Naïve Discriminative Learning:
Theoretical and Experimental Observations

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Outline

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Objectives

- Explain the mathematical foundations of Naïve Discriminative Learning (NDL) in one place and in a consistent way
- Highlight the theoretical similarities of NDL with linear/logistic regression and the single-layer perceptron
- Present some empirical simulations of stochastic NDL learners, in light of the theoretical insights

Naïve Discriminative Learning

- Baayen (2011); Baayen *et al.* (2011)
- Incremental learning equations for direct associations between cues and outcomes (Rescorla and Wagner 1972)
- Equilibrium conditions (Danks 2003)
- Implementation as R package `ndl` (Arppe *et al.* 2014)

Naive: cue-outcome associations estimated separately for each outcome (this independence assumption is similar to a naive Bayesian classifier)

Discriminative: cues predict outcomes based on total activation level = sum of direct cue-outcome associations

Learning: incremental learning of association strengths

The Rescorla-Wagner equations (1972)

Represent incremental associative learning and subsequent on-going adjustments to an accumulating body of knowledge.

Changes in cue-outcome association strengths:

- No change if a cue is not present in the input
- Increased if the cue and outcome co-occur
- Decreased if the cue occurs without the outcome
- If outcome can already be predicted well (based on all input cues), adjustments become smaller

Only results of incremental adjustments to the cue-outcome associations are kept – no need for remembering the individual adjustments, however many there are.

Danks (2003) equilibrium conditions

- Presume an ideal stable "adult" state, where all cue-outcome associations have been fully learnt – further data points should then have no impact on the cue-outcome associations
- Provide a convenient short-cut to calculating the final cue-outcome association weights resulting from incremental learning, using relatively simple matrix algebra
- Most learning parameters of the Rescorla-Wagner equations drop out of the Danks equilibrium equation
- Circumvent the problem that a simulation of an R-W learner does usually not converge to a stable state unless the learning rate is gradually decreased

Traditional vs. linguistic applications of R-W

- Traditionally: simple controlled experiments on item-by-item learning, with only a handful of cues and perfect associations
- Natural language: full of choices among multiple possible alternatives – phones, words, or constructions – which are influenced by a large number of contextual factors, and which often show weak to moderate tendencies towards one or more of the alternatives rather than a single unambiguous decision
- These messy, complex types of problems are a key area of interest in modeling and understanding language use
- Application of R-W in the form of a Naïve Discriminative Learner to such linguistic classification problems is sucessful in practice and can throw new light on research questions

Related work

- R-W vs. perceptron (Sutton and Barto 1981, p. 155f)
- R-W vs. least-squares regression (Stone 1986, p. 457)
- R-W vs. logistic regression (Gluck and Bower 1988, p. 234)
- R-W vs. neural networks (Dawson 2008)

☞ similarities are also mentioned by many other authors . . .

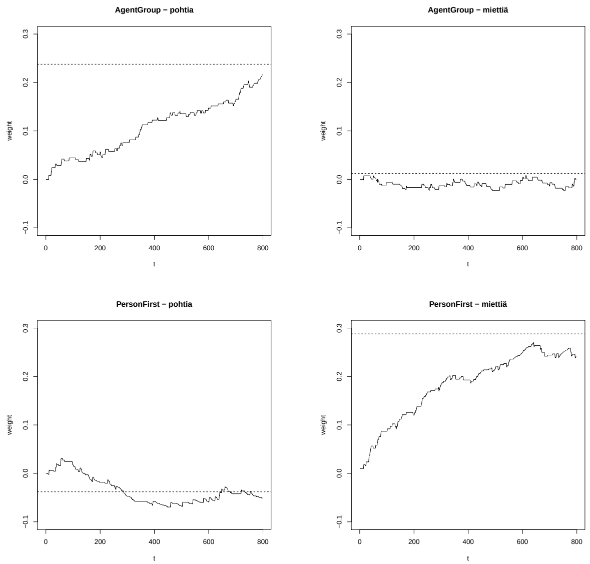
Outline

Simple vs. complex settings – QITL-1 revisited

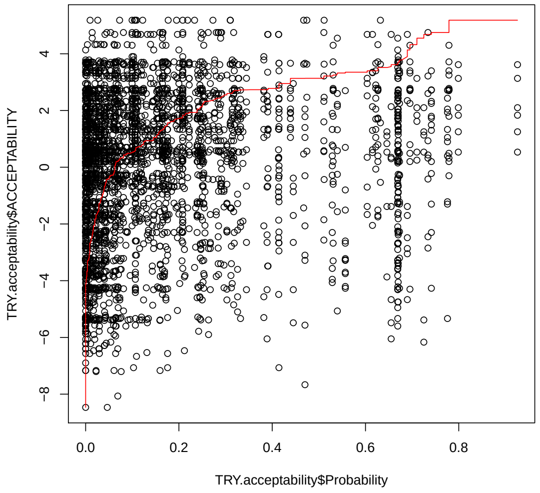
- Arppe and Järvikivi (2002, 2007)
- *Person* (FIRST PERSON SINGULAR or not) and *Countability* (COLLECTIVE or not) of AGENT/SUBJECT of Finnish verb synonym pair *miettiä* vs. *pohtia* 'think, ponder':

Forced-choice		Frequency	Acceptability	
Dispreferred	Preferred	(relative)	Unacceptable	Acceptable
∅	miettiä+SG1 pohtia+COLL	Frequent	∅	miettiä+SG1 pohtiaä+COLL
miettiä+COLL pohtia+SG1	∅	Rare	miettiä+COLL	pohtia+SG1

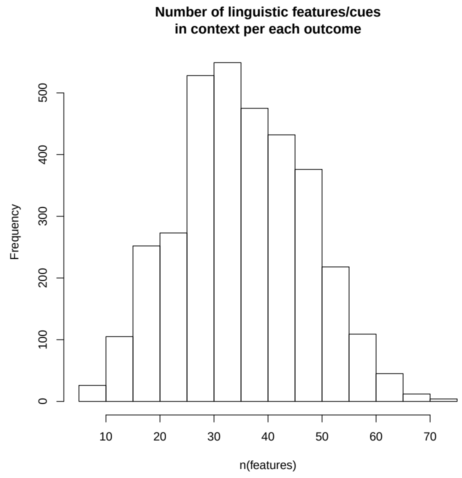
QITL-1 through the lens of NDL



QITL-1 through the lens of QITL-6
(courtesy of Dagmar Divjak)



Simple vs. complex settings – QITL-2 revisited



QITL-4 revisited – NDL vs. statistical classifiers

	$\lambda_{\text{prediction}}$	$\tau_{\text{classification}}$	accuracy
Polytomous logistic regression (One-vs-rest)	0.368	0.488	**0.645**
Polytomous mixed logistic regression (Poisson reformulation)			
• 1|Section	0.360	0.482	0.640
• 1|Author	0.358	0.481	0.640
• 1|Section + 1|Author	0.358	0.481	0.640
Support Vector Machine	0.340	0.466	0.629
Memory-Based Learning (TiMBL)	0.286	0.422	0.599
Random Forests	0.326	0.455	0.621
Naive Discriminative Learning	0.346	0.471	0.632

Table: Classification diagnostics for models fitted to the Finnish data set ($n = 3404$).

Outline

The Rescorla-Wagner equations

- Goal of naïve discriminative learner: predict an outcome O based on presence or absence of a set of cues $C_1, \dots, C_n$
- An event $(\mathbf{c}, o)$ is formally described by indicator variables

$$c_i = \begin{cases} 1 & \text{if } C_i \text{ is present} \\ 0 & \text{otherwise} \end{cases} \qquad o = \begin{cases} 1 & \text{if } O \text{ results} \\ 0 & \text{otherwise} \end{cases}$$

- Given cue-outcome associations $\mathbf{v} = (V_1, \dots, V_n)$ of learner, the activation level of the outcome O is

$$\sum_{j=1}^{n} c_j V_j$$

- Associations $\mathbf{v}^{(t)}$ as well as cue and outcome indicators $(\mathbf{c}^{(t)}, o^{(t)})$ depend on time step t

The Rescorla-Wagner equations

- Rescorla and Wagner (1972) proposed the R-W equations for the change in associations given an event $(\mathbf{c}, o)$:

$$\Delta V_i = \begin{cases} 0 & \text{if } c_i = 0 \\ \alpha_i \beta_1 \big(\lambda - \sum_{j=1}^{n} c_j V_j\big) & \text{if } c_i = 1 \wedge o = 1 \\ \alpha_i \beta_2 \big(0 - \sum_{j=1}^{n} c_j V_j\big) & \text{if } c_i = 1 \wedge o = 0 \end{cases}$$

with parameters

$\lambda > 0$	target activation level for outcome O
$\alpha_i > 0$	salience of cue C_i
$\beta_1 > 0$	learning rate for positive ovents $(o = 1)$
$\beta_2 > 0$	learning rate for negative ovents $(o = 0)$

The Widrow-Hoff rule

- The W-H rule (Widrow and Hoff 1960) is a widely-used simplification of the R-W equations:

$$\Delta V_i = \begin{cases} 0 & \text{if } c_i = 0 \\ \beta \big(1 - \sum_{j=1}^{n} c_j V_j\big) & \text{if } c_i = 1 \wedge o = 1 \\ \beta \big(0 - \sum_{j=1}^{n} c_j V_j\big) & \text{if } c_i = 1 \wedge o = 0 \end{cases}$$
$$= c_i \beta \big(o - \textstyle\sum_{j=1}^{n} c_j V_j\big)$$

with parameters

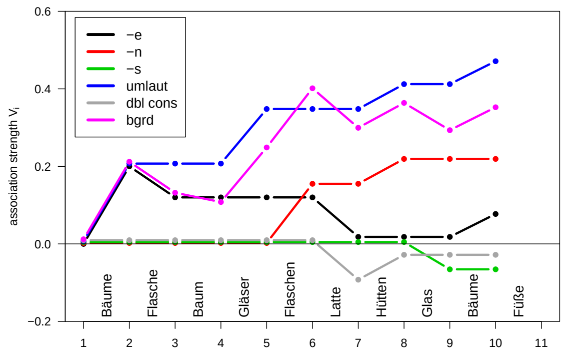
$\lambda = 1$	target activation level for outcome O
$\alpha_i = 1$	salience of cue C_i
$\beta_1 = \beta_2$ $= \beta > 0$	global learning rate for positive and negative events

A simple example: German noun plurals

t	word	o pl?	c_1 −e	c_2 −n	c_3 −s	c_4 umlaut	c_5 dbl cons	c_6 bgrd
1	Bäume	1	1	0	0	1	0	1
2	Flasche	0	1	0	0	0	0	1
3	Baum	0	0	0	0	0	0	1
4	Gläser	1	0	0	0	1	0	1
5	Flaschen	1	0	1	0	0	0	1
6	Latte	0	1	0	0	0	1	1
7	Hütten	1	0	1	0	1	1	1
8	Glas	0	0	0	1	0	0	1
9	Bäume	1	1	0	0	1	0	1
10	Füße	1	1	0	0	1	0	1

A simple example: German noun plurals

t	$\sum c_j V_j$	V_1	V_2	V_3	V_4	V_5	V_6
10	.882	.077	.217	-.070	.464	-.038	.340
Füße	1	1	0	0	1	0	1
	o	c_1	c_2	c_3	c_4	c_5	c_6

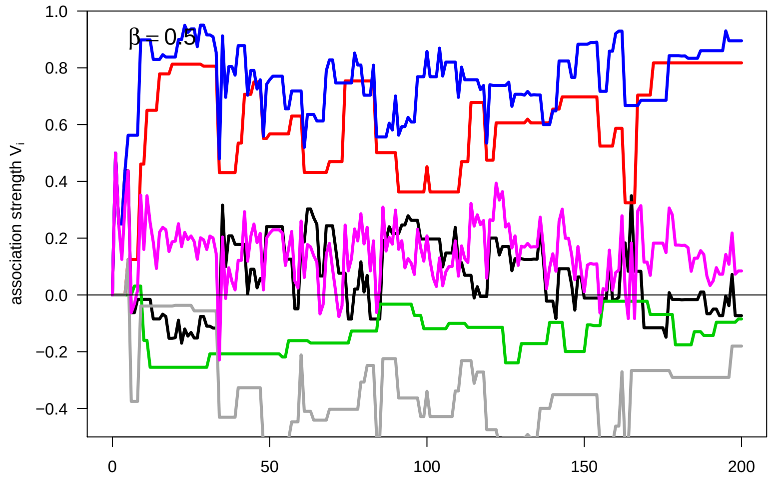


A stochastic NDL learner

- A specific event sequence $(\mathbf{c}^{(t)}, o^{(t)})$ will only be encountered in controlled experiments
- For applications in corpus linguistics, it is more plausible to assume that events are randomly sampled from a population of event tokens $(\mathbf{c}^{(k)}, o^{(k)})$ for $k = 1, \ldots, m$
  - ☞ event types listed repeatedly proportional to their frequency
- I.i.d. random variables $\mathbf{c}^{(t)} \sim \mathbf{c}$ and $o^{(t)} \sim o$
  - ☞ distributions of $\mathbf{c}$ and o determined by population
- NDL can now be trained for arbitrary number of time steps, even if population is small (as in our example)
  - study asymptotic behaviour of learners
  - convergence ➜ stable "adult" state of associations

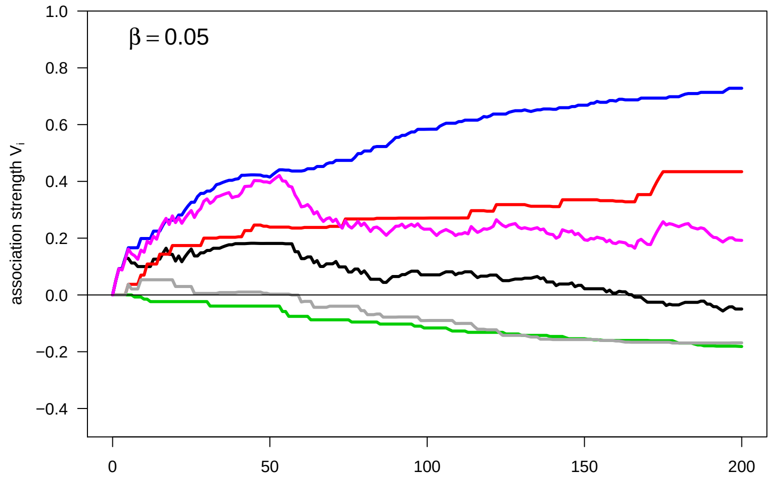
A stochastic NDL learner

Effect of the learning rate β



A stochastic NDL learner

Effect of the learning rate β



Outline

Expected activation levels

- Since we are interested in the general behaviour of a stochastic NDL, it makes sense to average over many individual learners to obtain expected associations $\mathrm{E}[V_j^{(t)}]$

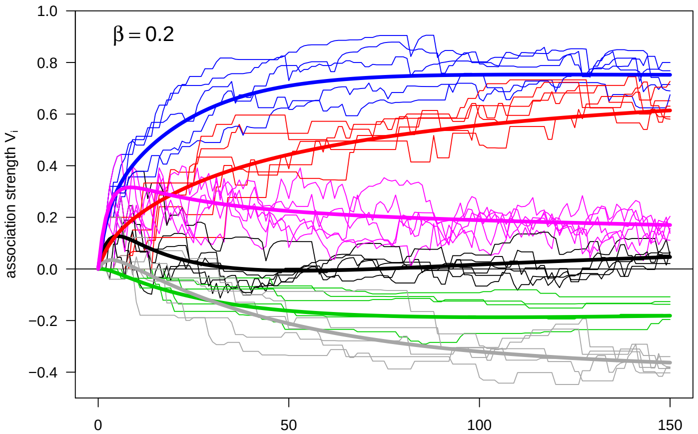
$$\mathrm{E}[V_j^{(t+1)}] = \mathrm{E}[V_j^{(t)}] + \mathrm{E}[\Delta V_j^{(t)}]$$

$$\begin{aligned}\mathrm{E}[\Delta V_j^{(t)}] &= \mathrm{E}\left[c_i \beta \big(o - \textstyle\sum_{j=1}^n c_j V_j^{(t)}\big)\right] \\ &= \beta \cdot \left(\Pr(C_i, O) - \textstyle\sum_{j=1}^n \Pr(C_i, C_j)\mathrm{E}[V_j^{(t)}]\right)\end{aligned}$$

- c_i and c_j are independent from $V_j^{(t)}$
- indicator variables: $\mathrm{E}[c_i o] = \Pr(C_i, O)$; $\mathrm{E}[c_i c_j] = \Pr(C_i, C_j)$

Expected activation levels

$$\mathrm{E}[\Delta V_j^{(t)}] = \beta \cdot \left(\Pr(C_i, O) - \textstyle\sum_{j=1}^n \Pr(C_i, C_j)\mathrm{E}[V_j^{(t)}]\right)$$



The Danks equilibrium

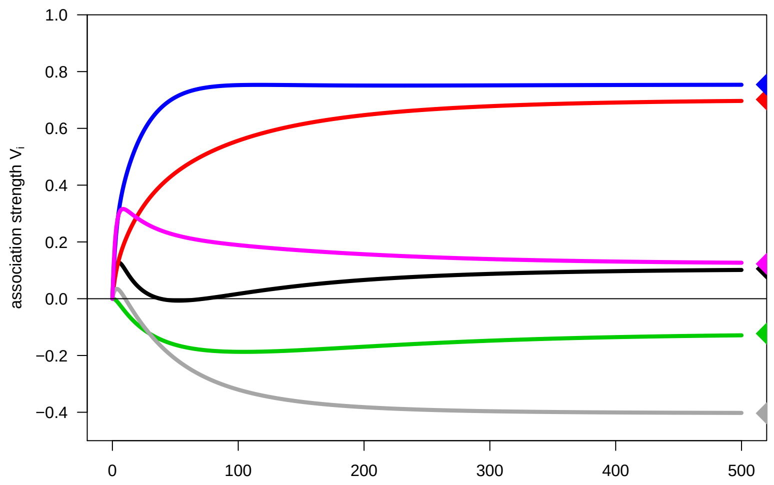
- If $\mathrm{E}[V_i^{(t)}]$ converges, the asymptote $V_i^* = \lim_{t\to\infty} \mathrm{E}[V_i^{(t)}]$ must satisfy the Danks equilibrium conditions $\mathrm{E}[\Delta V_i^*] = 0$, i.e.

$$\Pr(C_i, O) - \textstyle\sum_{j=1}^n \Pr(C_i, C_j) V_j^* = 0 \quad \forall i$$

(Danks 2003, p. 113)

- Now there is a clear interpretation of the Danks equilibrium as the stable average associations reached by a community of stochastic learners with input from the same population
  - ☞ allows us to compute the "adult" state of NDL without carrying out a simulation of the learning process

The Danks equilibrium



Matrix notation

$$\mathbf{X} = \begin{bmatrix} c_1^{(1)} & \cdots & c_n^{(1)} \\ c_1^{(2)} & \cdots & c_n^{(2)} \\ \vdots & & \vdots \\ c_1^{(m)} & \cdots & c_n^{(m)} \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} o^{(1)} \\ o^{(2)} \\ \vdots \\ o^{(m)} \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} V^{(1)} \\ \vdots \\ V^{(n)} \end{bmatrix}$$

$$\begin{bmatrix} \Pr(C_1, O) \\ \vdots \\ \Pr(C_n, O) \end{bmatrix} = \frac{1}{m}\mathbf{X}^T\mathbf{z} \qquad \begin{bmatrix} \Pr(C_1, C_1) & \cdots & \Pr(C_1, C_n) \\ \vdots & & \vdots \\ \Pr(C_n, C_1) & \cdots & \Pr(C_n, C_n) \end{bmatrix} = \frac{1}{m}\mathbf{X}^T\mathbf{X}$$

Danks equilibrium: $\mathbf{X}^T\mathbf{z} = \mathbf{X}^T\mathbf{X}\mathbf{w}^*$

Matrix notation: German noun plurals

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} V^{(1)} \\ \vdots \\ V^{(n)} \end{bmatrix}$$

$$\begin{bmatrix} .3 \\ .2 \\ .0 \\ .5 \\ .1 \\ .6 \end{bmatrix} = \frac{1}{m}\mathbf{X}^T\mathbf{z} \qquad \begin{bmatrix} .5 & .0 & .0 & .3 & .1 & .5 \\ .0 & .2 & .0 & .1 & .1 & .2 \\ .0 & .0 & .1 & .0 & .0 & .1 \\ .3 & .1 & .0 & .5 & .1 & .5 \\ .1 & .1 & .0 & .1 & .2 & .2 \\ .5 & .2 & .1 & .5 & .2 & 1 \end{bmatrix} = \frac{1}{m}\mathbf{X}^T\mathbf{X}$$

Outline

The single-layer perceptron (SLP)

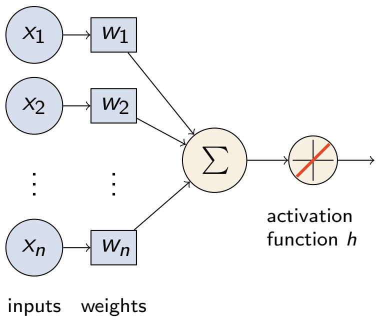
SLP (Rosenblatt 1958) is most basic feed-forward neural network

- numeric inputs $x_1, \ldots, x_n$
- output activation $h(y)$ based on weighted sum of inputs

$$y = \sum_{j=1}^{n} w_j x_j$$

- h = Heaviside step function in traditional SLP
- even simpler model: $h(y) = y$
- cost wrt. target output z:

$$E(\mathbf{w}, \mathbf{x}, z) = \left(z - \sum_{j=1}^{n} w_j x_j\right)^2$$



SLP training: the delta rule

- SLP weights are learned by gradient descent training: for a single training item $(\mathbf{x}, z)$ and learning rate $\delta > 0$

$$\begin{aligned}\Delta w_i &= -\delta \frac{\partial E(\mathbf{w}, \mathbf{x}, z)}{\partial w_i} \\ &= 2\delta x_i \left(z - \sum_{j=1}^{n} x_j w_j\right) \\ &= \beta c_i \left(o - \sum_{j=1}^{n} c_j V_j\right)\end{aligned}$$

- Perfect correspondence to W-H rule with

$$V_i = w_i \qquad c_i = x_i \qquad o = z \qquad \beta = 2\delta$$

Batch training

- Neural networks often use batch training, where all training data are considered at once instead of one item at a time
- The corresponding batch training cost is

$$\begin{aligned}E(\mathbf{w}) &= \frac{1}{m} \sum_{k=1}^{m} E(\mathbf{w}, \mathbf{x}^{(k)}, z^{(k)}) \\ &= \frac{1}{m} \sum_{k=1}^{m} \left(z^{(k)} - \sum_{j=1}^{n} w_j x_j^{(k)}\right)^2\end{aligned}$$

- Similar to stochastic NDL, batch training computes the expected weights $\mathrm{E}\left[\mathbf{w}^{(t)}\right]$ for an SLP with stochastic input
- Minimization of $E(\mathbf{w})$ = linear least-squares regression

Linear least-squares regression

- Matrix formulation of the linear least-squares problem:

$$\begin{aligned}E(\mathbf{w}) &= \frac{1}{m} \sum_{k=1}^{m} \left(z^{(k)} - \sum_{j=1}^{n} w_j x_j^{(k)}\right)^2 \\ &= \frac{1}{m} (\mathbf{z} - \mathbf{X}\mathbf{w})^T (\mathbf{z} - \mathbf{X}\mathbf{w})\end{aligned}$$

- Minimum of $E(\mathbf{w})$, the L_2 solution, must satisfy $\nabla E(\mathbf{w}^*) = \mathbf{0}$, which leads to the normal equations

$$\mathbf{X}^T \mathbf{z} = \mathbf{X}^T \mathbf{X} \mathbf{w}^*$$

- Normal equations = Danks equilibrium conditions
- Regression theory shows that batch training / stochastic NLP converges to the unique* solution of the L_2 problem

What have we learned?

$$\boxed{\begin{array}{ccccc} \text{stochastic} & = & \text{batch} & = & L_2 \text{ regression} \\ \text{NDL} & = & \text{SLP} & & \end{array}}$$

☞ These equivalences also hold for the general R-W equations with arbitrary values of α_i, β_1, β_2 and λ (see paper)

Outline

Effects of R-W parameters

$\beta > 0$: learning rate ➜ convergence of individual learners

$\lambda \neq 1$: linear scaling of associations / activation (obvious)

$\alpha_i \neq 1$: salience of cue C_i determines how fast associations are learned, but does not affect the final stable associations (same L_2 regression problem)

$\beta_1 \neq \beta_2$: different positive/negative learning rates *do* affect the stable associations; closely related to prevalence of positive and negative events in the population

What about logistic regression?

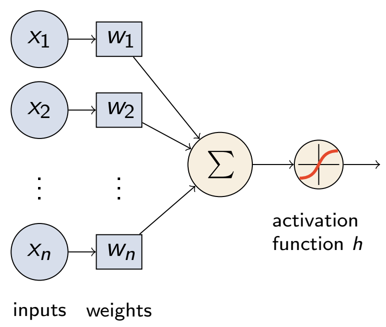
Logistic regression is the standard tool for predicting a categorical response from binary features

- can be expressed as SLP with probabilistic interpretation
- uses logistic activation function

$$h(y) = \frac{1}{1 + e^{-y}}$$

- and Bernoulli cost

$$E(\mathbf{w}, \mathbf{x}, z) = \begin{cases} -\log h(y) & \text{if } z = 1 \\ -\log(1 - h(y)) & \text{if } z = 0 \end{cases}$$



What about logistic regression?

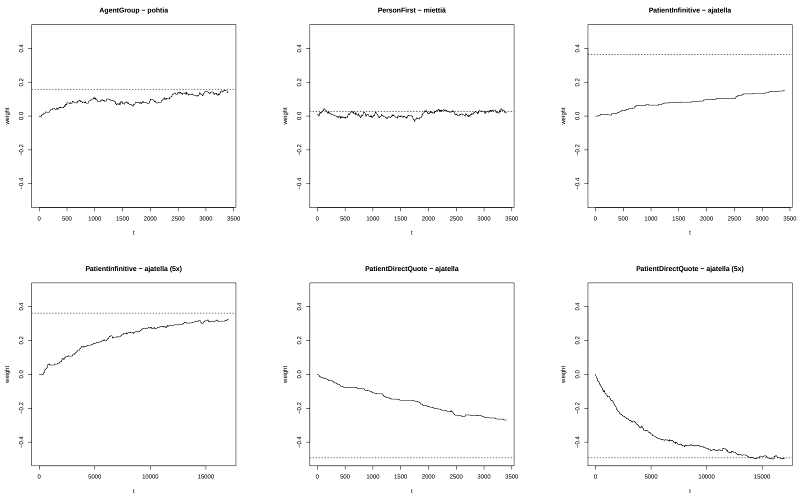
- Gradient descent training leads to delta rule that corresponds to a modified version of the R-W equations

$$\Delta V_i = \begin{cases} 0 & \text{if } c_i = 0 \\ \beta\left(1 - h\left(\sum_{j=1}^{n} c_j V_j\right)\right) & \text{if } c_i = 1 \land o = 1 \\ \beta\left(0 - h\left(\sum_{j=1}^{n} c_j V_j\right)\right) & \text{if } c_i = 1 \land o = 0 \end{cases}$$

- Same as original R-W, except that activation level is now transformed into probability $h(y)$
- But no easy way to analyze stochastic learning process (batch training $\neq$ expected value of single-item training)
- Less robust for highly predictable outcomes ➜ **w** diverges

Outline

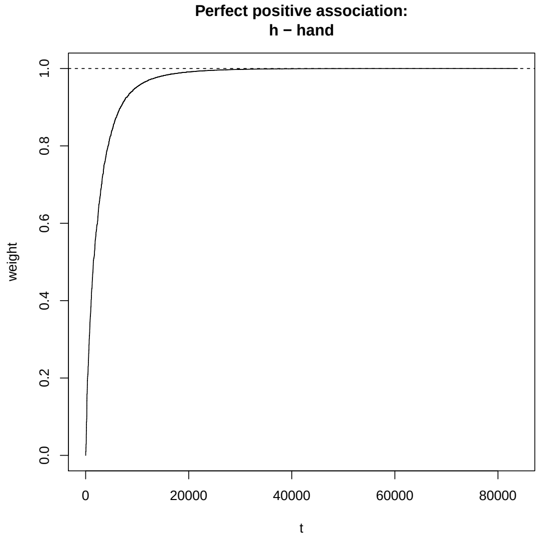
Some NDL simulation runs



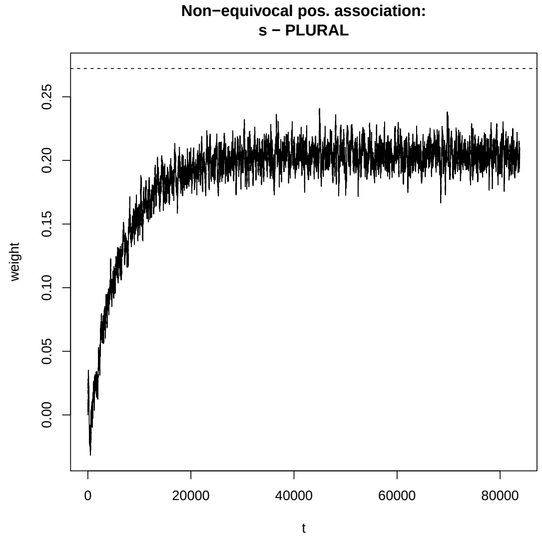
Convergence vs. non-convergence – artificial data

word form	frequency	outcomes	cues
hand	10	hand_NIL	h_a_n_d
hands	20	hand_PLURAL	h_a_n_d_s
land	8	land_NIL	l_a_n_d
lands	3	land_PLURAL	l_a_n_d_s
and	35	and_NIL	a_n_d
sad	18	sad_NIL	s_a_d
as	35	as_NIL	a_s
lad	102	lad_NIL	l_a_d
lad	54	lad_PLURAL	l_a_d
lass	134	lass_NIL	l_a_s_s

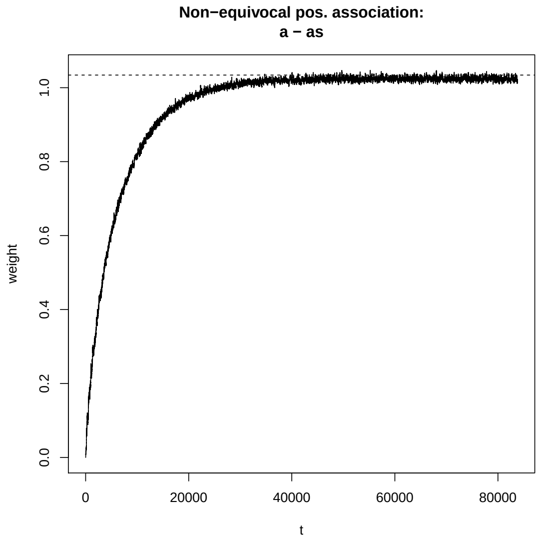
Perfect positive association ➜ convergence



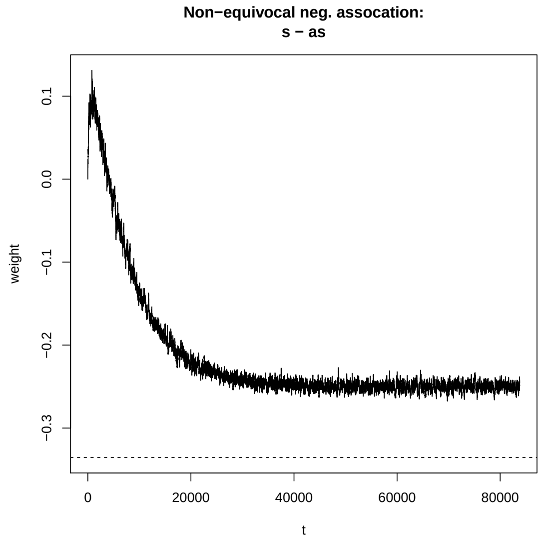
Moderate positive association ➜ non-convergence



Perfect positive association ➜ convergence



Moderate negative association ➜ non-convergence



Outline

Summary & next steps

stochastic	=	batch	=	L_2 regression
NDL	=	SLP		

- How many training steps are needed for a stochastic NDL learner to converge to the Danks equilibrium?
- Are there cases of non-convergence? If yes, why?
- Does NDL accuracy always improve with more cues and more training data? If not, why?
- How does logistic regression behave as incremental learner?
- Which sequences / patterns in the input data lead to significantly different behaviour from stochastic learner?

Acknowledgements 1/2

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Acknowledgements 2/2

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