Time stepping methods for SWE in SWEET

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The intention of this paper is to define an interface description for the different time stepping methods for the SWE as they are currently available in SWEET. This document focuses mainly on the SWE on the bi-periodic plane, but the same nomenclature is used for the SWE on the sphere.

To find a meaningful interface description, we first split up the non-linear SWE to its components. This is important since each of these components can be treated with a different time stepping method. Also, we provide some additional information on the SWE related to their time step limitations.

1 Shallow water equations on the plane

This section discusses the realization of different time stepping methods for the shallow water equations (SWE).

The full non-linear shallow water equations in advective formulation on the plane are given by

$$\frac{\partial \mathbf{U}}{\partial t} = \begin{bmatrix} & -\overline{\Phi}\partial_x & -\overline{\Phi}\partial_y \\ -\partial_x & & f \\ -\partial_y & -f & \end{bmatrix} .\mathbf{U} + \begin{bmatrix} -\nabla\left(\Phi' \boldsymbol{u}\right) \\ -u\partial_x u - v\partial_y u \\ -u\partial_x v - v\partial_y v \end{bmatrix} + D(\boldsymbol{U})$$

with the state variables U, the average geopotential $\overline{\Phi} = g\overline{h}$, and derivatives along dimensions $x$and y. The state variable is given by $U = (\Phi, u, v)$ which is the geopotential $\Phi = gh$ with h the height of the shallow water, u the velocity in direction i and v the velocity in direction y.

1.1 Splitting

The following notations split up the shallow-water equations in their advective formulation in their gravitational-related linear part $L_G(U)$, the Coriolis related part $L_C(U)$, the non-linear part which can be formulated to be part of the total derivative (advective terms) denoted by $N_A(U)$ and the divergence-related term $N_D(U)$. We also include a diffusion term $D(U)$ (which is typically linear) and is required to avoid accumulation of modes on the find grid.

$$\frac{\partial U}{\partial t} = L_G(U) + L_C(U) + N_A(U) + N_D(U) + D(U)$$

$$
\begin{aligned}
= & \underbrace{\begin{bmatrix} & -\overline{\Phi}\partial_x & -\overline{\Phi}\partial_y \\ -\partial_x & & 0 \\ -\partial_y & 0 & \end{bmatrix}}_{L_G} U + \underbrace{\begin{bmatrix} & & f \\ & -f & \end{bmatrix}}_{L_C} U \\
& + \underbrace{\begin{bmatrix} -\boldsymbol{u}.\nabla\Phi' \\ -u\partial_x u - v\partial_y u \\ -v\partial_y v - u\partial_x v \end{bmatrix}}_{N_A(U)} + \underbrace{\begin{bmatrix} -\Phi'\nabla.\boldsymbol{u} \\ \\ \end{bmatrix}}_{N_D(U)} \\
& + D(U)
\end{aligned}
$$

1.1.1 Wave equation (L_G)

The linear term L_G is directly related to a 1D wave equation

$$
\begin{aligned}
\frac{\partial^2 u}{\partial^2 t} &= c^2 \frac{\partial^2 u}{\partial^2 x} \\
\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) &= 0
\end{aligned}
$$

1.1.2 Harmonic oscillator (L_C)

In Fourier space all velocity component can be decoupled to a set of ODEs. Then, we can describe the evolution of Coriolis-induced effect with a harmonic oscillator

$$
\frac{\partial \boldsymbol{V}}{\partial t} = \begin{bmatrix} & f \\ -f & \end{bmatrix} \boldsymbol{V}
$$

1.1.3 Semi-Lagrangian formulation

The term $N_A(U)$ can be treated with a semi-Lagrangian formulation by using the total derivative on each state variable g

$$
\frac{dg}{dt} = \frac{\partial g}{\partial t} + u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y}
$$

1.1.4 Divergence-induced effect

The term $N_D(U)$

$$
\frac{\partial \Phi}{\partial t} = -\Phi'\nabla.\boldsymbol{u}
$$

1.1.5 Time step restrictions

This section provides some estimations on the time step restrictions for flows relevant to atmospheric simulations for the earth:

PDE terms	Time step restrictions and further information
$L_G(U)$	Time step limiting term. Frequently treated implicitly.
$L_C(U)$	Similar time step limitation compared to non-linear terms
$N_A(U)$	SL treatment allows running time steps which are about 6 to 10 times larger with reasonable accuracy
$N_R(U)$	??!?
$D(U)$	??!?

2 Time stepping methods

In the following, we give an overview of different time stepping methods and introduce a naming convention describing which terms are treated with which time stepping method.

2.1 Naming terms:

The naming convention for the terms is as follows:

PDE terms	Naming
$L_G(U)$	lg
$L_C(U)$	lc
$N_A(U)$	na
$N_D(U)$	nd
$L_G(U) + L_C(U)$	l(=lgc)
$N_A(U) + N_D(U)$	n(=ntd)
$L_G(U) + L_C(U) + N_T(U) + N_D(U)$	ln
$D(U)$	d

2.2 Naming time stepping methods

Time stepping method	Naming	Comment
Explicit Runge-Kutta	erk	1st / 2nd / 4th order
Implicit Runge-Kutta	irk	1st order and 2nd order Crank-Nicolson
Semi-Lagrangian	sl	1st / 2nd order
Exponential integration	exp	inf order

2.3 Behavior of accuracy

Accuracy	Naming
1, 2, 3, 4, ... order	1, 2, 3, 4, ...
exponentially fast decaying	exp
analytical solution	inf
Default	default

The default value should automatically select the accuracy in case that there's only one choice, e.g. *exp* for REXI or *inf* for the analytical solution.

2.4 Implementations in SWEET

We provide an overview of the different time stepping methods which are made available in SWEET for the rotational SWE on the bi-periodic plane.

	Order	$L_G(U)$	$L_C(U)$	$N_A(U)$	$N_D(U)$	$D(U)$
l_irk	1/2/4	X	X			
l_erk	1/2/4	X	X			
l_exp	exp	X	X			
lg_exp	exp	X				
l_direct	inf	X	X			
l_exp_na_sl	1/2	X	X	X		
ln_erk	1/2/4	X	X	X	X	
lg_exp_lc_implicit_na_sl_nr_settls	2	X	X	X	X	

The diffusion may be implemented as implicitly or explicitly with a lower order time stepping method.