Exponential integration for linear geostrophic balance

Martin Schreiber

2022-08-01

1 Introduction

1.1 Nonlinear PDE

We like to investigate an exponential time integration method for the linear parts of the SWE on the rotating sphere which are given by

$$\frac{\partial}{\partial t}\begin{pmatrix}\Phi\\ \vec{V}\end{pmatrix}=\underbrace{\begin{pmatrix}-\overline{\Phi}\nabla\cdot\vec{V}\\ -\nabla\Phi\end{pmatrix}}_{L_g=\text{linear gravity}}+\underbrace{\begin{pmatrix}\\ -f\vec{k}\times\vec{V}\end{pmatrix}}_{L_c=\text{linear Coriolis}}+\underbrace{\begin{pmatrix}-\vec{V}\cdot\nabla\Phi\\ -\vec{V}\cdot\nabla\vec{V}\end{pmatrix}}_{N_a=\text{nonlinear advection}}+\underbrace{\begin{pmatrix}-\Phi'\nabla\cdot\vec{V}\\ 0\end{pmatrix}}_{N_r=\text{nonlinear remainder}}.$$

We would like to treat $L=L_g+L_c$ exponentially by computing

$$\exp(\Delta t L)$$

in a computationally efficient way. Using spherical harmonics, it turned out that we can compute $\exp(L_g)$ directly in spectral space (see doc/rexi/rexi_for_swe_on_nonrotating_sphere), however, we need to incorporate the L_c part somehow and a direct solution in spectral space doesn't seem to feasible.

In this document, we'll investigate a particular method to overcome this problem.

1.2 Linear PDE

We continue by only focusing on the linear parts, hence

$$\begin{pmatrix}\Phi\\ \vec{V}\end{pmatrix}=\underbrace{\begin{pmatrix}-\overline{\Phi}\nabla\cdot\vec{V}\\ -\nabla\Phi\end{pmatrix}}_{L_g=\text{linear gravity}}+\underbrace{\begin{pmatrix}\\ -f\vec{k}\times\vec{V}\end{pmatrix}}_{L_c=\text{linear Coriolis}}.$$

In what follows, it's important to understand the stiffness of both terms with respect to the time step size. The L_g term is the most limiting part whereas the L_c term is quite the opposite of it. Hence, we could think about taking a very large time step for the L_g term with linear exponential integration and incorporate the L_c effect with a quadrature rule over this large time step (since it only moderately oscillates over this time step). Taking the perspecitve of the standard exponential integration, for a PDE of the form

$$U_t = LU + N(U)$$

with

$$U(t+\Delta t)=e^{\Delta t L}U(t)+\int_0^{\Delta t}e^{(\Delta t-\tau)L}N(U(t+\tau))d\tau.$$

and setting $L=L_g$ and $N(U)=L_c$, we can write this in exponential integration form as

$$U(t+\Delta t)=e^{\Delta t L_g}U(t)+\int_0^{\Delta t}e^{(\Delta t-\tau)L}L_c(U(t+\tau))d\tau$$

where the integral finally only acts on a very slowly varying function.

In what follows, we investigate different ways to approximate this integral with known exponential integrator methods.

1.3 ETDnRK methods

The particular exponential method which we use to approximate this equation are known as the exponential time differentiating Runge-Kutta methods [1].

TODO: Write more.

2 ODE form

2.1 Example

We first need to investigate the stability of the method to see whether the discretized exponential time integration method is feasible with the given terms. To do so, we use a simplified model given by

$$\tfrac{d}{dt}u = \underbrace{\lambda_1 u}_{L_g} + \underbrace{\lambda_2 u}_{L_c}$$

where we relate each part to one linear term of the SWE on the rotating sphere with L_g the fast (external) gravity modes and L_c the (linear) Coriolis effect. Here, we assume that we can ignore the non-commutative property of both operators.

2.2 Stability analysis

We like to do stability plots where we would have a 4 dimensional space (with complex valued $\lambda_{1/2}$). Therefore, we reduce it to a 2 dimensional plot with coordinates given by x and y by only focusing on the properties we're interested in. First, we fix $\lambda_1 = 1i$ to represent a highly oscillatory problem. Second, we choose $x = \Delta t$ as the time step size and $\lambda_2 = yi$ to vary over the stiffness of the 2nd term with results now given in a plot for the (x, y) coordinate. We are mainly interested in studying the problem of $\lambda_2 \ll \lambda_1$, hence study choose $y \in [-10^{-1}; 10^{-1}]$ and large time step sizes, hence $x \in [0; 60]$.

This leads to studying the stability of the following ODE

$$u(t + \Delta t) = e^{\Delta t \lambda_1} u(t) + \int_0^{\Delta t} e^{(\Delta t - \tau)\lambda_1} \lambda_2 (u(t + \tau)) d\tau$$

or

$$u(t + \Delta t) = e^{\Delta t \lambda_1} \left(u(t) + \lambda_2 \int_0^{\Delta t} e^{-\tau \lambda_1} u(t + \tau) d\tau \right)$$

which we don't use further, but which can provide a good basis of future work on this since we factored out λ_2, hence reduces likely the computational complexity.

We can now investigate the stability and errors of this scheme with results given in Fig. 1.

3 PDE investigation

Next, we perform studies with a full PDE and the above given splitting method. The benchmark we use is based on a standard test case for barotropic instability [2]. We use the same parameters, however with a perfectly (with respect to spherical harmonics) linear balanced initial conditions and the same perturbation with a Gaussian bump.

The reference solution is computed with a Runge-Kutta method

References

[1] David A. Cox, John Little, and Donal O'Shea. *Using Algebraic Geometry (Graduate Texts in Mathematics)*. 2008.

[2] Joseph Galewsky, Richard K. Scott, and Lorenzo M. Polvani. An initial-value problem for testing numerical models of the global shallow-water equations. *Tellus, Series A: Dynamic Meteorology and Oceanography*, 56(5):429–440, 2004.

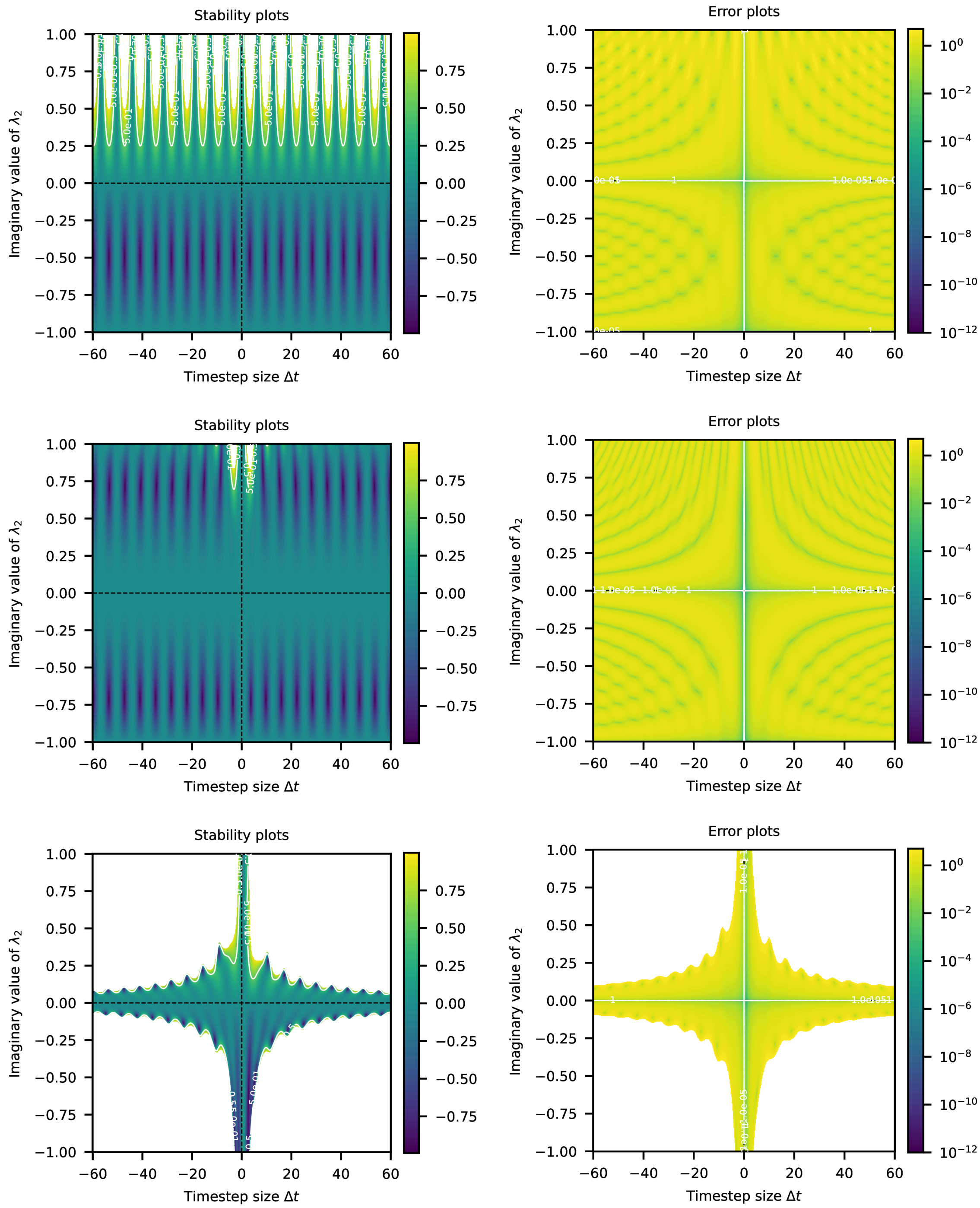

Figure 1: Stability and errors of exponential integration method for different orders of ETDnRK with a slow/fast oscillatory splitting.