Implicit solver for linear wave using the Helmholtz formulation

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Contents

This document briefly describes how to reformulate the linear wave equation into a Helmholtz equation since this was nowhere explained so far.

1 Linear wave equation

Let the linear wave equation, e.g., the shallow-wave equation, be given by

$$\begin{aligned} \Phi_t &= -\overline{\Phi}\nabla\cdot\boldsymbol{V} \\ \boldsymbol{V}_t &= -\nabla\Phi. \end{aligned}$$

In this context, $\Phi = gh$ relates to the geopotential with h the water surface height, $\boldsymbol{V} = (u, v)$ to the velocities and $\overline{\Phi}$ to the average geopotential.

2 Solver

2.1 Implicit time step

Next, we follow a standard approach for atmospheric simulations by using a backward Euler time stepping and a reformulation which reduces the linear system of equations to solve to a Helmholtz problem. With $L_x = -\sqrt{c}\partial_x$, we obtain

$$\begin{aligned} \frac{\Phi^{n+1}-\Phi^n}{\Delta t} &\approx -\overline{\Phi}\nabla\cdot\boldsymbol{V}^{n+1} \\ \frac{\boldsymbol{V}^{n+1}-\boldsymbol{V}^n}{\Delta t} &\approx -\nabla\Phi^{n+1}. \end{aligned}$$

The first step is to rewrite this to a divergence form using $\delta = \nabla \cdot \boldsymbol{V}$ to obtain

$$\begin{aligned}\frac{\Phi^{n+1} - \Phi^{n}}{\Delta t} &\approx -\overline{\Phi}\delta^{n+1} \\ \frac{\delta^{n+1} - \delta^{n}}{\Delta t} &\approx -\nabla^2 \Phi^{n+1}.\end{aligned}$$

A further rearrangement of the equations leads to

$$\begin{aligned}\Phi^{n+1} &\approx -\Delta t \overline{\Phi}\delta^{n+1} + \Phi^{n} \\ \delta^{n+1} &\approx -\Delta t \nabla^2 \Phi^{n+1} + \delta^{n}.\end{aligned}$$

We can now either **solve for the divergence** δ using the 1st line in the second one to obtain

$$\begin{aligned}\delta^{n+1} &\approx -\Delta t \nabla^2 \left(-\Delta t \overline{\Phi}\delta^{n+1} + \Phi^{n}\right) + \delta^{n} \\ \Delta t \nabla^2 \Phi^{n} - \delta^{n} &\approx \nabla^2 \Delta t^2 \overline{\Phi}\delta^{n+1} - \delta^{n+1} \\ \left(\Delta t^2 \overline{\Phi} \nabla^2 - I\right) \delta^{n+1} &\approx \Delta t \nabla^2 \Phi^{n} - \delta^{n} \\ \left(\nabla^2 - \frac{1}{\Delta t^2 \overline{\Phi}} I\right) \delta^{n+1} &\approx \frac{1}{\Delta t^2 \overline{\Phi}} \left(\Delta t \nabla^2 \Phi^{n} - \delta^{n}\right)\end{aligned}$$

or solve for the geopotential the 2nd line in the first one to get

$$\begin{aligned}\Phi^{n+1} &\approx -\Delta t \overline{\Phi} \left(-\Delta t \nabla^2 \Phi^{n+1} + \delta^{n}\right) + \Phi^{n} \\ \Delta t \overline{\Phi}\delta^{n} - \Phi^{n} &\approx \Delta t^2 \overline{\Phi} \nabla^2 \Phi^{n+1} - \Phi^{n+1} \\ \left(\overline{\Phi} \Delta t^2 \nabla^2 - I\right) \Phi^{n+1} &\approx \Delta t \overline{\Phi}\delta^{n} - \Phi^{n} \\ \left(\nabla^2 - \frac{1}{\Delta t^2 \overline{\Phi}} I\right) \Phi^{n+1} &\approx \frac{1}{\Delta t^2 \overline{\Phi}} \left(\Delta t \overline{\Phi}\delta^{n} - \Phi^{n}\right).\end{aligned}$$

2.2 Helmholtz equation

This follows the Helmholtz problem formulation of the form

$$\left(\nabla^2 + aI\right) x = b.$$

We should mention here, that the Helmholtz problem itself was originally given by $\left(\nabla^2 + k^2\right) x = b$, but in atmospheric simulations (and probably only there), the above equation was claimed to be a Helmholtz equation.

3 SWEET

Next, we discuss the particular implementation of the Helmholtz solver in SWEET. Note, that once the Helmholtz problem was solved for one variable, this solution can be used to compute the solution of the other variable explicitly.

3.1 Notes for SWEET's SWE_Sphere_TS_l_irk (DEPRECATED)

The 1st order method without Corliolis effect performs as follows:

1. Compute RHS:
$$rhs = \overline{\Phi}\delta^{n} - \frac{1}{\Delta t}\Phi^{n}.$$

2. Compute divergence

$$\left(aI + b\nabla^2\right)\Phi'^{n+1} = rhs$$

with

$$a = \frac{1}{\Delta t^2}$$
$$b = -\overline{\Phi}.$$

3. Final conversion:

$$\Phi^{n+1} = -\frac{1}{\Delta t}\Phi'^{n+1}$$
$$\Phi'^{n+1} = -\Delta t \Phi^{n+1}$$

Check:

$$\left(I\frac{1}{\Delta t^2} - \overline{\Phi}\nabla^2\right)\Phi'^{n+1} = \overline{\Phi}\delta^n - \frac{1}{\Delta t}\Phi^n$$
$$\left(I\frac{1}{\Delta t^2} - \overline{\Phi}\nabla^2\right)\left(-\Delta t\Phi^{n+1}\right) = \overline{\Phi}\delta^n - \frac{1}{\Delta t}\Phi^n$$
$$\left(I\frac{1}{\overline{\Phi}\Delta t^2} - \nabla^2\right)\Phi^{n+1} = \frac{1}{-\overline{\Phi}\Delta t}\left(\overline{\Phi}\delta^n - \frac{1}{\Delta t}\Phi^n\right)$$
$$\left(\nabla^2 - I\frac{1}{\overline{\Phi}\Delta t^2}\right)\Phi^{n+1} = \frac{1}{\overline{\Phi}\Delta t}\left(\overline{\Phi}\delta^n - \frac{1}{\Delta t}\Phi^n\right)$$
$$\left(\nabla^2 - I\frac{1}{\overline{\Phi}\Delta t^2}\right)\Phi^{n+1} = \frac{1}{\overline{\Phi}\Delta t^2}\left(\Delta t\overline{\Phi}\delta^n - \Phi^n\right)$$

3.2 Notes for SWEET's SWE_Sphere_TS_l_irk

The 1st order method without Corliolis effect performs as follows:

1. Compute RHS:

$$rhs = \delta^n - \Delta t\nabla^2\Phi^n$$

2. Compute divergence

$$\left(I + b\nabla^2\right)\delta^{n+1} = rhs$$

with

$$b = -\overline{\Phi}\Delta t^2.$$

Check:

$$\left(I + \left(-\overline{\Phi}\Delta t^2\right)\nabla^2\right)\delta^{n+1} = \delta^n - \Delta t\nabla^2\Phi^n$$
$$\left(\nabla^2 - \frac{1}{\Delta t^2\overline{\Phi}}I\right)\delta^{n+1} = \frac{1}{\overline{\Phi}\Delta t^2}\left(\Delta t\nabla^2\Phi^n - \delta^n\right)$$