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Let $A=\begin{pmatrix}1&0&-4\\0&5&4\\-4&4&3\end{pmatrix}$

Find the eigenvalues and corresponding eigenvectors of A.

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda&0&-4\\0&5-\lambda&4\\-4&4&3-\lambda\end{vmatrix}$$ Use the Rule of Sarrus to calculate eigenvalues.

$$=(\lambda-1)(\lambda-5)(\lambda-3)-16(\lambda-1)-16(\lambda-5)$$

$$=-\lambda^3+9\lambda^2+9\lambda-81=(3-\lambda)(\lambda-9)(\lambda+3)$$

Thus, the eigenvalues are $\lambda_1=3$ $\lambda_2=9$ $\lambda_3=-3$

Since all the eigenvalues are distinct, A can be diagonalized.

For $\lambda_1=3$

$$\begin{pmatrix}1-3&0&-4\\0&5-3&4\\-4&4&3-3\end{pmatrix}\to\begin{array}{c}R_1\\R_2\\R_3\end{array}\begin{pmatrix}-2&0&-4\\0&2&4\\-4&4&0\end{pmatrix}\xrightarrow[-2R_2+R_3]{-2R_1+R_3}\begin{pmatrix}-2&0&-4\\0&2&4\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&2\\0&1&2\\0&0&0\end{pmatrix}$$

$$\Rightarrow v_1=\begin{pmatrix}-2\\-2\\1\end{pmatrix}$$

For $\lambda_2=9$

$$\begin{pmatrix}1-9&0&-4\\0&5-9&4\\-4&4&3-9\end{pmatrix}\to\begin{pmatrix}-8&0&-4\\0&-4&4\\-4&4&-6\end{pmatrix}\to\begin{pmatrix}-8&0&-4\\0&-4&4\\0&-8&8\end{pmatrix}\to\begin{pmatrix}-8&0&-4\\0&-4&4\\0&6&0\end{pmatrix}\to\begin{pmatrix}-2&0&-1\\0&-1&1\\0&0&\end{pmatrix}$$

$$\Rightarrow v_2=\begin{pmatrix}1\\-2\\-2\end{pmatrix}$$

For $\lambda_3=-3$

$$\begin{pmatrix}1-(-3)&0&-4\\0&5-(-3)&4\\-4&4&3-(-3)\end{pmatrix}\to\begin{pmatrix}4&0&-4\\0&8&4\\-4&4&6\end{pmatrix}\xrightarrow[R_2-2R_3]{R_1+R_3}\begin{pmatrix}4&0&-4\\0&8&4\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&-1\\0&2&1\\0&0&0\end{pmatrix}$$

$$\Rightarrow v_3=\begin{pmatrix}-2\\1\\-2\end{pmatrix}$$

Is A similar to a diagonal matrix? If so, find a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Is P unique? Explain.

A is symmetric, if can be diagonalized. If the matrix A can be diagonalized, then it's possible to write $D = P^{-1}AP$

$P = (v_1, v_2, v_3) = \begin{pmatrix} -2 & 1 & -2 \\ -2 & -2 & 1 \\ 1 & -2 & -2 \end{pmatrix}$

the $P^{-1}AP = D$ where $P = \begin{pmatrix} -2 & 1 & -2 \\ -2 & -2 & 1 \\ 1 & -2 & -2 \end{pmatrix}$ ~~det P~~ =

~~P^{-1}~~ = Use python '.diagonal()' and 'inv()' and 'multi_dot()' to calculate the Diagonal Matrix D.

Then get the matrix([[3., 9., -3.]])

which means $P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix} = D$

P is not unique since the eigenvectors associated to an eigenvalue are not unique.

(c) Use python to calculate eigenvalues ~~of A^{-1}~~ where ainv = A^{-1}

w, v = linalg.eig(ainv)

Hence A^{-1} has eigenvalues $\frac{1}{3}, \frac{1}{9}, -\frac{1}{3}$.

Not necessary nor appropriate to use python for parts c and d. Use linear algebra to show these two results. -4

(d) Use python is .eig() and matrix_power to calculate the eigenvalues and eigenvectors of A^2.

I get the eigenvalues of A^2 are 9, 81 and 9.

And their associated eigenvectors are $v_1 = \begin{pmatrix} -.94280904 \\ -.23570226 \\ -0.23570226 \end{pmatrix}$

$v_2 = \begin{pmatrix} -0.33333333 \\ 0.66666667 \\ 0.66666667 \end{pmatrix}$ $v_3 = \begin{pmatrix} -0.26780105 \\ -0.74093291 \\ 0.61101239 \end{pmatrix}$

1D. (a) Let A be an $n\times n$ real matrix. 20/20

(i) Prove that the coefficient of λ^{n-1} in the characteristic polynomial of A is given by trA.

The characteristic polynomial of A is given by

$$\det(\lambda I_n - A) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n$$

$$= \begin{pmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{21} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{pmatrix}$$

Now, we can see that the expression involving λ^{n-1} in the characteristic polynomial arises from the product.

$$(\lambda - a_{11})(\lambda - a_{22})\dots(\lambda - a_{nn}) = \lambda^n - (a_{11} + a_{22} + \dots + a_{nn})\lambda^{n-1} + \dots$$

Thus, $a_1 = -(a_{11} + a_{22} + \dots + a_{nn}) = -trA$

(ii) Prove that trA is the sum of the eigenvalues of A

If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A, then $\lambda - \lambda_i$ $(i = 1, 2, \dots, n)$ are the factors of the characteristic polynomial.

$$\det(\lambda I_n - A) = \lambda_n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n$$
$$= (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$$
$$= \lambda^n - (\lambda_1 + \lambda_2 + \dots + \lambda_n)\lambda^{n-1} + \dots + (-1)^n \lambda_1\lambda_2\cdots\lambda_n.$$

Thus $a_1 = -\lambda_1 - \lambda_2 - \dots - \lambda_n = -trA$ ~~Thus, $a_1 = -\lambda_1 - \lambda_2 \dots - \lambda_n = -trA$.~~

Hence the trace of A is the sum of the eigenvalues of the characteristic polynomial of A.

(iii) Prove that the constant coefficient of A is $\pm$ the product of the eigenvalues of A.

In Eq. 1 above, the constant term is $a_n = (-1)^n\lambda_1\lambda_2\cdots\lambda_n$

We observe that

Also note that if $\lambda = 0$, then we have $(-1)^n \det A = \det(-A)$

$$= a_n$$
$$= (-1)^n \det A$$

Hence, $\det A$ is the product of the eigenvalues. $= (-1)^n\lambda_1\lambda_2\cdots\lambda_n$

1D (b) Let A be a 5×5 matrix. Suppose A has distinct eigenvalues, $-1, 1, -10, 5, 2$.

i) What is $\det A$? What is $\operatorname{tr} A$?

From part (a), we know that $\det A = -1\times1\times(-10)\times5\times2 = 100$

$$\operatorname{tr} A = -1+1+(-10)+5+2 = -3$$

(ii) ~~\$~~ If A and B are similar, what is $\det B$? Why?

Since A and B are similar, there is some invertible matrix ~~#~~ P such that $B = P^{-1}AP$. Hence,

$$\det B = \det(P^{-1}AP) = (\det P)^{-1}(\det A)(\det P) = \det A$$

Thus, $\det B = 100$.

(iii) Do you expect that all eigenvectors of A are mutually orthogonal? Why?

No, we can't say that all eigenvectors of A are mutually orthogonal because ~~we~~ A is not symmetric. We can expect that all the eigenvectors of A are linearly independent.

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(a) Prove the Cauchy-Schwarz Inequality: If u & v are any vectors in an inner product space V, then $\langle u,v\rangle^2 \le \|u\|^2\|v\|^2$.

Set u and v be vectors in an inner product space V.

Assume that $v\neq0$. Consider the orthogonal projection

$$z = u - \frac{\langle u,v\rangle}{\langle v,v\rangle}v. \Rightarrow u = z + \frac{\langle u,v\rangle}{\langle v,v\rangle}v$$

Since z and v are orthogonal, Apply the Pythagorean Theorem

$$\|u\|^2 = \|z\|^2 + \frac{\langle u,v\rangle^2}{\langle v,v\rangle^2}\|v\|^2 = \|z\|^2 + \frac{\langle u,v\rangle^2}{\|v\|^2} \geq \frac{\langle u,v\rangle^2}{\|v\|^2}$$

Hence $\langle u,v\rangle^2 \le \|u\|^2\|v\|^2$

b) Consider $\mathbb{R}^n$ with the standard inner product.

Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$.

Prove that $\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$

$\left(\sum_{i=1}^{n} u_i v_i\right)^2 = \langle u, v\rangle$ $\quad \sum_{i=1}^{n} u_i^2 = \langle u, u\rangle = \|u\|^2$ $\quad \sum_{i=1}^{n} v_i^2 = \langle v, v\rangle = \|v\|^2$

Thus, by the Cauchy-Schwarts Inequality, the inequality holds.

(c) Let V be the vector space of all continuous real-valued functions on the unit interval $[0, 1]$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$.

Prove $\left(\int_0^1 f(t)g(t)\,dt\right)^2 \leq \left(\int_0^1 f^2(t)\,dt\right)\left(\int_0^1 g^2(t)\,dt\right)$

Since $\left(\int_0^1 f(t)g(t)\,dt\right)^2 = \langle u, v\rangle$ $\quad \int_0^1 f^2(t)\,dt = \langle u, u\rangle = \|u\|^2$ and $\int_0^1 g^2(t)\,dt = \langle v, v\rangle = \|v\|^2$

Thus, by the Cauchy-Schwarts Inequality, the inequality holds.

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4C. Let W be the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product with basis. $S = \{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \quad u_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Transform S to an orthonormal basis $T = \{w_1, w_2, w_3\}$ using the Gram-Schmidt process.

The Gram-Schmidt ~~pr~~ is $\text{proj}_u(v) = \frac{\langle v, u \rangle}{\langle u, u \rangle} u$.

Define $w_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ ①

Then compute $v_2 = u_2 - \langle u_2, w_1 \rangle w_1$

$$= \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \left(\frac{2}{\sqrt{3}}\right) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

$\|v_2\| = \sqrt{\left(\frac{1}{9} + \frac{4}{9} + \frac{1}{9} + 1\right)} = \sqrt{\frac{15}{9}} = \sqrt{\frac{5}{3}}$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\frac{5}{3}}} \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \quad ②$$

Finally compute

$$v_3 = u_3 - \langle u_3, w_1 \rangle w_1 - \langle u_3, w_2 \rangle w_2$$

$$= \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \left(\frac{1}{\sqrt{3}}\right) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \left(\frac{2}{\sqrt{15}}\right) \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 + \frac{1}{3} - \frac{2}{15} \\ \frac{1}{3} + \frac{4}{15} \\ \cancel{\ } + \frac{1}{3} - \frac{2}{15} \\ -1 + 0 + \frac{6}{15} \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \\ \frac{1}{5} \\ -\frac{3}{5} \end{pmatrix}$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{35}} \times \begin{pmatrix} -4 \\ 3 \\ 1 \\ -3 \end{pmatrix} \quad ③$$

Hence, combine ①②③ we have the orthonormal basis

$$T = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \frac{1}{\sqrt{35}} \begin{pmatrix} -4 \\ 3 \\ 1 \\ -3 \end{pmatrix} \right.$$

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6E. Which of the following ~~matrices~~ matrices are positive-definite?

$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $\quad B = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$ $\quad C = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad \cancel{D}\ E = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}$

Recall that a symmetric matrix M is positive definite if and only if all the eigenvalues of M are positive.

So we can determine whether A, B and E are positive definite by examining their eigenvectors.

Use python's linalg.eig() function to calculate A, B and E's eigenvalues.

Thus get the eigenvalues of A are 1, 4 and 1., so A is positive definite.

The eigenvalues of B are 1.7639 and 6.2361. So B is also positive definite.

The eigenvalues of E are −0.6056 and 6.6056. So E is not positive definite.

fine to use python in this case.

To determine whether C is positive-definite.

~~Let~~ $\cancel{v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}}$ ~~be a nonzero vector.~~

Used the cholesky decomposition in python to valify whether C is positive-definite.

Used the np.linalg.cholesky() function. clever

This will raise LinAlgError if the matrix is not positive definite.

It turned out that C is not positive definite.