Exercise

Numerical simulation of physical systems:

Exercise 1 (Solution of the Heat Equation)
We consider a rod of length l, mathematically a one-dimensional domain $\Omega = (0 : l)$. The rod has a constant thermal conductivity k and is heated with a constant heat flux density q_Q. Therefore, $q_Q > 0$.

(a) Calculate the temperature distribution $u(x)$ for the stationary case from the heat equation

$$-k\frac{d^2u}{dx^2}(x) = q_Q$$

(b) Show that the temperature distribution in the stationary case can be expressed as

$$u(x) = -\frac{q_Q}{2k} \cdot x^2 + \frac{du}{dx}(0) \cdot x + u(0) \tag{1}$$

(c) Specify which expressions in this equation are material parameters and which are boundary conditions (what type of boundary conditions?).

(d) Consider now an experiment where both boundary conditions are Dirichlet boundary conditions, where the temperature at the ends of the rod is given by $u(0)$ and $u(l)$. Bring Equation (1) into a form where only these two boundary conditions appear.

(e) Due to the heating of the rod, it is possible that the temperature maximum of the rod is not at the rod ends but in between. Show that a temperature maximum between the rod ends can only occur under the condition

$$|u(l) - u(0)| < \frac{q_Q l^2}{2k}$$

What does this mean physically?