76/100

Big Data Analysis HW

Blake Hillier

Jan 28, 2020

18/20 **1C)**

- Note the basis of $P_2 = 1, x.x^2$. Each column of A is the coefficients of $L(1)$, $L(t)$, and $L(t^2)$. $\begin{matrix} L(1) = 2 + t + 0t^2 \\ L(t) = 0 + t + 0t^2 \\ L(t^2) = -1 - t + t^2 \end{matrix}$ so $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

- A has eigenvalues $1, 1,$ and 2 with the corresponding eigenvectors $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ Show Work. -2

- $P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

- $A^n = PD^nP^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2^n & 0 & 1-(2^n) \\ 2^n - 1 & 1 & 1-(2^n) \\ 0 & 0 & 1 \end{bmatrix}$
so $L^n = (a2^n + c(1-(2^n))) + (a(2^n - 1) + b + c(1-(2^n)))t + ct^2$ making $L^{100} = (a2^{100} + c(1-(2^{100}))) + (a(2^{100} - 1) + b + c(1-(2^{100})))t + ct^2$

18/20 **1D)**

- a)

  1. *Proof.* We can write the characteristic equation in the form $\prod_{i=1}^{n}(\lambda - a_{ii})$. To find the coefficient of λ^{n-1}, we simply select $n-1$ of the $(\lambda - a_{ii})$ to create λ^{n-1} and then multiply it by the remaining a_{ii} term. Repeat this for all combinations and we end up with $-\lambda^{n-1}\sum_{i=1}^{n} a_{ii} = -tr(A)\lambda^{n-1}$. □

     This is not the characteristic equation. just consider the matrix
     1 2
     1 2
     However it is the only part of the determinant which has a lambda^n-1 term. -1

  2. *Proof.* Similar to the last proof, we can write the characteristic equation in the form $\prod_{i=1}^{n}(\lambda - \lambda_i)$ where λ_i is the i-th eigenvalue of A. With the same logic as before, we find the coefficient of λ^{n-1} to be $-\sum_{i=1}^{n}\lambda_i = -tr(A)$. □

  3. *Proof.* Similar to the last two proofs, we can write the characteristic equation in the form $\prod_{i=1}^{n}(\lambda - \lambda_i)$ where λ_i is the i-th eigenvalue of A. With the same logic as before, we find the coefficient of λ^0 to be $(-1)^n\prod_{i=1}^{n}\lambda_i$. □

     you need to show that this is the determinant of A. -1

- b)

  1. $det(A) = -1 * 1 * -10 * 5 * 2 = 100$ and $trace(A) = -1 + 1 - 10 + 5 + 2 = -3$
  2. $det(B) = det(P^{-1}AP) = det(P^{-1})det(A)det(P) = det(A) = 100$

3. We cannot guarentee all eigenvectors of A are mutually orthogonal since A is not symmetric.

20/20

3B)

- *Proof.* Consider the orthogonal projectionA

$$
\begin{aligned}
x = u - \frac{u \cdot v}{\|v\|^2} v &\implies u = x + \frac{u \cdot v}{\|v\|^2} v \\
\|u\| &= \|x + \frac{u \cdot v}{\|v\|^2} v\| \\
\|u\|^2 &= \|x\|^2 + \frac{u \cdot v}{\|v\|^4}\|v\|^2 \quad \textit{by pythagorean theorem} \\
&= \|x\|^2 + \frac{(u \cdot v)^2}{\|v\|^2} \\
&\geq \frac{(u \cdot v)^2}{\|v\|^2} \\
\text{so} \quad \|u\|^2\|v\|^2 &\geq (u \cdot v)^2
\end{aligned}
$$

$\square$

I agree it is easy to see, but please write it out to some extent.

- *Proof.* We easily see the left side is the square of the inner product of u and v, while the right side is the square of the norms u and v, which is true by the Cauchy-Schwarz Inequality. $\square$

- *Proof.* This proof is identical to the proof above it. $\square$

20/20

3C)

Proof. We start by assuming $v \cdot w = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j = a^T C b$ is an inner product on V. This means $v \cdot v = a^T C a > 0$ when $v \neq 0$ and $v \cdot v = a^T C a = 0$ when $v = 0$ since S cannot contain a zero vector, making C positive definite.
Now assume C is positive definite. We need to check if this operation is symmetric, linear, and positive definite. It's easy to see it's symmetric by simply swaping a_i and b_j as well as the two summation symbols. It is also linear since $\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha a_i c_{ij} b_j = \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j$ for $\alpha \in \mathbb{R}$ and since $(a+d)^T C b = a^T C b + d^T C b$ where d is a vector of coefficients for $x = d_1 u_1 + \dots$.
Lastly, it is also positive definite by similar logic as the first section.
$\therefore \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j$ is an innerproduct iff C is positive definite. $\square$

Fifth problem?