Anti-aliasing for dummies

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This document gives a very brief introduction to the anti-aliasing strategies in SWEET and how to test for them. The aliasing problem shows up when evaluating non-linearities in physical space.

1 Problem formulation

Let our domain be given by the unit interval $\Omega = [0;1]$. Furthermore, let N be the physical resolution and K the number of spectral modes. We use $u(x) = \sin(k_1 x\pi)$ and $v(x) = \sin(k_2 x\pi)$ for particular frequencies k_1 and k_2 on our domain.

Using the identities

$$\cos(a+b) = \cos a \sin b - \sin a \sin b$$
$$\cos(a-b) = \cos a \sin b + \sin a \sin b$$

we can subtract both terms to get

$$\cos a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b).$$

Evaluating these non-linearities directly can be reformulated by using the product-to-sum rule:

$$\begin{aligned} f(x) &= u(x)v(x) \\ &= \sin(k_1 x)\sin(k_2 x) \\ &= \frac{1}{2}\cos\left(\left(k_1 - k_2\right)x\pi\right) - \frac{1}{2}\cos\left(\left(k_1 + k_2\right)x\pi\right) \end{aligned}$$

The frequency $k_1 - k_2$ is representable in our spectrum, since the frequency is lower than the max. representable frequency. This behaves differently for the 2nd term. If $k_1 + k_2 \geq K$, this cannot be represented in the spectrum anymore and would be truncated. Hence, using a convolution in spectral space to implement the non-linearities does not require any special treatment.

2 Errors with pseudo-spectrum

However, using a pseudo-spectral method and hence evaluating the non-linearities in physical space leads to aliasing effects. This is because the frequency $k_1 + k_2$ would be under sampled, hence violating the Shannon-Nyquist theorem. Therefore, this frequency must be "somehow" truncated off.

Solution The underlying idea is to use a resolution in physical space which allows to represent this frequency. During the transformation to spectral space, this therefore doesn't lead to an aliasing effect. Once in spectral space, the new modes are truncated. This can be realized as follows:

1. A physical resolution of $N = \frac{3}{2}M$ is used and the state variables in u and v are transformed from spectral space with M possible modes to a physical space or resolution N.

2. The non-linearities are evaluated in physical space at points x_i:

$$f_i = u_i v_i$$

   Since the maximum mode was $\max_{k_i} i = M$, the new resulting mode can be of max $2M$ and all modes between $\{M+1, \ldots M\}$ must be eliminated.

3. The results are transformed back to spectral space and we discuss the representation of this data in spectral space. First of all, we can exploit the representation in spectral space: There's a symmetry due to the physical values being real-valued only. Therefore, one half (requiring $M/2$ complex values) of the data in spectral space can be used to reconstruct the physical data and we focus on this half.
   Now, an anti-aliasing requires zeroing out half of these coefficients, leading to the requirement to extend the spectrum by $M/2/2 = M/4$ additional modes. Finally, this leads to a required spectrum size of

$$\begin{aligned} 2(M/2 + M/4) &= M + M/2 \\ &= M3/2 \\ &= N \end{aligned}$$

   which is also known as the 3/2 rule and requires to use a resolution of $N = M3/2$ in physical space.