a cyclically pure homomorphism of Noetherian rings containing $\mathbb{Q}$. If B is regular, then A is pseudo-rational.*

Proof. As in the proof of Theorem 1.4.4, we may reduce to the local case. Remark 1.2.14(1) yields that A is F-rational, whence pseudo-rational by Theorem 1.5.3. □

At present, no proof of this using Hochster-Huneke tight closure is known. Boutot actually proves a stronger version in the affine case, in which he only assumes that B itself has rational singularities. I gave a tight closure proof of this more general result under the additional assumption that B is Gorenstein ([71, §5.14]), but I do not know whether this also holds in the general case.

We already observed that a cyclically pure subring of a regular ring is in fact F-regular, which is stronger than being F-rational. According to Table (1.1), we expect quotient singularities therefore to be actually log-terminal. This was proven for a quotient modulo a finite group by Kawamata [47]. I will now discuss an extension of this result.

Definition 1.5.5 ($\mathbb{Q}$-Gorenstein Singularities). Let R be an equicharacteristic zero Noetherian local domain and put $X := \operatorname{Spec} R$. We say that R is $\mathbb{Q}$-*Gorenstein* if it is normal and some positive multiple of the canonical divisor K_X is Cartier; the least such positive multiple is called the *index* of R. If R is the homomorphic image of an excellent regular local ring (which is for instance the case if R is complete), then X admits an *embedded resolution of singularities* $f\colon Y \to X$ by [25]. If E_i are the irreducible components of the exceptional locus of f, then the canonical divisor K_Y is numerically equivalent to $f^*(K_X)+\sum a_iE_i$ (as $\mathbb{Q}$-divisors), for some $a_i \in \mathbb{Q}$. The rational number a_i is called the *discrepancy* of X along E_i; see [48, Definition 2.22]. If all $a_i > -1$, we call R *log-terminal* (in case we only have a weak inequality, we call R *log-canonical*).

If r is the index of the $\mathbb{Q}$-Gorenstein ring R, then $\mathcal{O}_X(rK_X) \cong \mathcal{O}_X$, where $X := \operatorname{Spec} R$ and K_X the canonical divisor of X. This isomorphism induces an R-algebra structure on

$$R^* := H^0(X, \mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \cdots \oplus \mathcal{O}_X((r-1)K_X)),$$

which is called the *canonical cover* of R; see [47]. To relate log-terminal singularities to F-singularities, we use the following characterization:

Proposition 1.5.6 ([47, Proposition 1.7]). *Let R be a homomorphic image of an excellent regular local ring (e.g., R is complete). If R has equal characteristic zero and is $\mathbb{Q}$-Gorenstein, then it has log-terminal singularities if and only if its canonical cover is rational.* □

Unfortunately, we cannot use this characterization directly to conclude that a cyclically pure local subring R of a regular ring S has log-terminal singularities. For starters, we do not know whether the assumptions imply that R is $\mathbb{Q}$-Gorenstein. We resolve this by simply adding this as an additional assumption on R. Now, by our previous discussion, R is F-regular—in fact, it is easy to show that is also *ultra-F-regular* in the sense that any ideal is equal to its ultra-closure in any localization of R. However, in order to show that R has log-terminal singularities, we would like to invoke Proposition 1.5.6, and so it would suffice to show that its canonical cover R^* is (ultra-)F-regular too, whence has rational singularities by Theorem 1.5.3. We know that F-regularity is preserved under étale extensions,[20] but the canonical cover $R \to R^*$ is only étale in codimension one (see for instance [80, 4.12]). It was Smith's brilliant observation that a strengthening of the F-regularity condition, however, is preserved under this type of maps.

Strong F-regularity

Let R be an equicharacteristic excellent normal local ring. If R has characteristic p, then we call it *strongly F-regular*, if for any non-zero c, there exists an $n := n(c)$ with the property that for any element $z \in R$ and any ideal $I \subseteq R$, if $c\,\mathrm{Frob}^n_R(z) \in \mathrm{Frob}^n_R(I)R$, then $z \in I$. In other words, for each given c, a single tight closure equation (1.2) of a sufficiently high power implies already ideal membership. In particular, a strongly F-regular ring is F-regular. The converse is conjectured to hold, but is currently only known in the graded case [53]. If this condition holds just for $c = 1$, then we call R *strongly F-pure*.

To make the definition in characteristic zero, we must allow non-standard powers of the ultra-Frobenius: let α be a positive element in $\mathbb{Z}_\natural$, the ultrapower of $\mathbb{Z}$ (in other words, α is an ultraproduct of positive integers α_w). We define $\mathrm{Frob}^\alpha_\infty$ as the the homomorphism $R \to \mathsf{L}(R)$ sending $x \in R$ to the ultraproduct of the $\mathrm{Frob}^{\alpha_w}_{R_w}(x_w)$, where R_w are approximations of R and $x_w \in R_w$ with ultraproduct equal to x (viewed as an element in $\mathsf{L}(R)$). One checks that this yields a well-defined homomorphism. We can now define similarly what it means for R to be *strongly F-regular* (respectively, *strongly F-pure*) in characteristic zero: for every non-zero c (for $c = 1$), there exists some positive $\alpha := \alpha(c) \in \mathbb{Z}_\natural$ with the property that for any element $z \in R$ and any ideal $I \subseteq R$, if $c\,\mathrm{Frob}^\alpha_\infty(z) \in \mathrm{Frob}^\alpha_\infty(I)\mathsf{L}(R)$, then $z \in I$. We have:

Proposition 1.5.7 ([80, Theorem 4.15] and [75, Proposition 7.8]). *Let $R \subseteq S$ be a finite extension of equicharacteristic excellent normal local rings which is étale in codimension one. If R is strongly F-regular, then so is S.* □

Theorem 1.5.8 ([75, Main Theorem B]). *Let $R \to S$ be a cyclically pure homomorphism of equicharacteristic zero excellent local rings with S regular*

[20] A finite extension $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ of equal characteristic zero Noetherian local rings is *étale* if it is flat, and *unramified*, meaning that $\mathfrak{m}S = \mathfrak{n}$.

and R a homomorphic image of a regular local ring. If R is $\mathbb{Q}$-Gorenstein, then it is log-terminal.

Proof. The regular local ring S is strongly F-regular since its ultra-Frobenius is flat (use the argument in the proof of Theorem 1.2.10). Moreover, it is easy to check that R, being a cyclically pure subring, is then also strongly F-regular. Therefore, its canonical cover R^* is strongly F-regular by Proposition 1.5.7. In particular, R^* is F-rational whence has rational singularities by Theorem 1.5.3. Proposition 1.5.6 implies then that R is log-terminal. □

Kawamata-Viehweg vanishing

As a final application of our methods, I discuss some vanishing theorems and a conjecture of Smith on quotients of Fano varieties. Let X be a connected projective scheme of finite type over $\mathbb{C}$; a *projective variety*, for short. Choose an ample line bundle $\mathcal{P}$ on X, and let S be *section ring* of the pair $(X, \mathcal{P})$, defined as the graded ring

$$S := \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{P}^n). \tag{1.11}$$

The section ring is a finitely generated, positively graded $\mathbb{C}$-algebra, which encodes both the projective variety, to wit, $X = \operatorname{Proj}(S)$, as well as the ample line bundle, to wit, $\mathcal{P} \cong \widetilde{S(1)}$. The *vertex* of S is the localization of S at its irrelevant maximal ideal $S_{>0}$. By a *vertex* of X, we then mean the vertex of some section ring associated to some ample line bundle. It is common wisdom that (global) properties of the projective variety are often already captured by the (local) properties of one of its vertices. Following Smith, we define:

Definition 1.5.9. Let X be a projective variety. We call X *globally F-regular* if it has some strongly F-regular vertex. We call X *globally F-pure* if it has some strongly F-pure vertex.

If R is globally F-regular (respectively, globally F-pure), then any vertex is strongly F-regular (respectively strongly F-pure); see [81, Theorem 3.10] or [72, Remark 6.3]. In particular, the vertex is Cohen-Macaulay, whence so is X. The main technical result, inspired by Smith's work, is:

Theorem 1.5.10 ([72, Theorem 6.5 and Remark 6.6]). *Let X be a projective variety over $\mathbb{C}$, let $i > 0$, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Each of the following two conditions implies the vanishing of $H^i(X, \mathcal{L})$:*

1. *X is globally F-pure and $H^i(X, \mathcal{L}^n) = 0$ for all $n \gg 0$;*
2. *X is globally F-regular and for some effective Cartier divisor D, all $H^i(X, \mathcal{L}^n(D)) = 0$ for $n \gg 0$.* □

Using this we can now derive the following vanishing theorems:

Theorem 1.5.11. *Let X be a projective variety over $\mathbb{C}$ and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{L}^{-1})$ vanishes for all $i < \dim X$, in the following two cases:*

(Kodaira Vanishing) *X is globally F-pure and Cohen-Macaulay, and $\mathcal{L}$ is ample;*

(Kawamata-Viehweg Vanishing) *X is globally F-regular, and $\mathcal{L}$ is big and numerically effective.*

Proof. To prove the first vanishing theorem, observe that by Serre duality ([22, III. Corollary 7.7]), the dual of $H^i(X, \mathcal{L}^{-n})$ is $H^{d-i}(X, \omega_X \otimes \mathcal{L}^n)$ where d is the dimension of X and ω_X the canonical sheaf on X. Because $\mathcal{L}$ is ample, the latter cohomology group vanishes for large n, and hence so does the first. The conclusion then follows from Theorem 1.5.10(1).

To prove the second vanishing theorem, fix some $i < d$. Because $\mathcal{L}$ is big and numerically effective, we can find an effective Cartier divisor D such that $\mathcal{L}^m(-D)$ is ample for all $m \gg 0$, by [48, Proposition 2.61]. Hence

$$H^i(X, (\mathcal{L}^{-m}(D))^n) = H^i(X, (\mathcal{L}^m(-D))^{-n}) = 0$$

for all sufficiently large m and n, where the vanishing follows from Serre duality and the fact that $\mathcal{L}^m(-D)$ is ample. Hence, for fixed m, Theorem 1.5.10(1) yields the vanishing of $H^i(X, \mathcal{L}^{-m}(D))$. Since this holds for all large m, Theorem 1.5.10(2) then gives $H^i(X, \mathcal{L}^{-1}) = 0$. □

Recall that a normal projective variety X is called *Fano*, if its anticanonical sheaf ω_X^{-1} is ample. The following was conjectured by Smith:

Theorem 1.5.12. *Any quotient of a smooth Fano variety by a reductive group (in the sense of Geometric Invariant Theory) admits Kawamata-Viehweg Vanishing.*

Proof. The key fact is that a Fano variety X is globally F-regular ([72, Theorem 7.1]). This rests on some deep result due to Hara [21], which itself was proven using Kodaira vanishing. Assuming this fact, let S be a section ring with strongly F-regular vertex. If G is a reductive group acting algebraically on X, then S^G is a section ring of the GIT quotient $X//G$. Since $S^G \subseteq S$ is split (see the proof of Theorem 1.4.4), whence cyclically pure, the vertex of S^G is strongly F-regular too, showing that $X//G$ is globally F-regular. The result now follows from Theorem 1.5.11. □

Remark 1.5.13. Because of the analogy with the notion of *Frobenius split* (see [81, Proposition 3.1]) and the fact that a Schubert variety has this property [55, Theorem 2], it is reasonable to expect that a Schubert variety is globally F-pure. This is known in characteristic p by [50]. In particular, if this result on Schubert varieties also holds in characteristic zero, then we get Kodaira Vanishing for any GIT quotient of a Schubert variety.

1.6 Mixed characteristic

Tight closure and big Cohen-Macaulay modules are two powerful tools, as we showed above, but unfortunately, neither one is currently available in mixed characteristic. Hochster [30, 31] has made some attempts to define a closure operation akin to tight closure in mixed characteristic, called *solid closure*,[21] and some amendments have been made by Brenner [6], but as of yet, the theory is not powerful enough to derive any new result in mixed characteristic. Some of the homological theorems are now known in mixed characteristic due to work of Roberts [59, 61], but the preferred method, through big Cohen-Macaulay modules is only available in dimension at most three by work of Heitmann [23] and Hochster [32].[22]

Neither has the ultraproduct method being able to prove any of the outstanding problems in mixed characteristic. At best, we obtain *asymptotic* results, meaning that a certain property holds if the residual characteristic of the ring is large with respect to other invariants associated to the problem. There are essentially two approaches, which I will now sketch briefly. From this, the inherent asymptotic nature of the results should then also become clear.

1.6.1 Protoproducts

In a first approach, we use a mixed characteristic analogue of Proposition 1.3.8, the celebrated Ax-Kochen-Ershov Principle [3, 18, 19]: for each w, let $\mathfrak{O}_w^{\mathrm{mix}}$ be a complete discrete valuation ring of mixed characteristic with residue field K_w of characteristic p_w. To each $\mathfrak{O}_w^{\mathrm{mix}}$, we associate a corresponding equicharacteristic complete discrete valuation ring with the same residue field, by letting $\mathfrak{O}_w^{\mathrm{eq}} := K_w[[t]]$, where t is a single indeterminate.

Theorem 1.6.1 (Ax-Kochen-Ershov). *If the residual characteristics p_w are unbounded, then the ultraproduct of the $\mathfrak{O}_w^{eq}$ is isomorphic (as a local ring) with the ultraproduct of the $\mathfrak{O}_w^{mix}$.* □

Let $\mathfrak{O}_\natural$ be this common ultraproduct. It is an equal characteristic zero (non-discrete) valuation ring with principal maximal ideal, such that $\mathfrak{O}_\natural/\mathfrak{I}_{\mathfrak{O}_\natural}$ is a complete discrete valuation ring. Fix a tuple of indeterminates x. Let $A_w^{\mathrm{eq}} := \mathfrak{O}_w^{\mathrm{eq}}[x]$, and let $A_\natural^{\mathrm{eq}}$ be their ultraproduct. Since $\mathfrak{O}_\natural \subseteq A_\natural^{\mathrm{eq}}$ and the x are algebraically independent in $A_\natural^{\mathrm{eq}}$, we have an inclusion $\mathfrak{O}_\natural[x] \subseteq A_\natural^{\mathrm{eq}}$. We call $\mathfrak{O}_\natural[x]$ the *protoproduct* of the A_w^{eq}; it is the subring of all ultraproducts $f_\natural$

[21] Solid closure also intended to provide an alternative approach in characteristic zero, avoiding any reference to reductions modulo p. A counterexample due to Roberts [60], however, has seriously undermined this approach.

[22] Some earlier attempts that alas led nowhere were made by Hochster in [26].

of elements $f_w \in A_w^{\mathrm{eq}}$ having bounded degree.[23] The inclusion $\mathfrak{O}_\natural[x] \subseteq A_\natural^{\mathrm{eq}}$ is almost a difference hull as in Definition 1.3.1, except that it is flat, but not faithfully flat ([73, Theorem 4.2]). Using instead the mixed characteristic discrete valuation rings, we get $A_w^{\mathrm{mix}} := \mathfrak{O}_w^{\mathrm{mix}}[x]$, whose ultraproduct $A_\natural^{\mathrm{mix}}$ contains $\mathfrak{O}_\natural[x]$ as a flat subring. A note of caution: the Ax-Kochen-Ershov principle is false in higher dimensions: $A_\natural^{\mathrm{eq}}$ and $A_\natural^{\mathrm{mix}}$ are no longer isomorphic. Therefore, the transfer between the A_w^{eq} and the A_w^{mix} is achieved via their common subring $\mathfrak{O}_\natural[x]$, and we may thus think of them as respectively the mixed and equal characteristic approximations of $\mathfrak{O}_\natural[x]$. That the latter is not Noetherian causes quite some headaches; for details, see [67, 73]. The main technical result is that any local ring R which is essentially of finite type over $\mathfrak{O}_\natural$ admits the analogue of a big Cohen-Macaulay algebra. This enables us to prove some non-Noetherian analogues of the homological conjectures over R, which then descend to its mixed characteristic approximations. Since the transfer requires the degree to be bounded, we can only get an asymptotic version: the residual characteristic has to be sufficiently large with respect to the degrees of the data involved. For instance, we get:

Theorem 1.6.2 (Asymptotic Monomial Conjecture, [73, Corollary 9.5]). *For each N, we can find a bound $\mu(N)$ with the following property. Let $\mathfrak{O}$ be a mixed characteristic discrete valuation ring and let R be a finite extension of the localization $S := \mathfrak{O}[x]_{(x_1,\dots,x_d)\mathfrak{O}[x]}$. If R is defined by at most N polynomials of degree at most N over S, then the tuple $(x_1,\dots,x_d)$, viewed as a system of parameters in R, is monomial, provided the residual characteristic of $\mathfrak{O}$ is at least $\mu(N)$.* □

1.6.2 Cataproducts

In the second approach, rather than subrings, we look for nice residue rings. Let $(R_w, \mathfrak{m}_w)$ be Noetherian local rings and let $R_\natural$ be their ultraproduct. We already observed that $R_\natural$ is hardly ever Noetherian, and hence the usual methods from commutative algebra do not apply. Nonetheless, Proposition 1.4.9 and the property of the Ax-Kochen-Ershov ring $\mathfrak{O}_\natural$ are not isolated events; there is often a Noetherian residue ring lurking in the background:

Proposition 1.6.3. *If the R_w have bounded embedding dimension,*[24] *then their* cataproduct *$R_\sharp := R_\natural/\mathfrak{I}_{R_\natural}$ is a complete Noetherian local ring.*

Proof (Sketch; see [76, Theorem 8.1.4] or [77, Lemma 5.6]). By Łos' Theorem, $R_\natural$ has a finitely generated maximal ideal. By saturatedness of ultraproducts, every Cauchy sequence in $R_\natural$ has a limit. Hence the Haussdorfication

---

[23] We may similarly view an affine algebra over a field of characteristic zero as the protoproduct of its approximations; this is the point of view in [76, Chapter 9].

[24] The *embedding dimension* of a local ring is the minimal number of generators of its maximal ideal.

of $R_\natural$, that is to say, $R_\sharp$, is complete. Now, a complete local ring with finitely generated maximal ideal is Noetherian by [54, Theorem 29.4]. □

Moreover, the R_w share many properties with their cataproduct, and a transfer result, albeit weaker than Theorem 1.4.3, holds (see, for instance, [77, Corollaries 8.3 and 8.7, and Theorem 8.10]). Suppose the R_w have mixed characteristic. If their residual characteristics are unbounded, then by Łos' Theorem, their cataproduct has equal characteristic zero (since the ultraproduct of the residue fields of the R_w is the residue field of $R_\sharp$). In the remaining case, almost all R_w have residual characteristic p, for some p, and in that case $R_\sharp$ has characteristic p if the *ramification indices* of the R_w are unbounded, that is to say, if for all N, we have $p \in \mathfrak{m}_w^N$ for almost all w, whence $p \in \mathfrak{I}_{R_\natural}$. So in either case, we get an equicharacteristic cataproduct with a tight closure operation and a balanced big Cohen-Macaulay algebra. However, neither construction descends to the components R_w (intuitively, an ultraproduct can only transfer finitely many information). I conclude with an application of the method, and a discussion how this could potentially lead to the full conjecture.

Given a Noetherian local ring R, let $F_\bullet$ be a complex of length s consisting of finite free R-modules. We say that the *rank* of $F_\bullet$ is at most r, if each free R-module F_i in $F_\bullet$ has rank at most r; we say that $F_\bullet$ has *homological complexity* at most l, if each homology group $H_i(F_\bullet)$ for $i > 0$ has length at most l, and $H_0(F_\bullet)$ has a minimal generator generating a submodule of length at most l.

Theorem 1.6.4 (Asymptotic Improved New Intersection Theorem, [77, Theorem 13.6]). *For each triple of non-negative integers (m, r, l), there exists a bound $\nu(m, r, l)$ with the following property. Let R be a mixed characteristic Noetherian local ring of embedding dimension at most m. If $F_\bullet$ is a finite free complex of rank at most r and homological complexity at most l, then its length is at least the dimension of R, provided the residual characteristic or the ramification index of R is at least $\nu(m, r, l)$.* □

If we can show that the above bound grows slowly enough, then we can even deduce the full version from this. The idea is to reach a contradiction from a minimal counterexample by increasing its ramification, but controlling the growth of the other data. Without proof, I quote:

Theorem 1.6.5 ([77, Theorem 13.8]). *If for each fixed (m, r) the bound $\nu(m, r, l)$ from the previous theorem grows sub-linearly in l, in the sense that there exists some $0 \leq \alpha := \alpha_{m,r} < 1$ and $c > 0$ such that $\nu(m, r, l) \leq c \cdot l^\alpha$ for all l, then the Improved New Intersection Theorem holds.* □

Index

References

1. Artin, M.: Algebraic approximation of structures over complete local rings. Inst. Hautes Études Sci. Publ. Math. **36**, 23–58 (1969)
2. Aschenbrenner, M., Schoutens, H.: Lefschetz extensions, tight closure and big Cohen-Macaulay algebras. Israel J. Math. **161**, 221–310 (2007)
3. Ax, J., Kochen, S.: Diophantine problems over local fields I, II. Amer. J. Math. **87**, 605–630, 631–648 (1965)
4. Becker, J., Denef, J., van den Dries, L., Lipshitz, L.: Ultraproducts and approximation in local rings I. Invent. Math. **51**, 189–203 (1979)
5. Becker, J., Denef, J., Lipshitz, L.: The approximation property for some 5-dimensional Henselian rings. Trans. Amer. Math. Soc. **276**(1), 301–309 (1983)
6. Brenner, H.: How to rescue solid closure. J. Algebra **265**, 579–605 (2003)
7. Brenner, H., Katzman, M.: On the arithmetic of tight closure. J. Amer. Math. Soc. **19**(3), 659–672 (electronic) (2006)
8. Brenner, H., Monsky, P.: Tight closure does not commute with localization (2007). ArXiv:0710.2913
9. Briançon, J., Skoda, H.: Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de C^n. C. R. Acad. Sci. Paris **278**, 949–951 (1974)
10. Bruns, W., Herzog, J.: Cohen-Macaulay Rings. Cambridge University Press, Cambridge (1993)
11. Chang, C., Keisler, H.: Model theory. North-Holland Publishing Co., Amsterdam (1973)
12. Denef, J., Lipshitz, L.: Ultraproducts and approximation in local rings II. Math. Ann. **253**, 1–28 (1980)
13. Denef, J., Schoutens, H.: On the decidability of the existential theory of $\mathbb{F}_p[[t]]$. In: Valuation theory and its applications, Vol. II (Saskatoon, 1999), *Fields Inst. Commun.*, vol. 33, pp. 43–60. Amer. Math. Soc. (2003)
14. van den Dries, L.: Algorithms and bounds for polynomial rings. In: Logic Colloquium, pp. 147–157 (1979)
15. Ein, L., Lazarsfeld, R., Smith, K.: Uniform bounds and symbolic powers on smooth varieties. Invent. Math. **144**, 241–252 (2001)
16. Eisenbud, D.: Commutative Algebra with a View toward Algebraic Geometry, *Graduate Texts in Mathematics*, vol. 150. Springer-Verlag, New York (1995)
17. Eklof, P.: Ultraproducts for algebraists. In: Handbook of Mathematical Logic, pp. 105–137. North-Holland Publishing (1977)
18. Eršhov, Y.: On the elementary theory of maximal normed fields I. Algebra i Logica **4**, 31–69 (1965)
19. Eršhov, Y.: On the elementary theory of maximal normed fields II. Algebra i Logica **5**, 8–40 (1966)

20. Evans, E., Griffith, P.: The syzygy problem. Ann. of Math. **114**, 323–333 (1981)
21. Hara, N.: A characterization of rational singularities in terms of injectivity of Frobenius maps. Amer. J. Math. **120**, 981–996 (1998)
22. Hartshorne, R.: Algebraic Geometry. Springer-Verlag, New York (1977)
23. Heitmann, R.: The direct summand conjecture in dimension three. Ann. of Math. **156**, 695–712 (2002)
24. Henkin, L.: Some interconnections between modern algebra and mathematical logic. Trans. Amer. Math. Soc. **74**, 410–427 (1953)
25. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. Ann. of Math. **79**, 109–326 (1964)
26. Hochster, M.: Big Cohen-Macaulay modules and algebras and embeddability in rings of Witt vectors. In: Proceedings of the conference on commutative algebra, Kingston 1975, *Queen's Papers in Pure and Applied Math.*, vol. 42, pp. 106–195 (1975)
27. Hochster, M.: Topics in the Homological Theory of Modules over Commutative Rings, *CBMS Regional Conf. Ser. in Math*, vol. 24. Amer. Math. Soc., Providence, RI (1975)
28. Hochster, M.: Cyclic purity versus purity in excellent Noetherian rings. Trans. Amer. Math. Soc. **231**, 463–488 (1977)
29. Hochster, M.: Canonical elements in local cohomology modules and the direct summand conjecture. J. Algebra **84**, 503–553 (1983)
30. Hochster, M.: Solid closure. In: Commutative algebra: syzygies, multiplicities, and birational algebra, *Contemp. Math.*, vol. 159, pp. 103–172. Amer. Math. Soc., Providence (1994)
31. Hochster, M.: Tight closure in equal characteristic, big Cohen-Macaulay algebras, and solid closure. In: Commutative algebra: syzygies, multiplicities, and birational algebra, *Contemp. Math.*, vol. 159, pp. 173–196. Amer. Math. Soc., Providence (1994)
32. Hochster, M.: Big Cohen-Macaulay algebras in dimension three via Heitmann's theorem. J. Algebra **254**, 395–408 (2002)
33. Hochster, M., Huneke, C.: Tightly closed ideals. Bull. Amer. Math. Soc. **18**(1), 45–48 (1988)
34. Hochster, M., Huneke, C.: Tight closure, invariant theory, and the Briançon-Skoda theorem. J. Amer. Math. Soc. **3**, 31–116 (1990)
35. Hochster, M., Huneke, C.: Infinite integral extensions and big Cohen-Macaulay algebras. Ann. of Math. **135**, 53–89 (1992)
36. Hochster, M., Huneke, C.: F-regularity, test elements, and smooth base change. Trans. Amer. Math. Soc. **346**, 1–62 (1994)
37. Hochster, M., Huneke, C.: Tight closure of parameter ideals and splitting in module-finite extensions. J. Alg. Geom. **3** (1994)
38. Hochster, M., Huneke, C.: Applications of the existence of big Cohen-Macaulay algebras. Adv. in Math. **113**, 45–117 (1995)
39. Hochster, M., Huneke, C.: Tight closure. In: Commutative Algebra, vol. 15, pp. 305–338 (1997)
40. Hochster, M., Huneke, C.: Tight closure in equal characteristic zero (2000). Preprint on http://www.math.lsa.umich.edu/˜hochster/tcz.ps.Z
41. Hochster, M., Huneke, C.: Comparison of symbolic and ordinary powers of ideals. Invent. Math. **147**, 349–369 (2002)
42. Hochster, M., Roberts, J.: Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay. Adv. in Math. **13**, 115–175 (1974)
43. Hodges, W.: Model Theory. Cambridge University Press, Cambridge (1993)
44. Huneke, C.: Tight Closure and its Applications, *CBMS Regional Conf. Ser. in Math*, vol. 88. Amer. Math. Soc. (1996)
45. Huneke, C., Lyubeznik, G.: Absolute integral closure in positive characteristic. Adv. Math. **210**(2), 498–504 (2007)
46. Huneke, C., Smith, K.: Tight closure and the Kodaira vanishing theorem. J. Reine Angew. Math. **484**, 127–152 (1997)

47. Kawamata, Y.: The cone of curves of algebraic varieties. Ann. of Math. **119**, 603–633 (1984)
48. Kollár, J., Mori, S.: Birational Geometry and Algebraic Varieties. Cambridge University Press, Cambridge (1998)
49. Kunz, E.: Characterizations of regular local rings of characteristic p. Amer. J. Math. **41**, 772–784 (1969)
50. Lauritzen, N., Raben-Pedersen, U., Thomsen, J.: Global F-regularity of Schubert varieties with applications to D-modules. J. Amer. Math. Soc. **19**(2), 345–355 (electronic) (2004)
51. Lipman, J., Sathaye, A.: Jacobian ideals and a theorem of Briançon-Skoda. Michigan Math. J. **28**, 199–222 (1981)
52. Lipman, J., Teissier, B.: Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals. Michigan Math. J. **28**, 97–116 (1981)
53. Lyubeznik, G., Smith, K.: Strong and weakly F-regularity are equivalent for graded rings. Amer. J. Math. **121**, 1279–1290 (1999)
54. Matsumura, H.: Commutative Ring Theory. Cambridge University Press, Cambridge (1986)
55. Mehta, V., Ramanathan, A.: Frobenius splitting and cohomology vanishing for Schubert varieties. Ann. of Math. **122**, 27–40 (1985)
56. Milne, J.: Etale Cohomology. 33. Princeton Math. (1980)
57. Peskine, C., Szpiro, L.: Dimension projective finie et cohomologie etale. Inst. Hautes Études Sci. Publ. Math. **42**, 47–119 (1972)
58. Popescu, D.: General Néron desingularization and approximation. Nagoya Math. J. **104**, 85–115 (1986)
59. Roberts, P.: Le théorème d'intersections. C. R. Acad. Sci. Paris **304**, 177–180 (1987)
60. Roberts, P.: A computation of local cohomology. In: Proceedings Summer Research Conference On Commutative Algebra, *Contemp. Math.*, vol. 159, pp. 351–356. Amer. Math. Soc., Providence (1994)
61. Roberts, P.: Multiplicities and Chern classes in local algebra, *Cambridge Tracts in Mathematics*, vol. 133. Cambridge University Press, Cambridge (1998)
62. Rotthaus, C.: On the approximation property of excellent rings. Invent. Math. **88**, 39–63 (1987)
63. Schmidt, K., van den Dries, L.: Bounds in the theory of polynomial rings over fields. A non-standard approach. Invent. Math. **76**, 77–91 (1984)
64. Schoutens, H.: Bounds in cohomology. Israel J. Math. **116**, 125–169 (2000)
65. Schoutens, H.: Uniform bounds in algebraic geometry and commutative algebra. In: Connections between model theory and algebraic and analytic geometry, *Quad. Mat.*, vol. 6, pp. 43–93. Dept. Math., Seconda Univ. Napoli, Caserta (2000)
66. Schoutens, H.: Lefschetz principle applied to symbolic powers. J. Algebra Appl. **2**, 177–187 (2003)
67. Schoutens, H.: Mixed characteristic homological theorems in low degrees. C. R. Acad. Sci. Paris **336**, 463–466 (2003)
68. Schoutens, H.: Non-standard tight closure for affine $\mathbb{C}$-algebras. Manuscripta Math. **111**, 379–412 (2003)
69. Schoutens, H.: A non-standard proof of the Briançon-Skoda theorem. Proc. Amer. Math. Soc. **131**, 103–112 (2003)
70. Schoutens, H.: Projective dimension and the singular locus. Comm. Algebra **31**, 217–239 (2003)
71. Schoutens, H.: Canonical big Cohen-Macaulay algebras and rational singularities. Illinois J. Math. **48**, 131–150 (2004)
72. Schoutens, H.: Log-terminal singularities and vanishing theorems via non-standard tight closure. J. Alg. Geom. **14**, 357–390 (2005)
73. Schoutens, H.: Asymptotic homological conjectures in mixed characteristic. Pacific J. Math. **230**, 427–468 (2007)

74. Schoutens, H.: Bounds in polynomial rings over Artinian local rings. Monatsh. Math. **150**, 249–261 (2007)
75. Schoutens, H.: Pure subrings of regular rings are pseudo-rational. Trans. Amer. Math. Soc. **360**, 609–627 (2008)
76. Schoutens, H.: Use of ultraproducts in commutative algebra (2008). In preparation
77. Schoutens, H.: Dimension theory for local rings of finite embedding dimension (in preparation). ArXiv:0809.5267v1
78. Smith, K.: Tight closure of parameter ideals. Invent. Math. **115**, 41–60 (1994)
79. Smith, K.: F-rational rings have rational singularities. Amer. J. Math. **119**, 159–180 (1997)
80. Smith, K.: Vanishing, singularities and effective bounds via prime characteristic local algebra. In: Algebraic geometry—Santa Cruz 1995, *Proc. Sympos. Pure Math.*, vol. 62, pp. 289–325. Amer. Math. Soc., Providence, RI (1997)
81. Smith, K.: Globally F-regular varieties: applications to vanishing theorems for quotients of Fano varieties. Michigan Math. J. **48**, 553–572 (2000)
82. Smith, K.: An introduction to tight closure. In: Geometric and combinatorial aspects of commutative algebra (Messina, 1999), *Lecture Notes in Pure and Appl. Math.*, vol. 217, pp. 353–377. Dekker, New York (2001)
83. Spivakovsky, M.: A new proof of D. Popescu's theorem on smoothing of ring homomorphisms. J. Amer. Math. Soc. **12**, 381–444 (1999)
84. Strooker, J.: Homological Questions In Local Algebra, *LMS Lect. Note Ser.*, vol. 145. Cambridge University Press (1990)
85. Swan, R.: Néron-Popescu desingularization (Spring 1995). Expanded notes from a Univ. of Chicago series of lectures
86. Wall, C.: Lectures on C^∞ stability and classification. In: Proceedings of Liverpool Singularities–Symposium I, *Lect. Notes in Math.*, vol. 192, pp. 178–206. Springer-Verlag (1971)
87. Weil, A.: Foundations of algebraic geometry. American Mathematical Society, Providence, R.I. (1962)