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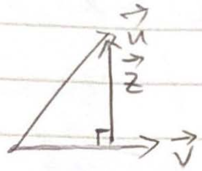
Yuxin Xin HW 1.

20/20 3B. (a) When $\vec{v}=0$. $0=0$ The theorem is trivially true.

So let us suppose $\vec{v}\neq 0$ let $\vec{z}$ be the orthogonal projection $\vec{z}=\vec{u}-\frac{\langle\vec{u},\vec{v}\rangle}{\langle\vec{v},\vec{v}\rangle}\vec{v}$. As we can see from the graph, $\vec{z}$ and $\vec{v}$ are orthogonal. We then apply Pythagorean Theorem to $\vec{u}=\vec{z}+\frac{\langle\vec{u},\vec{v}\rangle}{\langle\vec{v},\vec{v}\rangle}\vec{v}=$



$$\|\vec{u}\|^2=\|\vec{z}\|^2+\left|\frac{\langle\vec{u},\vec{v}\rangle}{\langle\vec{v},\vec{v}\rangle}\right|^2\|\vec{v}\|^2. \Rightarrow$$

$$\|\vec{z}\|^2+\frac{|\langle\vec{u},\vec{v}\rangle|^2}{(\|\vec{v}\|^2)^2}\|\vec{v}\|^2=\|\vec{z}\|^2+\frac{\langle u,v\rangle^2}{\|\vec{v}\|^2}$$

$$\geq\frac{\langle\vec{u},\vec{v}\rangle^2}{\|\vec{v}\|^2}.$$ Then we get $\|\vec{u}\|^2\|\vec{v}\|^2\geq\langle\vec{u},\vec{v}\rangle^2$

(b) Notice that $\sum_{i=1}^{n}u_iv_i=\vec{u}\cdot\vec{v}$ $\sum_{i=1}^{n}u_i^2=\|\vec{u}\|^2$. $\sum_{i=1}^{n}v_i^2=\|\vec{v}\|^2$

Thus, using Cauchy-Schwartz. we have $\|\vec{u}\|^2\|\vec{v}\|^2\geq\langle\vec{u},\vec{v}\rangle^2$, which equals $(\sum_{i=1}^{n}u_i^2)(\sum_{i=1}^{n}v_i)^2\geq(\sum_{i=1}^{n}u_iv_i)^2$.

(c) Notice that $(\int_0^1 f(t)g(t)\,dt)^2=\langle\vec{f},\vec{g}\rangle^2$

$\langle f,f\rangle=\int_0^1 f^2(t)\,dt=\|\vec{f}\|^2$. $\langle g,g\rangle=\int_0^1 g^2(t)\,dt=\|\vec{g}\|^2$

Thus we have from Cauchy-Schwartz:

$$(\int_0^1 f(t)g(t)\,dt)^2\leq(\int_0^1 f^2(t)\,dt)(\int_0^1 g^2(t)\,dt).$$

20/20 3C. let $\vec{x}=(a_1,a_2,\cdots,a_n)$ $\vec{y}=(b_1,b_2,\cdots,b_n)$. then we can write $(\vec{v},\vec{w})=\sum_{i=1}^{n}\sum_{j=1}^{n}a_ic_{ij}b_j$ as $\vec{x}^TC\vec{y}$.

① let us prove that $(\vec{v},\vec{w})$ defines an inner product on V. $\Rightarrow$ C is a positive definite matrix.

Recall that C is a positive definite matrix if $\vec{x}^TC\vec{x}>0$ for all $\vec{x}\in\mathbb{R}^n\setminus\vec{0}$. as $(\vec{v},\vec{w})$ defines an inner product on V, we have $(\vec{v},\vec{v})=\vec{x}^TC\vec{x}\geq 0$. Notice that $(\vec{v},\vec{v})=0$ only when $\vec{v}=0$, which means $\vec{x}=0$. Therefore, $\vec{x}^TC\vec{x}>0$ for all $\vec{x}\in\mathbb{R}^n\setminus\vec{0}$. C is positive definite.

② let us prove that C is positive definite $\Rightarrow$ $(\vec{v},\vec{w})$ defines an inner product on V. In order to define an inner product on V, one has to satisfy: ① $(\vec{v},\vec{v})\geq 0$ and $(\vec{v},\vec{v})=0$ only when $\vec{v}=0$. ② $(\vec{v},\vec{w})=(\vec{w},\vec{v})$.

③ $(\vec{u}+\vec{v},\vec{w}) = (\vec{u},\vec{w}) + (\vec{v},\vec{w})$ ④ $(r\vec{v},\vec{w}) = r(\vec{v},\vec{w})$

let us prove ①: $(\vec{v},\vec{v}) = \vec{x}^T C\vec{x}$ since C is positive definite, $\vec{x}^T C\vec{x} > 0$ for all $\vec{x} \in \mathbb{R}^n \setminus \vec{0}$ and $\vec{x}^T C\vec{x} = 0$ when $\vec{x} = 0$, which means $\vec{v} = 0$.

let us prove ②: $(\vec{v},\vec{w}) = \vec{x}^T C\vec{y} = \langle \vec{x}, C\vec{y} \rangle = \langle C\vec{x}, \vec{y} \rangle$
$= \langle \vec{y}, C\vec{x} \rangle = \vec{y}^T C\vec{x} = (\vec{w},\vec{v})$.

since C is symmetric.

since $\langle \vec{u},\vec{v} \rangle = \langle \vec{v},\vec{u} \rangle$

let us prove ③: let $\vec{u} = C_1\vec{u}_1 + C_2\vec{u}_2 + \cdots C_n\vec{u}_n$. then let $\vec{z} = (C_1, C_2, \cdots C_n)$. $(\vec{u}+\vec{v},\vec{w}) = (\vec{z}^T + \vec{x}^T)^T C\vec{y} =$
$\vec{z}^T C\vec{y} + \vec{x}^T C\vec{y} = (u,w) + (v,w)$.

let us prove ④: for any $r \in \mathbb{R}$. $(r\vec{v},\vec{w}) = r\vec{x}^T C\vec{y} = r(\vec{v},\vec{w})$

Thus, if C is positive definite, $(\vec{v},\vec{w})$ defines an inner product on V.

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6B. let $\vec{p}$ be the equilibrium percentages. then $M\vec{p} = \vec{p}$ since it reaches equilibrium. $(M-I)\vec{p} = 0$.

we have: $\left[\begin{array}{ccc|c} -0.5 & 0.25 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & 0 \\ 0 & 0.25 & -0.5 & 0 \\ 1 & 1 & 1 & 100 \end{array}\right] \Rightarrow \vec{p} = \begin{pmatrix} 25\% \\ 50\% \\ 25\% \end{pmatrix}$

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6E. Notice that $A = A^T$, $B = B^T$, $E = E^T$. So A, B, E are symmetric. Another way to define positive definite matrices is that positive definite matrices are symmetric matrices with all positive eigenvalues. Thus, it suffices to check only the eigenvalues of A, B, and E.

A: $\det\left(\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}\right) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$. $\lambda = 1, 1, 4$. > 0.

So A is positive definite.

B: $\det\left(\begin{vmatrix} 3-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix}\right) = \lambda^2 - 8\lambda + 11 = 0$ $\lambda = 4 \pm \sqrt{5} > 0$.

So B is positive definite.

E: $\det\left(\begin{vmatrix} 1-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}\right) = \lambda^2 - 6\lambda - 4 = 0$ $\lambda = 3 \pm \sqrt{13}$

$3 - \sqrt{13}$ is not positive.

So E is not positive definite.

let us now consider C. if we can show that there exists such $\vec{x}$ s.t. $\vec{x}^T C \vec{x} < 0$. then C is not positive definite.

let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. then $\vec{x}^T C \vec{x} = (x_1\ x_2\ x_3)\begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$= (x_1\quad 4x_1+2x_2\quad 5x_1+6x_2+3x_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 4x_1x_2 + 2x_2^2 + 5x_1x_3 + 6x_2x_3 + 3x_3^2$.

let $x_1 = 1$, $x_2 = 1$. $\Rightarrow 1+4+2+5x_3+6x_3+3x_3^2 < 0$.

$\Rightarrow 3x_3^2 + 11x_3 + 7 < 0$. we can see $x_3 = -1$ suffices.

Thus, there exists $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ such that $\vec{x}^T C \vec{x} < 0$.

So C is not positive definite.

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6F. (a) We first write g in matrix form: $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & -3 \end{pmatrix}$.

then we get the eigenvalues for the matrix:

$\det\left(\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -3-\lambda & 2 \\ 0 & 2 & -3-\lambda \end{vmatrix}\right) = (9+6\lambda+\lambda^2-4)(3-\lambda)$

$\lambda = 3, -1, -5$.

Thus, the desired quadratic form is $3y_1^2 - y_2^2 - 5y_3^2$.

As there are 3 eigenvalues, we know rank of g is 3.

There are 1 positive eigenvalue and 2 negatives.

So signature of g is -1.

(b) Since rank = 3. and g has 1 positive eigenvalue and 2 negative eigenvalues and $g > 0$. we know that the surface should be a hyperboloid of two sheets.

we can write it as: $\dfrac{y_1^2}{3} - \dfrac{y_2^2}{9} - \dfrac{y_3^2}{\frac{9}{5}} = 1$.