Calculus Problem Book:
Form V and Form VI

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Contents

Preface

The following short booklet contains a collection of problems with solutions that roughly follow the topics typically covered in the calculus sequence for Form V and Form VI mathematics classes at Ethical Culture Fieldston School. The hope is to provide a book of problems of varying difficulty that will provide non-routine exercises as well as to record general themes and approaches to tackling these types of problems in general. Ideally these problems can be useful to all courses that cover these topics. Problems are roughly categorized by difficulty level and notated via *, **, or *** in increasing difficulty. We have adopted problems from memory, and from years of working with the material. The following references were consulted for problems as well as other considerations such as narrative structure in introducing various topics and thematic development of the material. [FGI96], [Pat04], [Pis14], [more later].

CHAPTER 1

Functions and Other Miscellaneous Content

1. Set Notation and Numbers

Exercise 1   Sets, Unions and Intersections
Set $A \subset B$ iff every element of A in also in B. Set $A \supset B$ iff every element of B is also in A and two sets are equal iff both $A \supset B$ and $A \subset B$.

(1) If $A \subset B$ then what is the set $A \cap B$?

(2) If $A \subset B$ then what is the set $A \cup B$?

(3) If $A \cup B = B$, give an argument that explains why $A \subset B$.

Exercise 2   Complements
Given a set A, and a set B we can define the complement of that set, A^c or sometimes: $\overline{A}$, A', or $B - A$, as the elements in B but not in A. Notice A^c, A' and $\overline{A}$ do not reference the set B and are often used when no confusion can arise or there is some all encompassing universal set U that contains A.

(1) Given a universal set U and A, B contained in U. Show that $(A \cap B)^c = A^c \cup B^c$ by first showing that $(A \cap B)^c \subset A^c \cup B^c$. Then show that $(A \cap B)^c \supset A^c \cup B^c$. This property is often called DeMorgan's Law.
(2) Describe the elements that are contained in the set $(A - B) \cup (B - A)$. This set is called the symmetric difference of A and B, or $A \Delta B$.

** *Exercise 3   Working with Sets*

In the previous problem we defined the symmetric difference. The following exercises will work through some properties of this operation on sets. Recall that to show two sets A, B are equal you must show that $A \subset B$ ie. every element is A is also in B, as well as that $B \subset A$, ie. every element of B is also in A.

(1) Show that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ as sets. These are two different ways to think about the set that is the symmetric difference of two sets $A \Delta B$. Let A, B be arbitrary sets.
(2) Show that $A \Delta B = B \Delta A$ by showing that every element in the set on the left is in the set on the right and vice versa.
(3) Show that $A \Delta \varnothing = A$. Where $\varnothing$ is the empty set.
(4) Show that $A \Delta A = \varnothing$.

2. Working with Inequalities

3. Polynomials

Exercise 4 Polynomials I
Let $f(x) = x^2 - 3$ and $g(x) = x^3 - 2x$. Both f and g are polynomials. You can form new polynomials by adding, subtracting, multiplying, and composing.

(1) Find $(f + g)(x)$
(2) Find $(f - 2g)(x)$
(3) Find $(f \cdot g)(x)$
(4) Find $f \circ g)(x)$
(5) Explain why $(f/g)(x)$ is not a polynomial.

solution 4

* *Exercise 5 Defining Polynomials*
A polynomial f has the general form $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n$, where $n \in \mathbb{Z}$ and $a_i \in \mathbb{R}$. Show that adding, subtracting, or multiplying two polynomials always results in a polynomial. Explain why this is not the case for division.

solution 5

Exercise 6 Polynomials II
A function f is even if and only if $f(x) = f(-x)$ for all x in its domain. A function is odd if and only if $f(-x) = -f(x)$ for all x in its domain.

(1) Give an example of an even polynomial.
(2) Give an example of an odd polynomial.

4. Trigonometric Functions

5. Miscellaneous

CHAPTER 2

Limits and Continuity

1. Intuitive Notion

2. Epsilon and Delta

* *Exercise 7 Dirichlet Function*

The Dirichlet function D is define as $D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$. Two sets of numbers are *not seperable* if it is impossible to find an open interval around an element from one set that contains only elements from that set and not the other. Rational numbers and irrational numbers are not separable. Use this to argue that $\lim_{x \to a} D(x)$ does not exist for any value of a. *solution* 7

3. Continuity

* *Exercise 8 A Bounded Function around Zero*

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|$ for all x in the domain of f. Prove that f is continuous at $x = 0$.

solution 8

* *Exercise 9 Some Points Don't Move That Much*

Let $f : [0, 1] \to [0, 1]$ be a function such that $|f(x)| \leq |x|$ for all x in the domain of f. Prove that f is continuous at $x = 0$.

solution 9

4. Intermediate Value Theorem

* *Exercise 10 Degree 3 Polynomial with Real Roots*

Let $f(x) = (x-a)(x-b)+(x-b)(x-c)+(x-a)(x-c)$ where a, b, c are distinct real numbers. Prove that f must have 3 real distinct zeroes.

* *Exercise 11 Fixed points*

Let f be a continous function from the unit interval to itself. $f : [0, 1] \to [0, 1]$. Show that f must have at least one fixed point x_0. In other words, $f(x_0) = x_0$.

** *Exercise 12 Antipodes*

Two points on a sphere are antipodal if the line connecting them passes through the center of the sphere. They are 'directly opposite' one another, or maximally distant from one another while still being on the sphere.

(1) If you assume that the earth is a sphere and that temperature varies continuously from point to point on the earth, show that there must be a pair of antipodal points that are the same temperature at a given moment in time.

(2) If we also assume that air pressure varies continuously across the earth, show that there must be a pair of antipodal points on the surface of the earth that have the same temperature and air pressure at a given moment in time.

CHAPTER 3

The Derivative

1. Limit Definition and Properties of the Derivative

Exercise 13 Introducing a Discontinuity

Let $f(x) = |x|$. Use the limit definition of the derivative to show that $f'(x) = \dfrac{|x|}{x}$. Show using the definition of continuity how the derivative

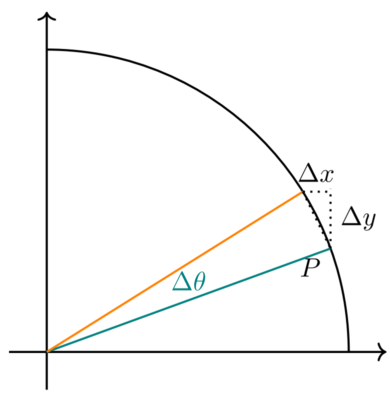
2. Tangent Line Problem

3. Higher Order Derivatives

4. Power Rule and others

* *Exercise 14 Unit Circle and Sine*

Let $P = (\cos\theta, \sin\theta)$. Use similar triangles to calculate an approximation for $\dfrac{\Delta y}{\Delta\theta}$ and explain how this shows what the derivative of $f(x) = \sin(x)$ is. Do the same for $g(x) = \cos(x)$ by looking at $\dfrac{\Delta x}{\Delta\theta}$.



solution 14

5. Implicit Differentiation

CHAPTER 4

Applications of the Derivative

CHAPTER 5

The Integral

CHAPTER 6

The Fundamental Theorem of Calculus

CHAPTER 7

Logarithms, Exponentials, and Inverses

CHAPTER 8

Methods of Integration

CHAPTER 9

Differential Equations

1. Slope Fields and Euler's Method

2. Separation of Variables

3. Modeling

4. Logistic Growth

5. Systems of Differential Equations

6. Reduce 2nd order equations to two 1st order equations

CHAPTER 10

More Applications of Integration

CHAPTER 11

Infinite Series

CHAPTER 12

Conics, Parametric Equations, Polar Coordinates

CHAPTER 13

Vectors and the Geometry of $\mathbb{R}^3$

CHAPTER 14

Vector Valued Functions

CHAPTER 15

The Partial Derivative and Function of Several Variables

CHAPTER 16

The Gradient and the Method of Lagrange Multipliers

CHAPTER 17

The Derivative of maps from $\mathbb{R}^n \to \mathbb{R}^m$

CHAPTER 18

Longer Problem Sets

Working with Parameters and the Derivative

In this exploration we will work with derivative to find tangent lines to a curve. We will also introduce working with parameters and see how this gives us a set of tools to solve more complex problems. We will end by using our skill with parameters to explore a little bit of the early history of Calculus by taking a look at Descartes' Method of Normals and Fermat's Method of Adequality.

First Let's Look at Tangent Lines to a Parabola.

(1) Let $f(x) = x^2 + 1$ and point $C(0, -2)$. Find the equation of the two lines that pass through C and are tangent to f.

(2) Let C vary its position along the y-axis. Let $C(0, k)$. Find the equations of the two lines that pass through C and are tangent to f. Notice your solutions will be in terms of the parameter k.

(3) Lastly, let's generalize and let C be any point in the plane $C(h, k)$. What are the equations of the two lines that pass through C and are tangent to f, in terms of the parameters h and k?

(4) Notice this gives you the equations for the lines through any point in the plane that are tangent to the given parabola. Demonstrate the ease with which you can find these lines that are tangent to the parabola, by finding them for $P(5, 3)$ and $Q(3, -5)$.

(5) What do these equations tell you about the family of lines tangent to the graph of f? What points on the plane have no lines through them that are tangent to the curve? How can you see this from the equations? Explain how this is different than just looking at $f'(x)$?

Descartes' Method Of Normals versus Fermat's Method. In this next section we will take a look at two early methods of finding slopes of lines tangent to a a curve.

An Algorithm to Find Roots

We will learn how to find the zeroes of a function $f(x)$ using Newton's Method, which involves using the derivative to find lines tangent to the curve and using these to successively better approximate the zeros of a function.

(1) First recognize that finding the zero of a line is straightforward. We will reduce finding the zeroes of an arbitrary differentiable function to finding the zeroes of a bunch of lines. Let $y - a = m(x - b)$ be a line through (b, a) with slope m. Find its x-intercept in terms of a, b and m.

(2) We can get a sense for how the method works by looking at a quadratic $f(x) = -x^2 + 5x$.

   (a) First let's solve for the zeroes by factoring or using the quadratic formula. What are the zeroes of f?

   (b) Let's use the point $(4, 4)$ on f, and call our initial x value $x_0 = 4$. Find the line tangent to f at $(4, 4)$. Let's call this line l_1.

   (c) What is the x-intercept of l_1? Call the x-coordinate of the zero of l_1, x_1.

   (d) Notice x_1 is not equal to a zero of f, but it is closer than x_0. Let's repeat this process but using $(x_1, f(x_1))$.

   (e) What is the line tangent to f at $(x_1, f(x_1))$? Find its x-intercept and call it x_2. It should be close to one of the zeroes at this point.

   (f) Give a sketch of each tangent line that you found in this problem and its x-intercept. Detail how this process works to approximate the zeroes of a function.

   (g) If your initial guess was $x_0 = 3$ Would the process converge to $x = 5$ in more or fewer steps. Explain without carrying out the calcuations.

   (h) What initial guess would cause the process to fail? Explain why.

(3) The power of Newton's Method is that the same process can be applied to any differentiable function to find its zeroes. One of its drawbacks is that it can involve a lot of calculation. Let's offload some of this to a spreadsheet, but first let's develop an expression to generate the sequence of x-coordinates, $\{x_0, x_1, x_2, \dots\}$ that should converge to the a zero of the given function.

   (a) Let x_0 be your first guess at a zero. Then as we saw before we are looking for the tangent line to $f(x)$ that passes through $x_0, f(x_0)$. x_1 is the x-intercept of that tangent line. Find an expression for x_1 in terms of $x_0, f(x_0), andf'(x_0)$. Generalize this to an expression that gives x_n.

(b) Look at the spreadsheet linked here.

(c) Recall how difficult it is to find where the sine curve intersects with various lines. We will find when $y = 2\sin(x)$ meets $y = x$ by letting $g(x) = 2\sin(x) - x$ and solving for its zeroes.

(d) Use the IVT to get a guess for where $g(x)$ has zeroes. Then use the spreadsheet to find these zeroes with a reasonabe first guess determined by your use of the IVT. Take a screen shot of your spreadsheet that shows the calculations that give each zero.

1. Instantaneous Rate of Turning

Efficient Foraging

An Application of Optimization[1]

Foraging animals, whose food is arranged in clumps must make a decision about how long to stay in a given clump of food before moving on in search of another. We will try to arrive at an optimal strategy to foraging given a few simplifying assumptions. Let us first assume that patches of food are distributed uniformly (equally spaced) and that they are equally abundant, meaning an animal's optimal time spend in one patch would be the same for any other patch.

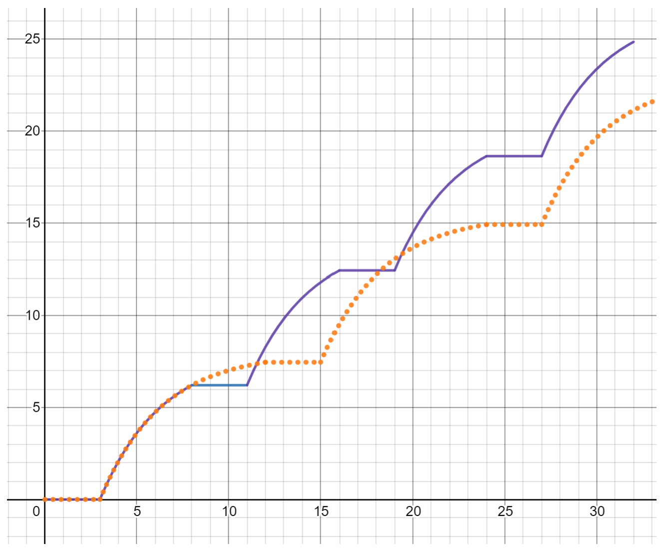
(1) Let us first try to model the rate at which resources can be gathered when arriving at a fresh patch.
   (a) What would a sketch of the amount of food gathered versus time look like assuming at time zero that no food had been gathered and the patch has a total of L units of food?
   (b) Explain why your sketch has the shape it has. What physical idea about the patch of food are you trying to capture? What criteria do you think would be common to all proposed models? Explain.
   (c) Does the graph $F(t) = L(1 - e^{-t/2})$ meet the criteria you gave in the previous question? Explain. Do you have a different function in mind? If so compare it to $F(t)$. What are its comparative strengths and weaknesses.

(2) Another factor to take into consideration is how long it takes to get from one patch of food to the next.
   (a) Let's look to extend our model from 1c by assuming it takes A seconds to travel from each patch to a new patch, and that the closest patch to home is also A seconds away. What does the graph of F versus time look like for an animal that leaves home and forages in one patch? Give a labeled sketch.
   (b) Give a piecewise defined function that will model the above situation. Use $f(t) = L(1 - e^{-t/2})$ as the food gathering profile for one patch and A as the time it takes to get to the patch.
   (c) Sketch and give a piecewise defined function that will model F versus time for an animal that leaves home and forages in two patches of food, where every patch is A seconds away and the animal spends B seconds in each patch. It may be easier to define $F(t)$ by using translates of $f(t)$.

(3) Let's take a moment to look at the various characteristics of our model so far.
   (a) What are some of the assumptions we are making in modeling this situation?
   (b) The model gives energy gathered(F) in terms of time spent foraging (t). What are the three other factors or parameters that affect this model?
   (c) Which of these parameters does the foraging animal have influence on?

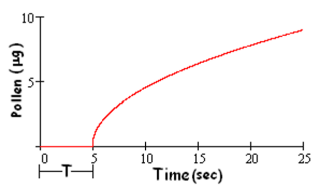
[1] adapted from Graves at NCSSM

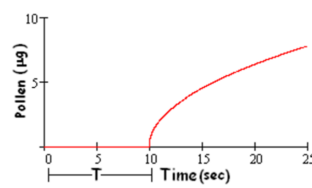
(d) What is the same and what is different in the two following foraging strategies?



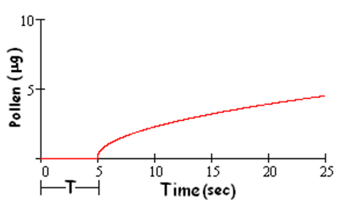
(4) Our main goal then is to model an animal's foraging behavior in choosing ______________________ given a certain density of food patches (time between patches), and richness of food patches (ie.energy gathering profile of a patch) so that it will maximize the food it collects in a given time.

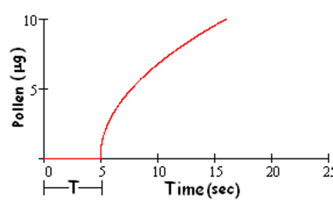
(a) Here we have varied the density of the food patches while keeping each patch's food gathering profile the same. For which situation will the animal stay longer in one food patch? Why?





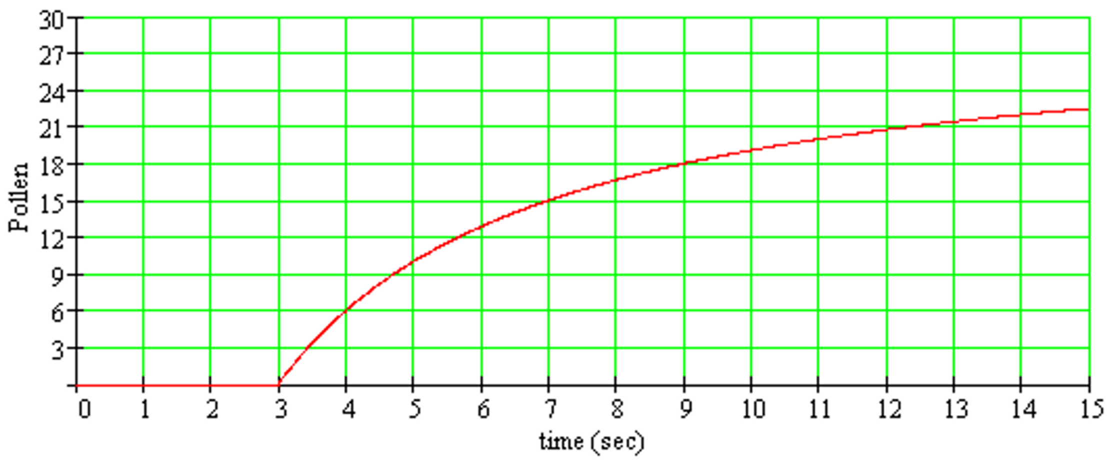
(b) Here we have varied the food gathering profile of each patch, while keeping the density the same. For which situation will the animal stay onlger in one food patch? Why?





(5) Let's get more familiar with these ideas with some concrete models of bees foraging for pollen.

(a) First, let's look at things graphically. Given the travel distance of 3 seconds and the food gathering profile given in the figure below, how much pollen is gathered over 12 seconds, if the bee stays on each flower 1 second versus 3 seconds.



(b) Assuming the food gathering profile is given by $f(t) = \sqrt{t}$, and a travel time of 3 seconds, let's look at the effect of various foraging times (time spent on a flower) on total collected pollen as well as average rate of pollen collected. Fill out the following table.

total time	foraging time	total pollen	avg rate of pollen collection
8	1		
10	2		
12	3		
12	1		
14	4		
15	2		

(c) Which foraging time seems optimal? Why?

(6) It may then be reasonable to assume that animals will choose the foraging time that maximizes the average rate of food gathering. Let $F(t)$ be one cycle of foraging that includes travel time A with a food gathering profile $f(t)$.

So $F(t) = \begin{cases} 0 & , \quad 0 \leq x < A \\ f(t-A) & , \quad A \leq x \end{cases}$

(a) The average rate of food gathering is given by $G(t) = \dfrac{F(t)}{t}$. Assuming $t > 0$, what equation must critical points of $G(t)$ satisfy? What does this look like on a graph of $F(t)$?

(b) Charnov in establishing the marginal value theorem says, "The predator should leave the patch it is presently in when the marginal capture rate in the patch drops to the average capture rate for the habitat." Explain how this is a restatement of what you found in the part a.

(7) Determine the optimal foraging time B for each of the following food gathering profiles and travel times.

(a) $F(t) = \begin{cases} 0 & , \quad 0 \leq t < 3 \\ \sqrt{t-3} & , \quad t \geq 3 \end{cases}$

(b) $F(t) = \begin{cases} 0 & , \quad 0 \leq t < A \\ K\sqrt{t-A} & , \quad t \geq A \end{cases}$

(c) $F(t) = \begin{cases} 0 & , \quad 0 \leq t < 3 \\ \sqrt[3]{t-3} & , \quad t \geq 3 \end{cases}$

(d) $F(t) = \begin{cases} 0 & , \quad 0 \leq t < A \\ \sqrt[n]{t-A} & , \quad t \geq A \end{cases}$

(8) In the previous set of questions what do n, A, and K represent in our foraging model, and how did changing these parameters affect the optimal foraging time? Explain.

2. When is Venus Bright in the Sky?

3. Rainbows

4. Constructing the Demand Curve

5. The Shape of Bee Hive Cells

6. The Art Gallery Problem

7. Getting to π using Integration by Parts

8. Predicting Peak Oil

9. SIR: A Model for the Spread of Infectious Disease

10. Modeling Air Resistance

11. A Model for Combat

12. A Model for Relationships

CHAPTER 19

Answers

Exercise 4

blah blah

Exercise 5

The expression $A \subset B$ is read, the set A is contained in the set B. Whereas the expression $A \supset B$ is read the set A contains the set B.

(1) $\mathbb{Z} \supset \mathbb{N}$
(2) $\mathbb{R} \supset \mathbb{Q}$
(3) $\mathbb{N} \subset \mathbb{Q}$

Exercise 8

blah blah

Exercise 8

blah blah

Exercise 9

blah blah

Exercise 10

Look at $f(a)$, $f(b)$ and $f(c)$. Since a, b, and c are distinct we can assume they are ordered $a < b < c$, this will then put you in a position to apply IVT.

Exercise 12

Both require looking at the difference in the function evaluated at a pair of antipodes.

Exercise 13

Exercise 14

blah blah

Bibliography

[FGI96] Fomin, Genkin, and Itenberg. *Mathematical Circles (Russian Experience)*. Vol. 7. Mathematical World. American Mathematical Society, 1996.

[Pat04] Patrick. *Introduction to Counting and Probability*. 2nd. AoPS Inc., 2004.

[Pis14] Hossein Pishro-Nik. *Introduction to Probability, Statistics, and Random Processes*. Kappa Research, 2014.