Chapter Five

LOGARITHMIC FUNCTIONS

Contents

5.1 LOGARITHMS AND THEIR PROPERTIES

What Is a Logarithm?

Suppose that a population grows according to the formula $P = 10^t$, where P is the colony size at time t, in hours. When will the population be 2500? We want to solve the following equation for t:

$$10^t = 2500.$$

In Section 4.2, we used a graphical method to approximate t. This time, we introduce a function that returns precisely the exponent of 10 we need.

Since $10^3 = 1000$ and $10^4 = 10{,}000$, and $1000 < 2500 < 10{,}000$, the exponent we are looking for is between 3 and 4. But how do we find the exponent exactly?

To answer this question, we define the *common logarithm function*, or simply the *log function*, written $\log_{10} x$, or $\log x$, as follows.

> If x is a positive number,
>
> $$\log x \text{ is the exponent of 10 that gives } x.$$
>
> In other words, if
>
> $$y = \log x \qquad \text{then} \qquad 10^y = x.$$

For example, $\log 100 = 2$, because 2 is the exponent of 10 that gives 100, or $10^2 = 100$.

To solve the equation $10^t = 2500$, we must find the power of 10 that gives 2500. Using the log button on a calculator, we can approximate this exponent. We find

$$\log 2500 \approx 3.398, \qquad \text{which means that} \qquad 10^{3.398} \approx 2500.$$

As predicted, this exponent is between 3 and 4. The precise exponent is $\log 2500$; the approximate value is 3.398. Thus, it takes roughly 3.4 hours for the population to reach 2500.

Example 1 Rewrite the following statements using exponents instead of logs.

(a) $\log 100 = 2$ (b) $\log 0.01 = -2$ (c) $\log 30 = 1.477$

Solution For each statement, we use the fact that if $y = \log x$ then $10^y = x$.

(a) $2 = \log 100$ means that $10^2 = 100$.
(b) $-2 = \log 0.01$ means that $10^{-2} = 0.01$.
(c) $1.477 = \log 30$ means that $10^{1.477} = 30$. (Actually, this is only an approximation. Using a calculator, we see that $10^{1.477} = 29.9916\ldots$ and that $\log 30 = 1.47712125\ldots$.)

Example 2 Rewrite the following statements using logs instead of exponents.

(a) $10^5 = 100{,}000$ (b) $10^{-4} = 0.0001$ (c) $10^{0.8} = 6.3096$.

Solution For each statement, we use the fact that if $10^y = x$, then $y = \log x$.

(a) $10^5 = 100{,}000$ means that $\log 100{,}000 = 5$.
(b) $10^{-4} = 0.0001$ means that $\log 0.0001 = -4$.
(c) $10^{0.8} = 6.3096$ means that $\log 6.3096 = 0.8$. (This, too, is only an approximation because $10^{0.8}$ actually equals $6.30957344\ldots$.)

Logarithms Are Exponents

Note that logarithms are just exponents! Thinking in terms of exponents is often a good way to answer a logarithm problem.

Example 3 Without a calculator, evaluate the following, if possible:

(a) $\log 1$ (b) $\log 10$ (c) $\log 1{,}000{,}000$
(d) $\log 0.001$ (e) $\log \dfrac{1}{\sqrt{10}}$ (f) $\log(-100)$

Solution

(a) We have $\log 1 = 0$, since $10^0 = 1$.
(b) We have $\log 10 = 1$, since $10^1 = 10$.
(c) Since $1{,}000{,}000 = 10^6$, the exponent of 10 that gives $1{,}000{,}000$ is 6. Thus, $\log 1{,}000{,}000 = 6$.
(d) Since $0.001 = 10^{-3}$, the exponent of 10 that gives 0.001 is -3. Thus, $\log 0.001 = -3$.
(e) Since $1/\sqrt{10} = 10^{-1/2}$, the exponent of 10 that gives $1/\sqrt{10}$ is $-\frac{1}{2}$. Thus $\log(1/\sqrt{10}) = -\frac{1}{2}$.
(f) Since 10 to any power is positive, -100 cannot be written as a power of 10. Thus, $\log(-100)$ is undefined.

Logarithmic and Exponential Functions Are Inverses

The operation of taking a logarithm "undoes" the exponential function; the logarithm and the exponential are inverse functions. For example, $\log(10^6) = 6$ and $10^{\log 6} = 6$. In particular,

For any N,
$$\log(10^N) = N$$
and for $N > 0$,
$$10^{\log N} = N.$$

Example 4 Evaluate without a calculator: (a) $\log\left(10^{8.5}\right)$ (b) $10^{\log 2.7}$ (c) $10^{\log(x+3)}$

Solution Using $\log(10^N) = N$ and $10^{\log N} = N$, we have:

(a) $\log\left(10^{8.5}\right) = 8.5$ (b) $10^{\log 2.7} = 2.7$ (c) $10^{\log(x+3)} = x + 3$

You can check the first two results on a calculator.

Properties of Logarithms

In Chapter 4, we saw how to solve exponential equations such as $100 \cdot 2^t = 337{,}000{,}000$, graphically. To use logarithms to solve these equations, we use the properties of logarithms, which are justified on page 184.

Properties of the Common Logarithm

- By definition, $y = \log x$ means $10^y = x$.
- In particular,
$$\log 1 = 0 \quad \text{and} \quad \log 10 = 1.$$
- The functions 10^x and $\log x$ are inverses, so they "undo" each other:
$$\begin{aligned} \log(10^x) &= x \qquad \text{for all } x, \\ 10^{\log x} &= x \qquad \text{for } x > 0. \end{aligned}$$
- For a and b both positive and any value of t,
$$\log(ab) = \log a + \log b$$
$$\log\left(\frac{a}{b}\right) = \log a - \log b$$
$$\log(b^t) = t \cdot \log b.$$

We can now use logarithms to solve the equation that we solved graphically in Section 4.2.

Example 5 Solve $100 \cdot 2^t = 337{,}000{,}000$ for t.

Solution Dividing both sides of the equation by 100 gives
$$2^t = 3{,}370{,}000.$$
Taking logs of both sides gives
$$\log\left(2^t\right) = \log(3{,}370{,}000).$$
Since $\log(2^t) = t \cdot \log 2$, we have
$$t \log 2 = \log(3{,}370{,}000),$$
so, solving for t, we have
$$t = \frac{\log(3{,}370{,}000)}{\log 2} = 21.684.$$
In Example 2 on page 151, we found the graphical approximation of between 21 and 22 days as the time for the Yonkers fine to exceed the city's annual budget.

The Natural Logarithm

When e is used as the base for exponential functions, computations are easier with the use of another logarithm function, called log base e. The log base e is used so frequently that it has its own notation: $\ln x$, read as the *natural log of* x. We make the following definition:

For $x > 0$,

$$\ln x \text{ is the power of } e \text{ that gives } x$$

or, in symbols,

$$\ln x = y \quad \text{means} \quad e^y = x,$$

and y is called the **natural logarithm** of x.

Just as the functions 10^x and $\log x$ are inverses, so are e^x and $\ln x$. The function $\ln x$ has similar properties to the common log function:

Properties of the Natural Logarithm

- By definition, $y = \ln x$ means $x = e^y$.
- In particular,
$$\ln 1 = 0 \quad \text{and} \quad \ln e = 1.$$
- The functions e^x and $\ln x$ are inverses, so they "undo" each other:
$$\begin{aligned} \ln(e^x) &= x \qquad \text{for all } x \\ e^{\ln x} &= x \qquad \text{for } x > 0. \end{aligned}$$
- For a and b both positive and any value of t,
$$\begin{aligned} \ln(ab) &= \ln a + \ln b \\ \ln\left(\frac{a}{b}\right) &= \ln a - \ln b \\ \ln(b^t) &= t \cdot \ln b. \end{aligned}$$

Example 6 Solve for x:

(a) $5e^{2x} = 50$ (b) $3^x = 100$.

Solution (a) We first divide both sides by 5 to obtain

$$e^{2x} = 10.$$

Taking the natural log of both sides, we have

$$\begin{aligned} \ln(e^{2x}) &= \ln 10 \\ 2x &= \ln 10 \\ x &= \frac{\ln 10}{2} \approx 1.151. \end{aligned}$$

(b) Taking natural logs of both sides,

$$\begin{aligned} \ln(3^x) &= \ln 100 \\ x \ln 3 &= \ln 100 \\ x &= \frac{\ln 100}{\ln 3} \approx 4.192. \end{aligned}$$

For more practice with logarithms, see the Skills Review on page 219.

Misconceptions and Calculator Errors Involving Logs

It is important to know how to use the properties of logarithms. It is equally important to recognize statements that are *not* true. Beware of the following:

- $\log(a+b)$ is not the same as $\log a + \log b$
- $\log(a-b)$ is not the same as $\log a - \log b$
- $\log(ab)$ is not the same as $(\log a)(\log b)$
- $\log\left(\frac{a}{b}\right)$ is not the same as $\frac{\log a}{\log b}$
- $\log\left(\frac{1}{a}\right)$ is not the same as $\frac{1}{\log a}$.

There are no formulas to simplify either $\log(a+b)$ or $\log(a-b)$. Also the expression $\log 5x^2$ is not the same as $2 \cdot \log 5x$, because the exponent, 2, applies only to the x and not to the 5. However, it is correct to write

$$\log 5x^2 = \log 5 + \log x^2 = \log 5 + 2\log x.$$

Using a calculator to evaluate expressions like $\log(\frac{17}{3})$ requires care. On some calculators, entering $\log 17/3$ gives 0.410, which is incorrect. This is because the calculator assumes that you mean $(\log 17)/3$, which is not the same as $\log(17/3)$. Notice also that

$$\frac{\log 17}{\log 3} \approx \frac{1.230}{0.477} \approx 2.579,$$

which is not the same as either $(\log 17)/3$ or $\log(17/3)$. Thus, the following expressions are all different:

$$\log\frac{17}{3} \approx 0.753, \qquad \frac{\log 17}{3} \approx 0.410, \qquad \text{and} \qquad \frac{\log 17}{\log 3} \approx 2.579.$$

Justification of $\log(a \cdot b) = \log a + \log b$ and $\log(a/b) = \log a - \log b$

If a and b are both positive, we can write $a = 10^m$ and $b = 10^n$, so $\log a = m$ and $\log b = n$. Then, the product $a \cdot b$ can be written

$$a \cdot b = 10^m \cdot 10^n = 10^{m+n}.$$

Therefore $m+n$ is the power of 10 needed to give $a \cdot b$, so

$$\log(a \cdot b) = m + n,$$

which gives

$$\boxed{\log(a \cdot b) = \log a + \log b.}$$

Similarly, the quotient a/b can be written as

$$\frac{a}{b} = \frac{10^m}{10^n} = 10^{m-n}.$$

Therefore $m-n$ is the power of 10 needed to give a/b, so

$$\log\left(\frac{a}{b}\right)=m-n,$$

and thus

$$\boxed{\log\left(\frac{a}{b}\right)=\log a-\log b.}$$

Justification of $\log(b^t)=t\cdot\log b$

Suppose that b is positive, so we can write $b=10^k$ for some value of k. Then

$$b^t=(10^k)^t.$$

We have rewritten the expression b^t so that the base is a power of 10. Using a property of exponents, we can write $(10^k)^t$ as 10^{kt}, so

$$b^t=(10^k)^t=10^{kt}.$$

Therefore kt is the power of 10 which gives b^t, so

$$\log(b^t)=kt.$$

But since $b=10^k$, we know $k=\log b$. This means

$$\log(b^t)=(\log b)t=t\cdot\log b.$$

Thus, for $b>0$ we have

$$\boxed{\log\left(b^t\right)=t\cdot\log b.}$$

Exercises and Problems for Section 5.1

Skill Refresher

Without using logs or a calculator, solve the equations in Exercises S1–S10 if possible.

S1. $10^x=1{,}000{,}000$

S2. $10^t=0.01$

S3. $e^z=\sqrt{e^3}$

S4. $10^x=1$

S5. $e^w=0$

S6. $e^{3x}=\dfrac{1}{e^5}$

S7. $\sqrt{e^{9t}}=e^7$

S8. $10^{-x}=-1{,}000$

S9. $10^{2t}=\sqrt[4]{0.1}$

S10. $e^{3x}=\sqrt[3]{e^5}$

Exercises

Rewrite the statements in Exercises 1–6 using exponents instead of logs.

1. $\log 19=1.279$

2. $\log 4=0.602$

3. $\ln 26=3.258$

4. $\ln(0.646)=-0.437$

5. $\log P=t$

6. $\ln q=z$

Rewrite the statements in Exercises 7–10 using logs.

7. $10^8=100{,}000{,}000$

8. $e^{-4}=0.0183$

9. $10^v=\alpha$

10. $e^a=b$

11. Evaluate without a calculator.

(a) $\log 1000$ **(b)** $\log\sqrt{1000}$ **(c)** $\log\left(10^0\right)$

(d) $\log\sqrt{10}$ **(e)** $\log\left(10^5\right)$ **(f)** $\log\left(10^2\right)$

(g) $\log\left(\dfrac{1}{\sqrt{10}}\right)$ **(h)** $10^{\log 100}$ **(i)** $10^{\log 1}$

(j) $10^{\log(0.01)}$

12. Evaluate without a calculator.

(a) $\ln 1$ **(b)** $\ln e^0$ **(c)** $\ln e^5$

(d) $\ln\sqrt{e}$ **(e)** $e^{\ln 2}$ **(f)** $\ln\left(\dfrac{1}{\sqrt{e}}\right)$

Solve the equations in Exercises 13–18 using logs.

13. $2^x = 11$

14. $(1.45)^x = 25$

15. $e^{0.12x} = 100$

16. $10 = 22(0.87)^q$

17. $48 = 17(2.3)^w$

18. $2/7 = (0.6)^{2t}$

Problems

19. Express the following in terms of x without logs.

(a) $\log 100^x$ **(b)** $1000^{\log x}$ **(c)** $\log 0.001^x$

20. Express the following in terms of x without natural logs.

(a) $\ln e^{2x}$ **(b)** $e^{\ln(3x+2)}$

(c) $\ln\left(\frac{1}{e^{5x}}\right)$ **(d)** $\ln\sqrt{e^x}$

21. Evaluate the following pairs of expressions without using a calculator. What do you notice?

(a) $\log(10 \cdot 100)$ and $\log 10 + \log 100$
(b) $\log(100 \cdot 1000)$ and $\log 100 + \log 1000$
(c) $\log\left(\frac{10}{100}\right)$ and $\log 10 - \log 100$
(d) $\log\left(\frac{100}{1000}\right)$ and $\log 100 - \log 1000$
(e) $\log(10^2)$ and $2\log 10$
(f) $\log(10^3)$ and $3\log 10$

22. **(a)** Write the general formulas reflected in what you observed in Problem 21.
(b) Apply these formulas to rewrite $\log\left(\frac{AB}{C}\right)^p$ at least two different ways.

23. True or false?

(a) $\log AB = \log A + \log B$
(b) $\frac{\log A}{\log B} = \log A - B$
(c) $\log A \log B = \log A + \log B$
(d) $p \cdot \log A = \log A^p$
(e) $\log\sqrt{x} = \frac{1}{2}\log x$
(f) $\sqrt{\log x} = \log(x^{1/2})$

Use properties of logarithms to solve for x in Problems 24–29. Assume a, b, M, and N are constants.

24. $\log(3 \cdot 2^x) = 8$

25. $\ln(25(1.05)^x) = 6$

26. $\ln(ab^x) = M$

27. $\log(MN^x) = a$

28. $\ln(3x^2) = 8$

29. $\log(5x^3) = 2$

30. A graph of $P = 25(1.075)^t$ is given in Figure 5.1.

(a) What is the initial value of P (when $t = 0$)? What is the percent growth rate?
(b) Use the graph to estimate the value of t when $P = 100$.
(c) Use logs to find the exact value of t when $P = 100$.

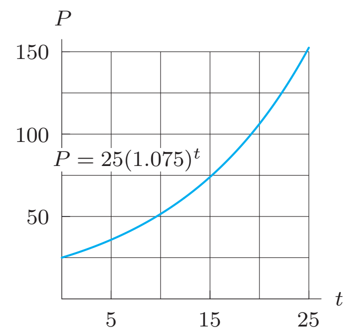


Figure 5.1

31. A graph of $Q = 10e^{-0.15t}$ is given in Figure 5.2.

(a) What is the initial value of Q (when $t = 0$)? What is the continuous percent decay rate?
(b) Use the graph to estimate the value of t when $Q = 2$.
(c) Use logs to find the exact value of t when $Q = 2$.

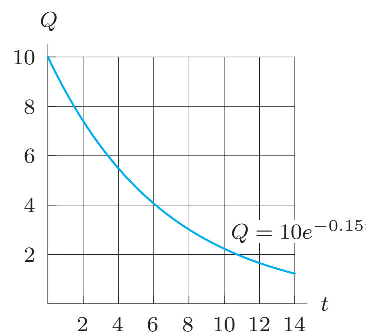


Figure 5.2

32. Let $u = \log 2$ and $v = \log 3$. Evaluate the following expressions in terms of u and/or v. For example, $\log 9 = \log(3^2) = 2\log 3 = 2v$.

(a) $\log 6$ **(b)** $\log 0.08$ **(c)** $\log\sqrt{\frac{3}{2}}$ **(d)** $\log 5$

33. Without using a calculator, write the following quantities in terms of $\log 15$ and/or $\log 5$.

(a) $\log 3$ **(b)** $\log 25$ **(c)** $\log 75$

34. Find a possible formula for the exponential function S in Figure 5.3, if $R(x) = 5.1403(1.1169)^x$.

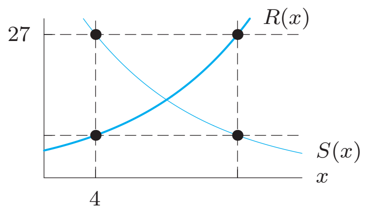


Figure 5.3

In Problems 35–51, solve the equations exactly for x or t.

35. $91 = 46(1.1)^x$

36. $84(0.74)^t = 38$

37. $e^{0.044t} = 6$

38. $200 \cdot 2^{t/5} = 355.$

39. $e^{x+4} = 10$

40. $e^{x+5} = 7 \cdot 2^x$

41. $0.4(\frac{1}{3})^{3x} = 7 \cdot 2^{-x}$

42. $\log_3 3^{5x+1} = 2$

43. $400e^{0.1x} = 500e^{0.08x}$

44. $6000 \left(\frac{1}{2}\right)^{t/15} = 1000$

45. $e^{x+4} = 10$

46. $ab^x = c$

47. $Pe^{kx} = Q$

48. $58e^{4t+1} = 30$

49. $\log(2x + 5) \cdot \log(9x^2) = 0$

50. $\log(1 - x) - \log(1 + x) = 2$

51. $\log(2x + 5) \cdot \log(9x^2) = 0$

52. Solve each of the following equations exactly for x.

(a) $e^{2x} + e^{2x} = 1$ **(b)** $2e^{3x} + e^{3x} = b$

53. If we square a positive quantity n, we double its log: $\log\left(n^2\right) = 2\log n$. Briefly describe what happens to the log when we double a positive quantity n.

54. Consider the exponential function $Q = r \cdot s^t$. Letting $q = \ln Q$, show that q is a linear function of t by writing it in the form $q = b + mt$. State the values of m and b.

55. The *arithmetic mean* of two numbers is half their sum, and the *geometric mean* is the square root of their product. Using log properties, show how we can think of the log of the geometric mean of v and w as the arithmetic mean of two related numbers, p and q.

56. Because they are so large, it is impossible to compare directly the two numbers[1]

$$A = 2^{2^{2^{83}}} \quad \text{and} \quad B = 3^{3^{3^{52}}}.$$

Instead, first simplify $\ln(\ln A)$ and $\ln(\ln B)$, and then use a calculator to determine which is larger, A or B.

57. Both of these numbers are slightly larger than 1:

$$A = 5^{3^{-47}} \quad \text{and} \quad B = 7^{5^{-32}}.$$

Which is larger? Explain your reasoning.

5.2 LOGARITHMS AND EXPONENTIAL MODELS

The log function is often useful when answering questions about exponential models. Logarithms "undo" exponentials—because the log is the inverse of the exponential function—so logs are used to solve many exponential equations.

Example 1 In Example 3 on page 151 we solved the equation $200(0.886^t) = 25$ graphically, where t is in thousands of years. We found that a 200 microgram sample of carbon-14 decays to 25 micrograms in approximately 17,200 years. Now solve $200(0.886)^t = 25$ using logarithms.

Solution First, isolate the power on one side of the equation

$$200(0.886^t) = 25$$
$$0.886^t = 0.125.$$

[1] Adapted from Robert P. Munafo's website on large numbers. These numbers are given as examples of Class-4 numbers. A Class-3 number is too large to be evaluated exactly on a computer; a Class-4 number is one whose *logarithm* is too large to be evaluated exactly. See http://www.mrob.com/pub/index.html, accessed April 7, 2008.

Take the log of both sides, and use the fact that $\log(0.886^t) = t\log 0.886$. Then

$$\log(0.886^t) = \log 0.125$$
$$t\log 0.886 = \log 0.125,$$

so

$$t = \frac{\log 0.125}{\log 0.886} \approx 17.180 \text{ thousand years.}$$

This answer is close to the value we found from the graph, 17,200.

Example 2 The US population, P, in millions, is currently growing according to the formula[2]

$$P = 299e^{0.009t},$$

where t is in years since 2006. When is the population predicted to reach 350 million?

Solution We want to solve the following equation for t:

$$299e^{0.009t} = 350.$$

Dividing by 299 gives

$$e^{0.009t} = \frac{350}{299},$$

so $0.009t$ is the power of e which gives $350/299$. Thus, by the definition of the natural log,

$$0.009t = \ln\left(\frac{350}{299}\right).$$

Solving for t and evaluating $\ln(350/299)$ on a calculator gives

$$t = \frac{\ln(350/299)}{0.009} = 17.5 \text{ years.}$$

The US population is predicted to reach 350 million during the year 2024.

Example 3 The population of City A begins with 50,000 people and grows at 3.5% per year. The population of City B begins with a larger population of 250,000 people but grows at the slower rate of 1.6% per year. Assuming that these growth rates hold constant, will the population of City A ever catch up to the population of City B? If so, when?

Solution If t is time measured in years and P_A and P_B are the populations of these two cities, then

$$P_A = 50{,}000(1.035)^t \qquad \text{and} \qquad P_B = 250{,}000(1.016)^t.$$

We want to solve the equation

$$50{,}000(1.035)^t = 250{,}000(1.016)^t.$$

We first get the exponential terms together by dividing both sides of the equation by $50{,}000(1.016)^t$:

$$\frac{(1.035)^t}{(1.016)^t} = \frac{250{,}000}{50{,}000} = 5.$$

Since $\dfrac{a^t}{b^t} = \left(\dfrac{a}{b}\right)^t$, this gives

$$\left(\frac{1.035}{1.016}\right)^t = 5.$$

[2]Based on data from www.census.gov and www.cia.gov/cia/publications/factbook, accessed July 31, 2006.

Taking logs of both sides and using $\log b^t = t \log b$, we have

$$\log\left(\frac{1.035}{1.016}\right)^t = \log 5$$
$$t \log\left(\frac{1.035}{1.016}\right) = \log 5$$
$$t = \frac{\log 5}{\log(1.035/1.016)} \approx 86.865.$$

Thus, the cities' populations will be equal in just under 87 years. To check this, notice that when $t = 86.865$,

$$P_A = 50{,}000(1.035)^{86.865} = 992{,}575$$

and

$$P_B = 250{,}000(1.016)^{86.865} = 992{,}572.$$

The answers are not exactly equal because we rounded off the value of t. Rounding can introduce significant errors, especially when logs and exponentials are involved. Using $t = 86.86480867$, the computed values of P_A and P_B agree to three decimal places.

Doubling Time

Eventually, any exponentially growing quantity doubles, or increases by 100%. Since its percent growth rate is constant, the time it takes for the quantity to grow by 100% is also a constant. This time period is called the *doubling time*.

Example 4 (a) Find the time needed for the turtle population described by the function $P = 175(1.145)^t$ to double its initial size.

(b) How long does this population take to quadruple its initial size? To increase by a factor of 8?

Solution (a) The initial size is 175 turtles; doubling this gives 350 turtles. We need to solve the following equation for t:

$$175(1.145)^t = 350$$
$$1.145^t = 2$$
$$\log\left(1.145^t\right) = \log 2$$
$$t \cdot \log 1.145 = \log 2$$
$$t = \frac{\log 2}{\log 1.145} \approx 5.119 \text{ years.}$$

We check this by noting that

$$175(1.145)^{5.119} = 350,$$

which is double the initial population. In fact, at any time it takes the turtle population about 5.119 years to double in size.

(b) Since the population function is exponential, it increases by 100% every 5.119 years. Thus it doubles its initial size in the first 5.119 years, quadruples its initial size in two 5.119 year periods, or 10.238 years, and increases by a factor of 8 in three 5.119 year periods, or 15.357 years. We check this by noting that

$$175(1.145)^{10.238} = 700,$$

or 4 times the initial size, and that

$$175(1.145)^{15.357} = 1400,$$

or 8 times the initial size.

Example 5 A population doubles in size every 20 years. What is its continuous growth rate?

Solution We are not given the initial size of the population, but we can solve this problem without that information. Let the symbol P_0 represent the initial size of the population.We have $P = P_0 e^{kt}$. After 20 years, $P = 2P_0$, and so

$$\begin{aligned} P_0 e^{k \cdot 20} &= 2P_0 \\ e^{20k} &= 2 \\ 20k &= \ln 2 \qquad \text{Taking } \ln \text{ of both sides} \\ k &= \frac{\ln 2}{20} \approx 0.03466. \end{aligned}$$

Thus, the population grows at the continuous rate of 3.466% per year.

Example 6 Interest rates in Brazil have fluctuated widely since 1995: investments in Brazil in different years were expected to double in value at wildly different rates.[3] The Brazilian currency is the real.

(a) An investment purchased in May 1995 has a value, V, in reals, t years later, given by

$$V = 100{,}000 \cdot 2^t.$$

(i) How much was the investment worth in May 1996? 1997? 1998?
(ii) What is the doubling time?

(b) An investment purchased in January 2009 is expected to have value, Z, in reals, t years later, given by

$$Z = 100{,}000 \cdot 2^{t/6}.$$

(i) How much is it expected to be worth in January 2015? 2021?
(ii) What is the doubling time?

(c) Suppose an investment's value t years after 2011 is predicted to be $100{,}000 \cdot 2^{t/n}$.
(i) How much is the investment expected to be worth after n years?
(ii) What is its doubling time?

Solution (a) (i) May 1996 is one year after the investment began, so $t = 1$, and

$$V = 100{,}000 \cdot 2^1 = 200{,}000.$$

May 1997 is two years after the investment began, so $t = 2$, and

$$V = 100{,}000 \cdot 2^2 = 400{,}000.$$

May 1998 is three years after the investment began, so $t = 3$, and

$$V = 100{,}000 \cdot 2^3 = 800{,}000.$$

(ii) The value of the investment, which started at 100,000 reals, doubles each year, so the doubling time is 1 year. Since the formula is

$$V = 100{,}000 \cdot 2^t = 100{,}000(1 + 1)^t,$$

the growth rate is $1 = 100\%$ per year.

[3] www.latin-focus.com/latinfocus/countries/brazil/brainter.htm, accessed January 10, 2010.

(b) (i) January 2015 is six years after the investment begins, so $t = 6$, and $t/6 = 1$, giving

$$Z = 100{,}000 \cdot 2^1 = 200{,}000.$$

January 2021 is twelve years after the investment begins, so $t = 12$, and $t/6 = 2$, giving

$$Z = 100{,}000 \cdot 2^2 = 400{,}000.$$

(ii) Note that the value of the investment doubles every 6 years, so the doubling time is 6 years. Since $Z = 100{,}000 \cdot 2^{t/6}$, when $t = 6$, the growth factor is $2^{6/6} = 2$, as expected.

(c) (i) After n years, the investment is worth $100{,}000 \cdot 2^{n/n} = 100{,}000 \cdot 2^1 = 200{,}000$.

(ii) We see that the value of the investment doubles after the first n years, so the doubling time is n years. Algebraically, we see that when $t = n$, the growth factor becomes

$$2^{t/n} = 2^{n/n} = 2^1 = 2,$$

so we see that the investment doubles in n years.

Half-Life

Just as an exponentially growing quantity doubles in a fixed amount of time, an exponentially decaying quantity decreases by a factor of 2 in a fixed amount of time, called the *half-life* of the quantity.

Example 7 Carbon-14 decays radioactively at a constant annual rate of 0.0121%. Show that the half-life of carbon-14 is about 5728 years.

Solution We are not given an initial amount of carbon-14, but we can solve this problem without that information. Let the symbol Q_0 represent the initial quantity of carbon-14 present. The growth rate is -0.000121 because carbon-14 is decaying. So the growth factor is $b = 1 - 0.000121 = 0.999879$. Thus, after t years the amount left will be

$$Q = Q_0(0.999879)^t.$$

We want to find how long it takes for the quantity to drop to half its initial level. Thus, we need to solve for t in the equation

$$\frac{1}{2}Q_0 = Q_0(0.999879)^t.$$

Dividing each side by Q_0, we have

$$\frac{1}{2} = 0.999879^t.$$

Taking logs

$$\begin{aligned} \log\frac{1}{2} &= \log\left(0.999879^t\right) \\ \log 0.5 &= t \cdot \log 0.999879 \\ t &= \frac{\log 0.5}{\log 0.999879} \approx 5728.143. \end{aligned}$$

Thus, no matter how much carbon-14 there is initially, after about 5728 years, half will remain.

Similarly, we can determine the growth rate given the half-life or doubling time.

Example 8 The quantity, Q, of a substance decays according to the formula $Q = Q_0 e^{-kt}$, where t is in minutes. The half-life of the substance is 11 minutes. What is the value of k?

Solution We know that after 11 minutes, $Q = \frac{1}{2}Q_0$. Thus, solving for k, we get

$$\begin{aligned} Q_0 e^{-k\cdot 11} &= \frac{1}{2}Q_0 \\ e^{-11k} &= \frac{1}{2} \\ -11k &= \ln\frac{1}{2} \\ k &= \frac{\ln(1/2)}{-11} \approx 0.06301, \end{aligned}$$

so $k = 0.063$ per minute. This substance decays at the continuous rate of 6.301% per minute.

Converting Between $Q = ab^t$ and $Q = ae^{kt}$

Any exponential function can be written in either of the two forms:

$$Q = ab^t \qquad \text{or} \qquad Q = ae^{kt}.$$

If $b = e^k$, so $k = \ln b$, the two formulas represent the same function.

Example 9 Convert the exponential function $P = 175(1.145)^t$ to the form $P = ae^{kt}$.

Solution Since the new formula represents the same function, we want $P = 175$ when $t = 0$. Thus, substituting $t = 0$ gives $175 = ae^{k(0)} = a$, so $a = 175$. The parameter a in both functions represents the initial population. For all t,

$$175(1.145)^t = 175(e^k)^t,$$

so we must find k such that

$$e^k = 1.145.$$

Therefore k is the power of e that gives 1.145. By the definition of ln, we have

$$k = \ln 1.145 \approx 0.1354.$$

Therefore,

$$P = 175e^{0.1354t}.$$

Example 10 Convert the formula $Q = 7e^{0.3t}$ to the form $Q = ab^t$.

Solution Using the properties of exponents,

$$Q = 7e^{0.3t} = 7(e^{0.3})^t.$$

Using a calculator, we find $e^{0.3} \approx 1.3499$, so

$$Q = 7(1.3499)^t.$$

Example 11 Assuming t is in years, find the continuous and annual percent growth rates in Examples 9 and 10.

Solution In Example 9, the annual percent growth rate is 14.5% and the continuous percent growth rate per year is 13.54%. In Example 10, the continuous percent growth rate is 30% and the annual percent growth rate is 34.99%.

Example 12 Find the continuous percent growth rate of $Q = 200(0.886)^t$, where t is in thousands of years.

Solution Since this function describes exponential decay, we expect a negative value for k. We want

$$e^k = 0.886.$$

Solving for k gives

$$k = \ln(0.886) = -0.12104.$$

So we have $Q = 200e^{-0.12104t}$ and the continuous growth rate is -12.104% per thousand years.

Exponential Growth Problems That Cannot Be Solved by Logarithms

Some equations with the variable in the exponent cannot be solved using logarithms.

Example 13 With t in years, the population of a country (in millions) is given by $P = 2(1.02)^t$, while the food supply (in millions of people that can be fed) is given by $N = 4+0.5t$. Determine the year in which the country first experiences food shortages.

Solution The country starts to experience shortages when the population equals the number of people that can be fed—that is, when $P = N$. We attempt to solve the equation $P = N$ by using logs:

$$\begin{aligned} 2(1.02)^t &= 4 + 0.5t \\ 1.02^t &= 2 + 0.25t \qquad \text{Dividing by 2} \\ \log 1.02^t &= \log(2 + 0.25t) \\ t \log 1.02 &= \log(2 + 0.25t). \end{aligned}$$

Unfortunately, we cannot isolate t, so, this equation cannot be solved using logs. However, we can approximate the solution of the original equation numerically or graphically, as shown in Figure 5.4. The two functions, P and N, are equal when $t \approx 199.381$. Thus, it will be almost 200 years before shortages occur.

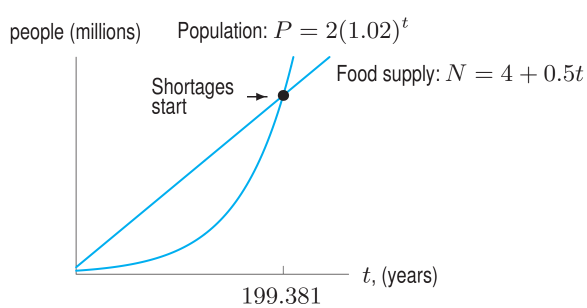


Figure 5.4: Finding the intersection of linear and exponential graphs

Exercises and Problems for Section 5.2

Skill Refresher

For Exercises S1–S4, simplify the expression if possible.

S1. $10^{-\log 5x}$

S2. $e^{-3\ln t}$

S3. $t\ln e^{t/2}$

S4. $10^{2+\log x}$

In Exercises S5–S10, solve for x.

S5. $4^x = 9$

S6. $e^x = 8$

S7. $2e^x = 13$

S8. $e^{7x} = 5e^{3x}$

S9. $\log(2x+7) = 2$

S10. $\log(2x) = \log(x+10)$

Exercises

For Exercises 1–2, write the exponential function in the form $y = ab^t$. Find b accurate to four decimal places. If t is measured in years, give the percent annual growth or decay rate and the continuous percent growth or decay rate per year.

1. $y = 25e^{0.053t}$

2. $y = 100e^{-0.07t}$

For Exercises 3–4, write the exponential function in the form $y = ae^{kt}$. Find k accurate to four decimal places. If t is measured in years, give the percent annual growth rate and the continuous percent growth rate per year.

3. $y = 6000(0.85)^t$

4. $y = 5(1.12)^t$

In Exercises 5–8, convert to the form $Q = ab^t$.

5. $Q = 4e^{7t}$

6. $Q = 0.3e^{0.7t}$

7. $Q = \dfrac{14}{5}e^{0.03t}$

8. $Q = e^{-0.02t}$

In Exercises 9–12, convert to the form $Q = ae^{kt}$.

9. $Q = 12(0.9)^t$

10. $Q = 16(0.487)^t$

11. $Q = 14(0.862)^{1.4t}$

12. $Q = 721(0.98)^{0.7t}$

In Exercises 13–20, give the starting value a, the growth rate r, and the continuous growth rate k.

13. $Q = 230(1.182)^t$

14. $Q = 0.181\left(e^{0.775}\right)^t$

15. $Q = 0.81(2)^t$

16. $Q = 5\cdot 2^{t/8}$

17. $Q = 12.1\cdot 10^{-0.11t}$

18. $Q = 40e^{(t-5)/12}$

19. $Q = 2e^{(1-3t/4)}$

20. $Q = 2^{-(t-5)/3}$

Problems

Find the doubling time in Exercises 21–24.

21. A population growing according to $P = P_0e^{0.2t}$.

22. A city is growing by 26% per year.

23. A bank account is growing by 2.7% per year.

24. A company's profits are increasing by an annual growth factor of 1.12.

Find the half-lives of the substances in Exercises 25–27.

25. Tritium, which decays at a rate of 5.471% per year.

26. Einsteinium-253, which decays at a rate of 3.406% per day.

27. A radioactive substance that decays at a continuous rate of 11% per minute.

28. You place \$800 in an account that earns 4% annual interest, compounded annually. How long will it be until you have \$2000?

29. **(a)** What annual interest rate, compounded continuously, is equivalent to an annual rate of 8%, compounded annually?
(b) What annual interest rate, compounded annually, is equivalent to an annual rate of 6%, compounded continuously?

30. A population grows from 11000 to 13000 in three years. Assuming the growth is exponential, find the:
(a) Annual growth rate **(b)** Continuous growth rate
(c) Why are your answers to parts (a) and (b) different?

31. A \$5000 investment earns 7.2% annual interest, and an \$8000 investment earns 5.4%, both compounded annually. How long will it take for the smaller investment to catch up to the larger one?

32. A \$9000 investment earns 5.6% annual interest, and a \$4000 investment earns 8.3%, both compounded continuously. When will the smaller catch up to the larger?

33. A population doubles in size every 15 years. Assuming exponential growth, find the

(a) Annual growth rate **(b)** Continuous growth rate

34. A population increases from 5.2 million at an annual rate of 3.1%. Find the continuous growth rate.

35. The half-life of nicotine in the body is 2 hours. What is the continuous decay rate?

36. If 17% of a radioactive substance decays in 5 hours, what is the half-life of the substance?

37. Total power generated by wind worldwide doubles every 3 years.[4] In 2008, world wind-energy generating capacity was about 90 thousand megawatts. Find the continuous growth rate and give a formula for wind generating capacity W (in thousand megawatts) as a function of t, number of years since 2008.

38. Sketch the exponential function $y = u(t)$ given that it has a starting value of 0.8 and a doubling time of 12 years. Label the axes and indicate the scale.

39. A town has 5000 people in year $t = 0$. Calculate how long it takes for the population P to double once, twice, and three times, assuming that the town grows at a constant rate of

(a) 500 people per year.
(b) 5% per year.

40. **(a)** Estimate the doubling time of the exponential function shown in Figure 5.5.
(b) Use the doubling time to find the continuous percent growth rate and give a formula for the function.

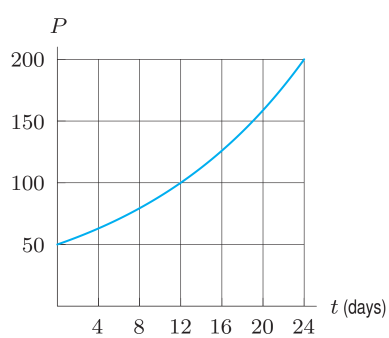


Figure 5.5

[4]World Wind Energy Report 2008.

41. **(a)** The quantity of caffeine in the body after drinking a cup of coffee is shown in Figure 5.6. Estimate the half-life of caffeine.
(b) Use the half-life to find the continuous percent decay rate and give a formula for Q as function of t.

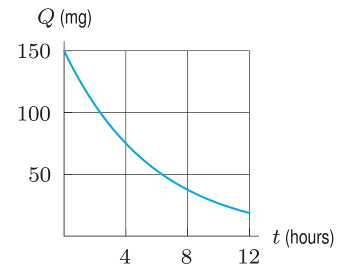


Figure 5.6

42. The temperature, H, in °F, of a cup of coffee t hours after it is set out to cool is given by the equation:

$$H = 70 + 120(1/4)^t.$$

(a) What is the coffee's temperature initially (that is, at time $t = 0$)? After 1 hour? 2 hours?
(b) How long does it take the coffee to cool down to 90°F? 75°F?

43. Use algebra to show that the time it takes for a quantity growing exponentially to double is independent of the starting quantity and the time. To do this, let d represent the time it takes for P to double. Show that if P becomes $2P$ at time $t + d$, then d depends only on the growth factor b, but not on the starting quantity a and time t. (Assume $P \neq 0$.)

44. Prices climb at a constant 3% annual rate.

(a) By what percent will prices have climbed after 5 years?
(b) How long will it take for prices to climb 25%?

45. The growth of an animal population, P, is described by the function $P = 300 \cdot 2^{t/20}$.

(a) How large is this population in year $t = 0$? $t = 20$?
(b) When does this population reach 1000?

46. Find values for a, b, k, s where

$$f(t) = ab^t = ae^{kt} = a \cdot 2^{t/s},$$

given that $f(-20) = 5$ and $f(40) = 30$.

47. **(a)** Find the time required for an investment to triple in value if it earns 4% annual interest, compounded continuously.
(b) Now find the time required assuming that the interest is compounded annually.

48. The world's population is aging. The approximate world population age 80 or older[5] is given in Table 5.1.

(a) Find a formula for P, the number of people in the world age 80 or older, in millions, as a function of time, t, in years since 2005. Use the form $P = ab^t$. What is the annual percent rate of increase?
(b) Convert to the form $P = ae^{kt}$. What is the continuous percent increase per year?
(c) Find the doubling time.

Table 5.1

t (year)	2005	2006	2007	2008	2009
P (millions)	89.144	92.175	95.309	98.550	101.901

49. Technetium-99m is a radioactive substance used to diagnose brain diseases. Its half-life is approximately 6 hours. Initially you have 200 mg of technetium-99m.

(a) Write an equation that gives the amount of technetium-99m remaining after t hours.
(b) Determine the number of hours needed for your sample to decay to 120 mg.
(c) Determine the concavity of the graph that models the half-life of technetium-99m using average rates of change over intervals of length 2 between $t = 0$ and $t = 6$.

50. The US census projects the population of the state of Washington using the function $N(t) = 5.4e^{0.013t}$, where $N(t)$ is in millions and t is in years since 1995.

(a) What is the population's continuous growth rate?
(b) What is the population of Washington in year $t = 0$?
(c) How many years is it before the population triples?
(d) In what year does this model indicate a population of only one person? Is this reasonable or unreasonable?

51. In 1991, the body of a man was found in melting snow in the Alps of Northern Italy. An examination of the tissue sample revealed that 46% of the carbon-14 present in his body at the time of his death had decayed. The half-life of carbon-14 is approximately 5728 years. How long ago did this man die?

52. A manager at Saks Fifth Avenue wants to estimate the number of customers to expect on the last shopping day before Christmas. She collects data from three previous years, and determines that the crowds follow the same general pattern. When the store opens at 10 am, 500 people enter, and the total number in the store doubles every 40 minutes. When the number of people in the store reaches 10,000, security guards need to be stationed at the entrances to control the crowds. At what time should the guards be commissioned?

53. Figure 5.7 shows the graphs of the exponential functions f and g, and the linear function, h.

(a) Find formulas for f, g, and h.
(b) Find the exact value(s) of x such that $f(x) = g(x)$.
(c) Estimate the value(s) of x such that $f(x) = h(x)$.

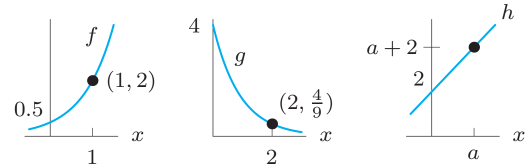


Figure 5.7

Problems 54–55 involve the Rule of 70, which gives quick estimates of the doubling time of an exponentially growing quantity. If $r\%$ is the annual growth rate of the quantity, then the Rule of 70 says

$$\text{Doubling time in years} \approx \frac{70}{r}.$$

54. Use the Rule of 70 to estimate how long it takes a $1000 investment to double if it grows at the following annual rates: 1%, 2%, 5%, 7%, 10%. Compare with the actual doubling times.

55. Using natural logs, solve for the doubling time for $Q = ae^{kt}$. Use your result to explain why the Rule of 70 works.

56. A person's blood alcohol content (BAC) is a measure of how much alcohol is in the blood stream. When a person stops drinking, the BAC declines over time as the alcohol is metabolized. The BAC, Q, of a person t minutes after the he stops drinking is given by

$$Q = Q_0 e^{-t/\tau},$$

where Q_0 is the person's initial BAC and τ is known as the *elimination time*. How long does it take for a person's BAC to drop from 0.10 to 0.04 if the elimination time is 2.5 hours?

57. The size of a population, P, of toads t years after it is introduced into a wetland is given by

$$P = \frac{1000}{1 + 49(1/2)^t}.$$

(a) How many toads are there in year $t = 0$? $t = 5$? $t = 10$?

[5] UN Department of Economic and Social Affairs, 2009.

(b) How long does it take for the toad population to reach 500? 750?

(c) What is the maximum number of toads that the wetland can support?

58. Write the exponential function $y = ab^t$ in the form

$$y = e^{k(t-t_0)}.$$

Give k and t_0 in terms of a and b.

59. *Gompertz functions* can be used to model population growth.[6] Solve $f(t) = 3$ for t for the particular Gompertz function

$$f(t) = 6e^{-0.5e^{-0.1t}}.$$

60. **(a)** Rewrite the equation $23\,(1.36)^t = 85$ in the form $e^{k+rt} = e^s$. State the values of the constants k, r, and s.

(b) Solve the original equation in terms of k, r, and s, then give a numerical approximation.

61. **(a)** Rewrite the equation $1.12^t = 6.3$ in the form $10^{vt} = 10^w$. State the values of the constants v and w.

(b) Solve the original equation in terms of v and w, then give a numerical approximation.

5.3 THE LOGARITHMIC FUNCTION

The Graph, Domain, and Range of the Common Logarithm

In Section 5.1 we defined the log function (to base 10) for all positive numbers. In other words,

Domain of $\log x$ is all positive numbers.

By considering its graph in Figure 5.8, we determine the range of $y = \log x$. The log graph crosses the x-axis at $x = 1$, because $\log 1 = \log(10^0) = 0$. The graph climbs to $y = 1$ at $x = 10$, because $\log 10 = \log(10^1) = 1$. In order for the log graph to climb to $y = 2$, the value of x must reach 100, or 10^2, and in order for it to climb to $y = 3$, the value of x must be 10^3, or 1000. To reach the modest height of $y = 20$ requires x to equal 10^{20}, or 100 billion billion! The log function increases so slowly that it often serves as a benchmark for other slow-growing functions. Nonetheless, the graph of $y = \log x$ eventually climbs to any value we choose.

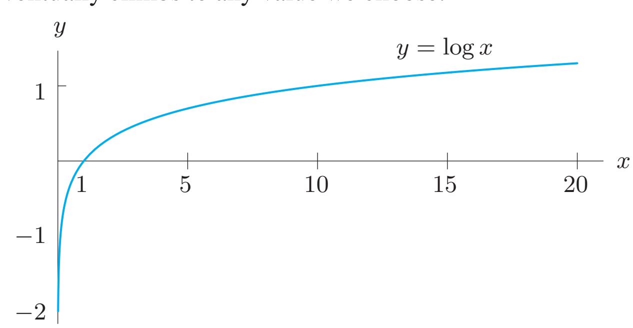


Figure 5.8: The log function grows very rapidly for $0 < x < 1$ and very slowly for $x > 1$. It has a vertical asymptote at $x = 0$ but never touches the y-axis

Although x cannot equal zero in the log function, we can choose $x > 0$ to be as small as we like. As x decreases toward zero, the values of $\log x$ get large and negative. For example,

$$\begin{aligned}\log 0.1 &= \log 10^{-1} = -1,\\ \log 0.01 &= \log 10^{-2} = -2,\\ &\vdots\\ \log 0.0000001 &= \log 10^{-7} = -7,\end{aligned}$$

[6] See http://en.wikipedia.org/wiki/Gompertz_curve, accessed April 13, 2008.

and so on. So small positive values of x give exceedingly large negative values of y. The graph has a vertical asymptote at $x = 0$ and

$$\text{Range of } \log x \text{ is all real numbers.}$$

The log function is increasing and its graph is concave down, since its rate of change is decreasing.

Graphs of the Inverse Functions $y = \log x$ and $y = 10^x$

The fact that $y = \log x$ and $y = 10^x$ are inverses means that their graphs are related. Looking at Tables 5.2 and 5.3, we see that the point $(0.01, -2)$ is on the graph of $y = \log x$ and the point $(-2, 0.01)$ is on the graph of $y = 10^x$. In general, if the point (a, b) is on the graph of $y = \log x$, the point (b, a) is on the graph of $y = 10^x$. Thus, the graph of $y = \log x$ is the graph of $y = 10^x$ with x- and y-axes interchanged. If the x- and y-axes have the same scale, this is equivalent to reflecting the graph of $y = 10^x$ across the diagonal line $y = x$. See Figure 5.9.

Table 5.2 *Log function*

x	$y = \log x$
0.01	-2
0.1	-1
1	0
10	1
100	2
1000	3

Table 5.3 *Exponential function*

x	$y = 10^x$
-2	0.01
-1	0.1
0	1
1	10
2	100
3	1000

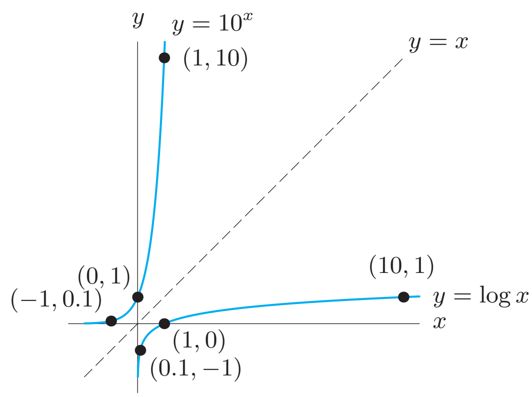


Figure 5.9: The functions $y = \log x$ and $y = 10^x$ are inverses of one another

Graph of Natural Logarithm

In addition to similar algebraic properties, the natural log and the common log have similar graphs.

Example 1 Graph $y = \ln x$ for $0 < x < 10$.

Solution Values of $\ln x$ are in Table 5.4. Like the common log, the natural log is only defined for $x > 0$ and has a vertical asymptote at $x = 0$. The graph is slowly increasing and concave down.

Table 5.4 *Values of* $\ln x$ *(rounded)*

x	$\ln x$
0	Undefined
1	0
2	0.7
e	1
3	1.1
4	1.4
$\vdots$	$\vdots$

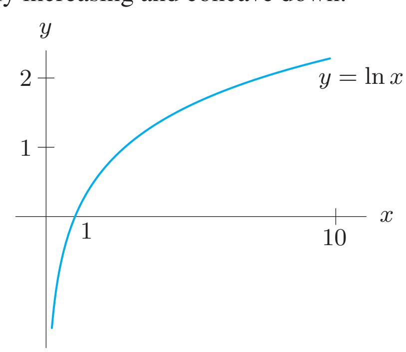


Figure 5.10: Graph of the natural logarithm

The functions $y = \ln x$ and $y = e^x$ are inverses. If the scales on the axes are the same, their graphs are reflections of one another across the line $y = x$. See Figure 5.11. For example, the vertical asymptote of the logarithm is the reflection of the horizontal asymptote of the exponential. On page 201, we see how to write asymptotes in limit notation.

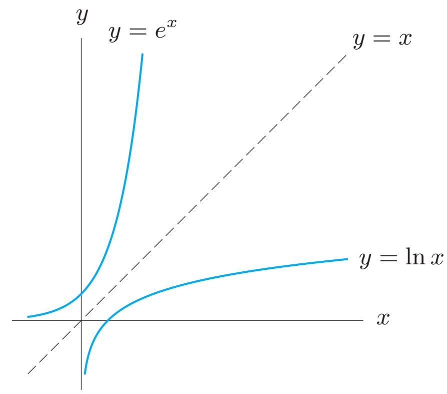


Figure 5.11: The functions $y = \ln x$ and $y = e^x$ are inverses of one another

Chemical Acidity

Logarithms are useful in measuring quantities whose magnitudes vary widely, such as acidity (pH), sound (decibels), and earthquakes (the Richter scale). In chemistry, the acidity of a liquid is expressed using pH. The acidity depends on the hydrogen ion concentration in the liquid (in moles per liter); this concentration is written $[\mathrm{H}^+]$. The greater the hydrogen ion concentration, the more acidic the solution. The pH is defined as:

$$\mathrm{pH} = -\log[\mathrm{H}^+].$$

Example 2 The hydrogen ion concentration of seawater is $[\mathrm{H}^+] = 1.1 \cdot 10^{-8}$. Estimate the pH of seawater. Then check your answer with a calculator.

Solution We want to estimate $\mathrm{pH} = -\log(1.1 \cdot 10^{-8})$. Since $1.1 \cdot 10^{-8} \approx 10^{-8}$ and $\log 10^{-8} = -8$, we know that

$$\mathrm{pH} = -\log(1.1 \cdot 10^{-8}) \approx -(-8) = 8.$$

Using a calculator, we have

$$\mathrm{pH} = -\log(1.1 \cdot 10^{-8}) = 7.959.$$

Example 3 A vinegar solution has a pH of 3. Determine the hydrogen ion concentration.

Solution Since $3 = -\log[\mathrm{H}^+]$, we have $-3 = \log[\mathrm{H}^+]$. This means that $10^{-3} = [\mathrm{H}^+]$. So the hydrogen ion concentration is 10^{-3} moles per liter.

Logarithms and Orders of Magnitude

We often compare sizes or quantities by computing their ratios. If A is twice as tall as B, then

$$\frac{\text{Height of } A}{\text{Height of } B} = 2.$$

If one object is 10 times heavier than another, we say it is an *order of magnitude* heavier. If one quantity is two factors of 10 greater than another, we say it is two orders of magnitude greater, and so on. For example, the value of a dollar is two orders of magnitude greater than the value of a penny, because we have

$$\frac{\$1}{\$0.01} = 100 = 10^2.$$

The order of magnitude is the logarithm of their ratio.

Example 4 The sound intensity of a refrigerator motor is 10^{-11} watts/cm^2. A typical school cafeteria has sound intensity of 10^{-8} watts/cm^2. How many orders of magnitude more intense is the sound of the cafeteria?

Solution To compare the two intensities, we compute their ratio:

$$\frac{\text{Sound intensity of cafeteria}}{\text{Sound intensity of refrigerator}} = \frac{10^{-8}}{10^{-11}} = 10^{-8-(-11)} = 10^3.$$

Thus, the sound intensity of the cafeteria is 1000 times greater than the sound intensity of the refrigerator. The log of this ratio is 3. We say that the sound intensity of the cafeteria is three orders of magnitude greater than the sound intensity of the refrigerator.

Decibels

The intensity of audible sound varies over an enormous range. The range is so enormous that we consider the logarithm of the sound intensity. This is the idea behind the *decibel* (abbreviated dB). To measure a sound in decibels, the sound's intensity, I, is compared to the intensity of a standard benchmark sound, I_0. The intensity of I_0 is defined to be 10^{-16} watts/cm^2, roughly the lowest intensity audible to humans. The comparison between a sound intensity I and the benchmark sound intensity I_0 is made as follows:

$$\boxed{\text{Noise level in decibels} = 10 \cdot \log\left(\frac{I}{I_0}\right).}$$

For instance, let's find the decibel rating of the refrigerator in Example 4. First, we find how many orders of magnitude more intense the refrigerator sound is than the benchmark sound:

$$\frac{I}{I_0} = \frac{\text{Sound intensity of refrigerator}}{\text{Benchmark sound intensity}} = \frac{10^{-11}}{10^{-16}} = 10^5.$$

Thus, the refrigerator's intensity is 5 orders of magnitude more than I_0, the benchmark intensity. We have

$$\text{Decibel rating of refrigerator} = 10 \cdot \underbrace{\text{Number of orders of magnitude}}_{5} = 50 \text{ dB}.$$

Note that 5, the number of orders of magnitude, is the log of the ratio I/I_0. We use the log function because it "counts" the number of powers of 10. Thus if N is the decibel rating, then

$$N = 10 \log\left(\frac{I}{I_0}\right).$$

Example 5

(a) If a sound doubles in intensity, by how many units does its decibel rating increase?
(b) Loud music can measure 110 dB whereas normal conversation measures 50 dB. How many times more intense is loud music than normal conversation?

Solution

(a) Let I be the sound's intensity before it doubles. Once doubled, the new intensity is $2I$. The decibel rating of the original sound is $10\log(I/I_0)$, and the decibel rating of the new sound is $10\log(2I/I_0)$. The difference in decibel ratings is given by

$$\begin{aligned}
\text{Difference in decibel ratings} &= 10\log\left(\frac{2I}{I_0}\right) - 10\log\left(\frac{I}{I_0}\right) \\
&= 10\left(\log\left(\frac{2I}{I_0}\right) - \log\left(\frac{I}{I_0}\right)\right) && \text{Factoring out 10} \\
&= 10\cdot\log\left(\frac{2I/I_0}{I/I_0}\right) && \text{Using the property } \log a - \log b = \log(a/b) \\
&= 10\cdot\log 2 && \text{Canceling } I/I_0 \\
&\approx 3.010\text{ dB.} && \text{Because } \log 2 \approx 0.3
\end{aligned}$$

Thus, if the sound intensity is doubled, the decibel rating goes up by approximately 3 dB.

(b) If I_M is the sound intensity of loud music, then

$$10\log\left(\frac{I_M}{I_0}\right) = 110\text{ dB.}$$

Similarly, if I_C is the sound intensity of conversation, then

$$10\log\left(\frac{I_C}{I_0}\right) = 50\text{ dB.}$$

Computing the difference of the decibel ratings gives

$$10\log\left(\frac{I_M}{I_0}\right) - 10\log\left(\frac{I_C}{I_0}\right) = 60.$$

Dividing by 10 gives

$$\begin{aligned}
\log\left(\frac{I_M}{I_0}\right) - \log\left(\frac{I_C}{I_0}\right) &= 6 \\
\log\left(\frac{I_M/I_0}{I_C/I_0}\right) &= 6 && \text{Using the property } \log b - \log a = \log(b/a) \\
\log\left(\frac{I_M}{I_C}\right) &= 6 && \text{Canceling } I_0 \\
\frac{I_M}{I_C} &= 10^6. && \log x = 6 \text{ means that } x = 10^6
\end{aligned}$$

So $I_M = 10^6 I_C$, which means that loud music is 10^6 times, or one million times, as intense as normal conversation.

Asymptotes and Limit Notation

In Section 4.3 we saw that the graph of an exponential function has a horizontal asymptote. In Figure 5.9 on page 198, we see that $y = 10^x$ has horizontal asymptote $y = 0$, because

$$\text{as } x \to -\infty, \quad 10^x \to 0.$$

Correspondingly, as x gets closer to zero, $y = \log x$ takes on larger and larger negative values. We write

$$\text{as } x \to 0^+, \qquad \log x \to -\infty.$$

The notation $x \to 0^+$ is read "x approaches zero from the right" and means that we are choosing smaller and smaller positive values of x—that is, we are sliding toward $x = 0$ through small positive values. We say the graph of the log function $y = \log x$ has a *vertical asymptote* of $x = 0$.

To describe vertical asymptotes in general, we use the notation

$$x \to a^+$$

to mean that x slides toward a from the right (that is, through values larger than a) and

$$x \to a^-$$

to mean that x slides toward a from the left (that is, through values smaller than a).

If $f(x) \to \infty$ as $x \to a^+$, we say the *limit* of $f(x)$ as x approaches a from the right is infinity,[7] and write

$$\lim_{x\to a^+} f(x) = \infty.$$

If $f(x) \to \infty$ as $x \to a^-$, we write

$$\lim_{x\to a^-} f(x) = \infty.$$

If both $\lim_{x\to a^+} f(x) = \infty$ and $\lim_{x\to a^-} f(x) = \infty$, we say the limit of $f(x)$ as x approaches a is infinity, and write

$$\lim_{x\to a} f(x) = \infty.$$

Similarly, we can write

$$\lim_{x\to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a} f(x) = -\infty.$$

We summarize the information about both horizontal and vertical asymptotes:

> Let $y = f(x)$ be a function and let a be a finite number.
>
> - The graph of f has a **horizontal asymptote** of $y = a$ if
>
> $$\lim_{x\to\infty} f(x) = a \quad \text{or} \quad \lim_{x\to-\infty} f(x) = a \quad \text{or both.}$$
>
> - The graph of f has a **vertical asymptote** of $x = a$ if
>
> $$\lim_{x\to a^+} f(x) = \infty \ \text{ or } \ \lim_{x\to a^+} f(x) = -\infty \ \text{ or } \ \lim_{x\to a^-} f(x) = \infty \ \text{ or } \ \lim_{x\to a^-} f(x) = -\infty.$$

Notice that the process of finding a vertical asymptote is different from the process for finding a horizontal asymptote. Vertical asymptotes occur where the function values grow larger and larger, either positively or negatively, as x approaches a finite value (i.e. where $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$). Horizontal asymptotes are determined by whether the function values approach a finite number as x takes on large positive or large negative values (i.e., as $x \to \infty$ or $x \to -\infty$).

Exercises and Problems for Section 5.3

Skill Refresher

For Exercises S1–S2, evaluate without a calculator.

S1. $\log 0.0001$

S2. $\dfrac{\log 100^6}{\log 100^2}$

[7] Some authors say that these limits do not exist.

For Exercises S3–S4, rewrite the exponential equation in equivalent logarithmic form.

S3. $10^5 = 100{,}000$

S4. $e^2 = 7.389$

For Exercises S5–S6, rewrite the logarithmic equation in equivalent exponential form.

S5. $-\ln x = 12$

S6. $\log(x+3) = 2$

For Exercises S7–S8, if possible, write the expression using sums and/or differences of logarithmic expressions that do not contain the logarithms of products, quotients or powers.

S7. $\ln(x(7-x)^3)$

S8. $\ln\left(\frac{xy^2}{z}\right)$

For Exercises S9–S10, rewrite the expression as a single logarithm.

S9. $\ln x^3 + \ln x^2$

S10. $\frac{1}{3}\log 8 - \frac{1}{2}\log 25$

Exercises

1. What is the equation of the asymptote of the graph of $y = 10^x$? Of the graph of $y = 2^x$? Of the graph of $y = \log x$?

2. What is the equation for the asymptote of the graph of $y = e^x$? Of the graph of $y = e^{-x}$? Of the graph of $y = \ln x$?

3. Without a calculator, match the functions $y = 10^x$, $y = e^x$, $y = \log x$, $y = \ln x$ with the graphs in Figure 5.12.

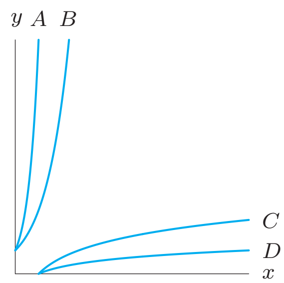


Figure 5.12

4. Without a calculator, match the functions $y = 2^x$, $y = e^{-x}$, $y = 3^x$, $y = \ln x$, $y = \log x$ with the graphs in Figure 5.13.

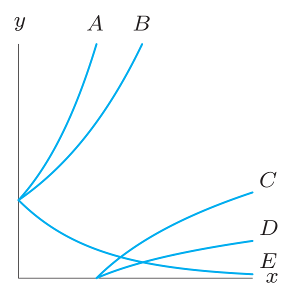


Figure 5.13

Graph the functions in Problems 5–8. Label all asymptotes and intercepts.

5. $y = 2\cdot 3^x + 1$

6. $y = -e^{-x}$

7. $y = \log(x-4)$

8. $y = \ln(x+1)$

In Problems 9–10, graph the function. Identify any vertical asymptotes. State the domain of the function.

9. $y = 2\ln(x-3)$

10. $y = 1 - \ln(2-x)$

11. What is the value (if any) of the following?
(a) 10^{-x} as $x \to \infty$ (b) $\log x$ as $x \to 0^+$

12. What is the value (if any) of the following?
(a) e^x as $x \to -\infty$ (b) $\ln x$ as $x \to 0^+$

13. Find **(a)** $\lim_{x\to 0^+} \log x$ **(b)** $\lim_{x\to 0^-} \ln(-x)$

In Exercises 14–18, find the hydrogen ion concentration, $[H^+]$, for the substances.[8] [Hint: $\text{pH} = -\log[H^+]$.]

14. Lye, with a pH of 13.

15. Battery acid, with a pH of 1.

16. Baking soda, with a pH of 8.3.

17. Tomatoes, with a pH of 4.5.

18. Hydrochloric acid, with a pH of 0.

[8] Data from www.miamisci.org/ph/hhoh.html, accessed November, 2001.

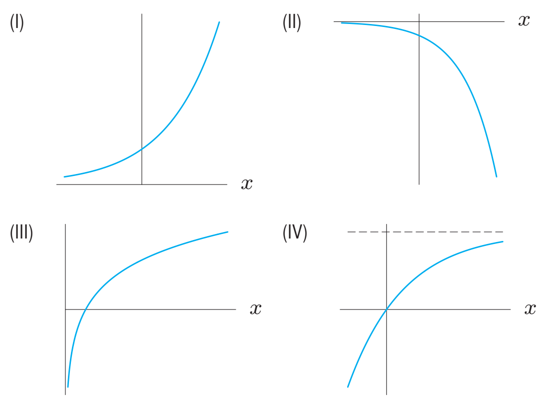
Problems

19. Immediately following the gold medal performance of the US women's gymnastic team in the 1996 Olympic Games, an NBC commentator, John Tesh, said of one team member: "Her confidence and performance have grown logarithmically." He clearly thought this was an enormous compliment. Is it a compliment? Is it realistic?

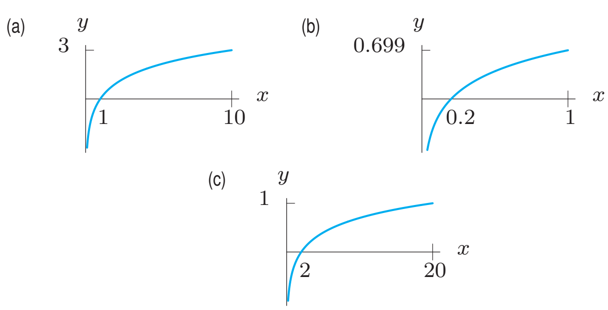
20. Match the statements (a)–(d) with the functions (I)–(IV).

(a) $\lim_{x\to 0^+} f(x) = -\infty$ **(b)** $\lim_{x\to 0^-} f(x) = 0$

(c) $\lim_{x\to \infty} f(x) = \infty$ **(d)** $\lim_{x\to -\infty} f(x) = 0$



21. Match the graphs (a)–(c) to one of the functions $r(x)$, $s(x)$, $t(x)$ whose values are in the tables.



x	2	4	10
$r(x)$	1	1.301	1.699

x	0.5	5	10
$s(x)$	-0.060	0.379	0.699

x	0.1	2	100
$t(x)$	-3	0.903	6

In Problems 22–26, sound in decibels is measured by comparing the sound intensity, I, to a benchmark sound I_0 with intensity 10^{-16} watts/cm^2. Then,

$$\text{Noise level in decibels } = 10\log(I/I_0).$$

22. The noise level of a whisper is 30 dB. Compute the sound intensity of a whisper.

23. Death of hearing tissue begins to occur at a noise level of 180 dB. Compute the sound's intensity at this noise level.

24. Denver Broncos fans recently broke the world record for "loudest roar" at a sports event.[9] A crowd of 76,000 fans reached a noise level of 128.7 dB. The previous world record of 125.4 dB was held by soccer fans in Dublin, Ireland. How many times more intense was the roar of the crowd of Denver Broncos fans than the roar of the soccer fans in Ireland?

25. Sound A measures 30 decibels and sound B is 5 times as loud as sound A. What is the decibel rating of sound B to the nearest integer?

26. **(a)** Let D_1 and D_2 represent the decibel ratings of sounds of intensity I_1 and I_2, respectively. Using log properties, find a simplified formula for the difference between the two ratings, $D_2 - D_1$, in terms of the two intensities, I_1 and I_2.

(b) If a sound's intensity doubles, how many decibels louder does the sound become?

Problems 27–30 use the *Richter scale* for the strength of an earthquake. The strength, W, of the seismic waves of an earthquake are compared to the strength, W_0, of the seismic waves of a standard earthquake. The Richter scale rating, M, is

$$M = \log\left(\frac{W}{W_0}\right).$$

27. In 2008 the Sichuan earthquake in China had a Richter-scale rating of 7.9. How many times more powerful were the seismic waves of the Sichuan earthquake than standard seismic waves?

28. In 1986 the worst nuclear power plant accident in history occurred in Chernobyl, Ukraine. The explosion resulted in seismic waves with a Richter scale rating of 3.5. How many times stronger were the seismic waves of the Chernobyl disaster than standard seismic waves?

[9] See http://www.encyclopedia.com/doc/1G1-65629077.html, accessed January 8, 2010.

29. Let M_1 and M_2 be the magnitude of two earthquakes whose seismic waves are of sizes W_1 and W_2, respectively. Using log properties, find a simplified formula for the difference $M_2 - M_1$ in terms of W_1 and W_2.

30. The 1989 earthquake in California had a rating of 7.1 on the Richter scale. How many times larger than the California earthquake were the seismic waves in the March 2005 earthquake off the coast of Sumatra, which measured 8.7 on the Richter scale? Give your answer to the nearest integer.

31. **(a)** Using the definition of pH on page 199, find the concentrations of hydrogen ions in solutions with

(i) $\text{pH} = 2$ (ii) $\text{pH} = 4$ (iii) $\text{pH} = 7$

(b) A high concentration of hydrogen ions corresponds to an acidic solution. From your answer to part (a), decide if solutions with high pHs are more or less acidic than solutions with low pHs.

32. **(a)** A 12-oz cup of coffee contains about $2.41 \cdot 10^{18}$ hydrogen ions. What is the concentration (moles/liter) of hydrogen ions in a 12-oz cup of coffee? [Hint: One liter equals 30.3 oz. One mole of hydrogen ions equals $6.02 \cdot 10^{23}$ hydrogen ions.]

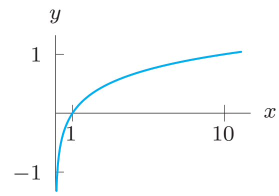
(b) Based on your answer to part (a) and the formula for pH, what is the pH of a 12-oz cup of coffee?

33. **(a)** The pH of lemon juice is about 2.3. What is the concentration of hydrogen ions in lemon juice?

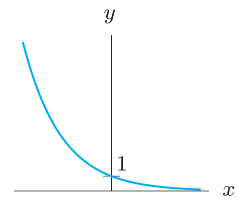
(b) A person squeezes 2 oz of lemon juice into a cup. Based on your answer to part (a), how many hydrogen ions does this juice contain?

In Problems 34–39, find possible formulas for the functions using logs or exponentials.

34.



35.



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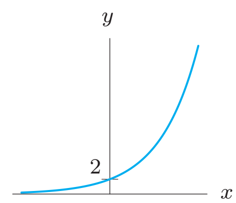

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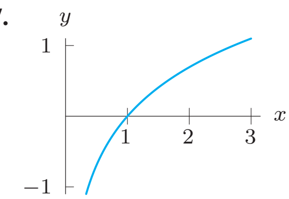

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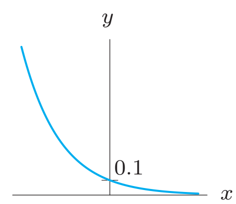

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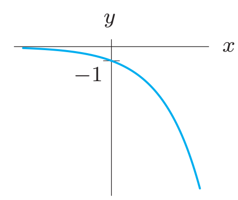

40. Give the domain of $y = \dfrac{1}{2 - \sqrt{7 - e^{2t}}}$.

5.4 LOGARITHMIC SCALES

The Solar System and Beyond

Table 5.5 gives the distance from the sun to a number of different astronomical objects. The planet Mercury is $58{,}000{,}000$ km from the sun, the earth is $149{,}000{,}000$ km from the sun, and Pluto is $5{,}900{,}000{,}000$ km, or almost 6 billion kilometers from the sun. The table also gives the distance to Proxima Centauri, the star closest to the sun, and to the Andromeda Galaxy, the spiral galaxy closest to our own galaxy, the Milky Way.

Table 5.5 *Distance from the sun to various astronomical objects*

Object	Distance (million km)	Saturn	1426
Mercury	58	Uranus	2869
Venus	108	Neptune	4495
Earth	149	Pluto	5900
Mars	228	Proxima Centauri	$4.1 \cdot 10^7$
Jupiter	778	Andromeda Galaxy	$2.4 \cdot 10^{13}$

Linear Scales

We can represent the information in Table 5.5 graphically in order to get a better feel for the distances involved. Figure 5.14 shows the distance from the sun to the first five planets on a *linear scale*, which means that the evenly spaced units shown in the figure represent equal distances. In this case, each unit represents 100 million kilometers.

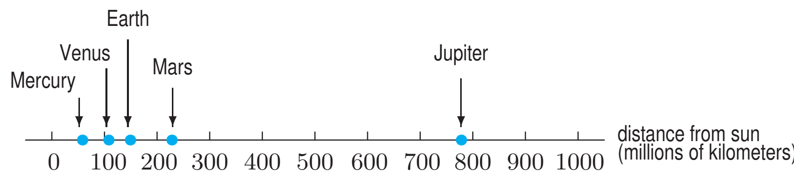

Figure 5.14: The distance from the sun of the first five planets (in millions of kilometers)

The drawback of Figure 5.14 is that the scale is too small to show all of the astronomical distances described by the table. For example, to show the distance to Pluto on this scale would require over six times the space on the page. Even worse, assuming that each 100 million km unit on the scale measures half an inch on the printed page, we would need 3 miles of paper to show the distance to Proxima Centauri!

You might conclude that we could fix this problem by choosing a larger scale. In Figure 5.15 each unit on the scale is 1 billion kilometers. Notice that all five planets shown by Figure 5.14 are crowded into the first unit of Figure 5.15; even so, the distance to Pluto barely fits. The distances to the other objects certainly don't fit. For instance, to show the Andromeda Galaxy, Figure 5.15 would have to be almost 200,000 miles long. Choosing an even larger scale will not improve the situation.

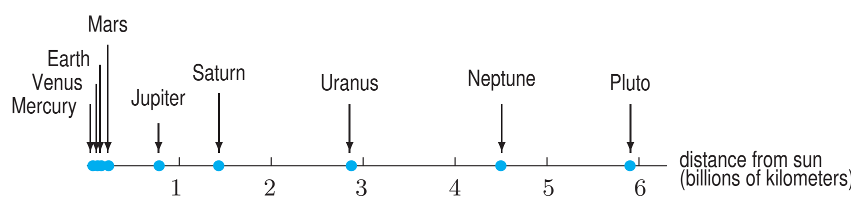

Figure 5.15: The distance to all nine planets (in billions of kilometers)

Logarithmic Scales

We conclude that the data in Table 5.5 cannot easily be represented on a linear scale. If the scale is too small, the more distant objects do not fit; if the scale is too large, the less distant objects are indistinguishable. The problem is not that the numbers are too big or too small; the problem is that the numbers vary too greatly in size.

We consider a different type of scale on which equal distances are not evenly spaced. All the objects from Table 5.5 are represented in Figure 5.16. The nine planets are still cramped, but it is possible to tell them apart. Each tick mark on the scale in Figure 5.16 represents a distance ten times larger than the one before it. This kind of scale is called *logarithmic*.

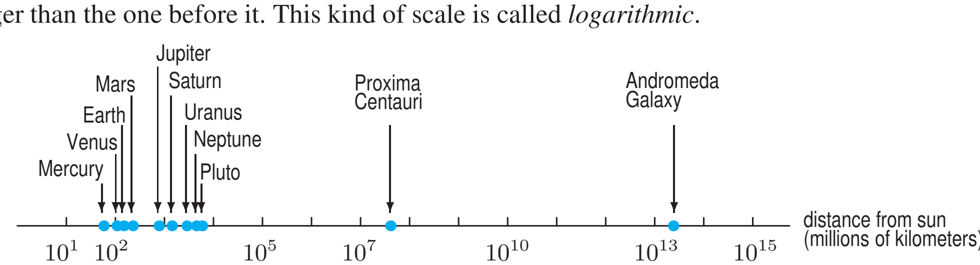

Figure 5.16: The distance from the sun (in millions of kilometers)

How Do We Plot Data on a Logarithmic Scale?

A logarithmic scale is marked with increasing powers of 10: 10^1, 10^2, 10^3, and so on. Notice that even though the distances in Figure 5.16 are not evenly spaced, the exponents are evenly spaced. Therefore the distances in Figure 5.16 are spaced according to their logarithms.

In order to plot Mercury's distance from the sun, 58 million kilometers, we use the fact that

$$10 < 58 < 100,$$

so Mercury's distance is between 10^1 and 10^2, as shown in Figure 5.16. To plot Mercury's distance more precisely, calculate $\log 58 = 1.763$, so $10^{1.763} = 58$, and use 1.763 to represent Mercury's position. See Figure 5.17.

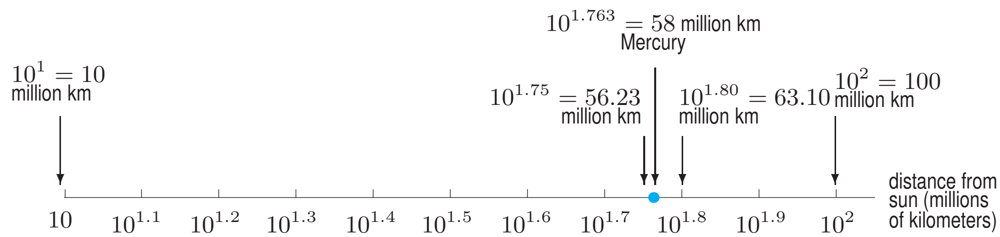

Figure 5.17: Mercury's distance, 58 million kilometers, falls between $10^{1.75} = 56.23$ and $10^{1.80} = 63.10$ million kilometers

Example 1 Where should Saturn be on the logarithmic scale? What about the Andromeda Galaxy?

Solution Saturn's distance is 1426 million kilometers, so we want the exponent of 10 that gives 1426, which is

$$\log 1426 \approx 3.154119526.$$

Thus $10^{3.154} \approx 1426$, so we use 3.154 to indicate Saturn's distance.

Similarly, the distance to the Andromeda Galaxy is $2.4 \cdot 10^{13}$ million kilometers, and since

$$\log(2.4 \cdot 10^{13}) \approx 13.38,$$

we use 13.38 to represent the galaxy's distance. See Figure 5.18.

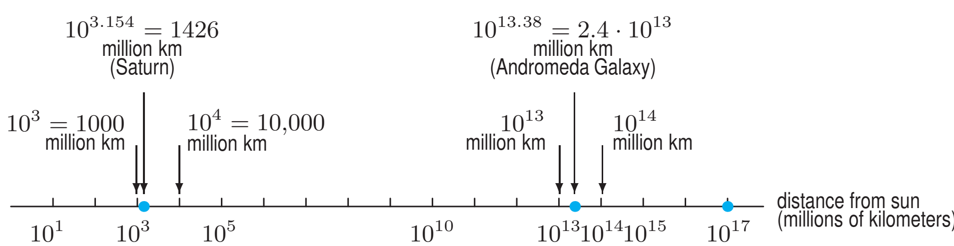

Figure 5.18: Saturn's distance is $10^{3.154}$ and the Andromeda Galaxy's distance is $10^{13.38}$

Logs of Small Numbers

The history of the world, like the distance to the stars and planets, involves numbers of vastly different sizes. Table 5.6 gives the ages of certain events[10] and the logarithms of their ages. The logarithms have been used to plot the events in Figure 5.19.

[10] *CRC Handbook,* 75^{th} ed., sec. 14-8.

Table 5.6 *Ages of various events in earth's history and logarithms of the ages*

Event	Age (millions of years)	log (age)	Event	Age (millions of years)	log (age)
Man emerges	1	0	Rise of dinosaurs	245	2.39
Ape-man fossils	5	0.70	Vertebrates appear	570	2.76
Rise of cats, dogs, pigs	37	1.57	First plants	2500	3.40
Demise of dinosaurs	67	1.83	Earth forms	4450	3.65

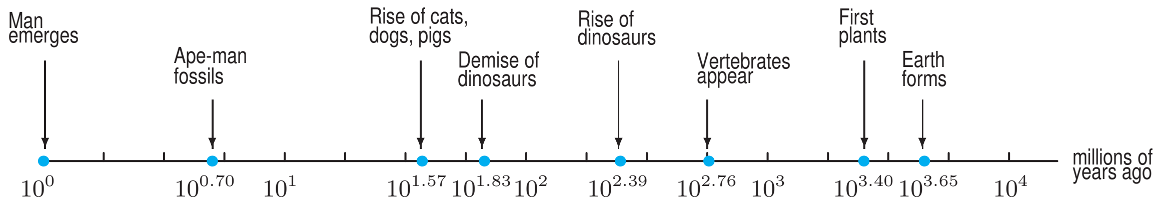

Figure 5.19: Logarithmic scale showing the ages of various events (in millions of years ago)

The events described by Table 5.6 all happened at least 1 million years ago. How do we indicate events which occurred less than 1 million years ago on the log scale?

Example 2 Where should the building of the pyramids be indicated on the log scale?

Solution The pyramids were built about 5000 years ago, or

$$\frac{5000}{1{,}000{,}000} = 0.005 \text{ million years ago.}$$

Notice that 0.005 is between 0.001 and 0.01, that is,

$$10^{-3} < 0.005 < 10^{-2}.$$

Since

$$\log 0.005 \approx -2.30,$$

we use -2.30 for the pyramids. See Figure 5.20.

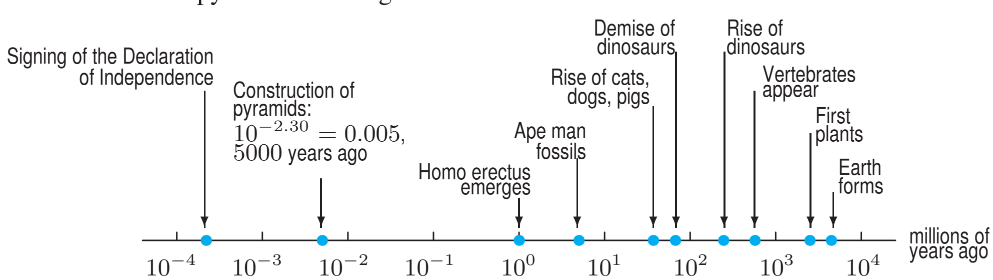

Figure 5.20: Logarithmic scale showing the ages of various events. Note that events that are less than 1 million years old are indicated by negative exponents

Another Way to Label a Log Scale

In Figures 5.19 and 5.20, the log scale has been labeled so that exponents are evenly spaced. Another way to label a log scale is with the values themselves instead of the exponents. This has been done in Figure 5.21.

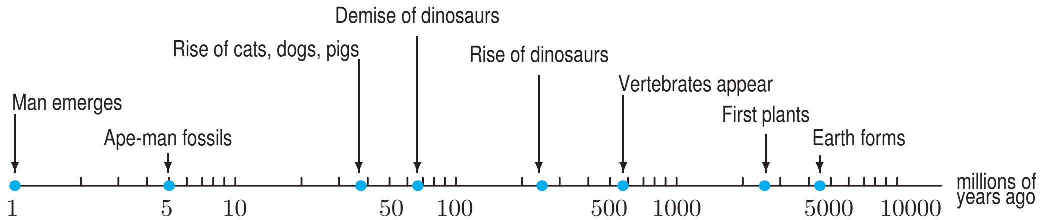

Figure 5.21: Axis labeled using actual values, not logs

Notice the characteristic way that the labels and tick marks "pile up" on each interval. The even spacing between exponents on log scales leads to uneven spacing in values. Although the values 10, 20, 30, 40, and 50 are evenly spaced, their corresponding exponents are not: $\log 10 = 1$, $\log 20 = 1.30$, $\log 30 = 1.48$, $\log 40 = 1.60$, and $\log 50 = 1.70$. Therefore, when we label an axis according to values on a scale that is spaced according to exponents, the labels get bunched up.

Log-Log Scales

Table 5.7 shows the average metabolic rate in kilocalories per day (kcal/day) for animals of different weights.[11] (A kilocalorie is the same as a standard nutritional calorie.) For instance, a 1-lb rat consumes about 35 kcal/day, whereas a 1750-lb horse consumes almost 9500 kcal/day.

Table 5.7 *The metabolic rate (in kcal/day) for animals of different weights*

Animal	Weight (lbs)	Rate (kcal/day)
Rat	1	35
Cat	8	166
Human	150	2000
Horse	1750	9470

It is not practical to plot these data on an ordinary set of axes. The values span too broad a range. However, we can plot the data using log scales for both the horizontal (weight) axis and the vertical (rate) axis. See Figure 5.22. Figure 5.23 shows a close-up view of the data point for cats to make it easier to see how the labels work. Once again, notice the characteristic piling up of labels and gridlines. This happens for the same reason as in Figure 5.21.

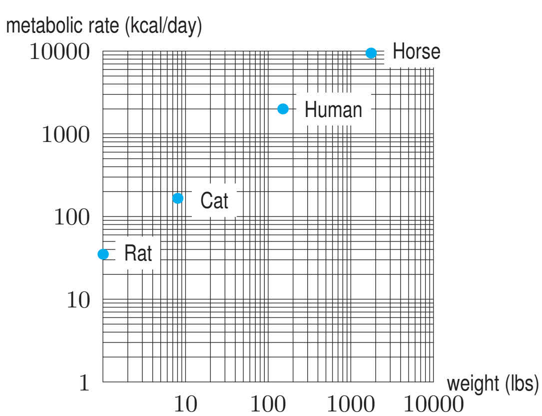

Figure 5.22: Metabolic rate (in kcal/hr) plotted against body weight

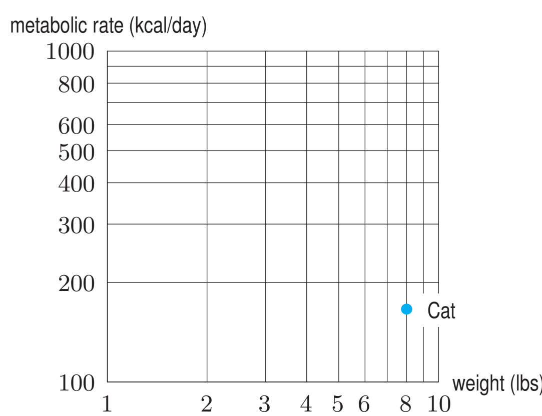

Figure 5.23: A close-up view of the Cat data point

[11] *The New York Times*, January 11, 1999.

Using Logs to Fit an Exponential Function to Data

In Section 1.6 we used linear regression to find the equation for a line of best fit for a set of data. What if the data do not lie close to a straight line, but instead approximate the graph of some other function? In this section we see how logarithms help us fit data with an exponential function of the form $Q = a \cdot b^t$.

Sales of Compact Discs

Table 5.8 shows the fall in the sales of vinyl long-playing records (LPs) and the rise of compact discs (CDs) during for the years 1982 through 1993.[12]

Table 5.8 *CD and LP sales*

t, years since 1982	c, CDs (millions)	l, LPs (millions)
0	0	244
1	0.8	210
2	5.8	205
3	23	167
4	53	125
5	102	107
6	150	72
7	207	35
8	287	12
9	333	4.8
10	408	2.3
11	495	1.2

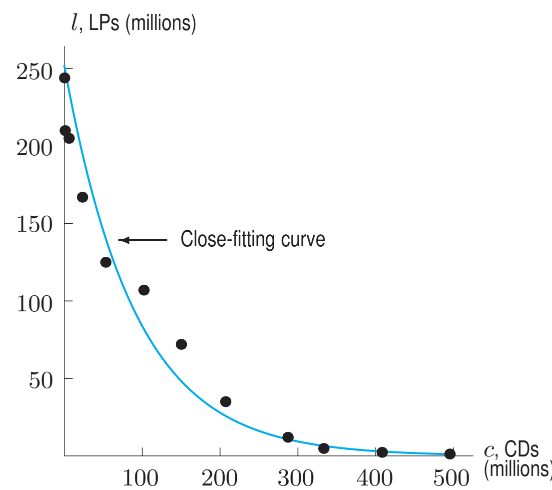

Figure 5.24: The number of LPs sold, l, as a function of number of CDs sold, c

From Table 5.8, we see that as CD sales rose dramatically during the 1980s and early 1990s, LP sales declined equally dramatically. Figure 5.24 shows the number of LPs sold in a given year as a function of the number of CDs sold that year.

Using a Log Scale to Linearize Data

In Section 5.4, we saw that a log scale allows us to compare values that vary over a wide range. Let's see what happens when we use a log scale to plot the data shown in Figure 5.24. Table 5.9 shows values $\log l$, where l is LP sales. These are plotted against c, CD sales, in Figure 5.25. Notice that plotting the data in this way tends to *linearize* the graph—that is, make it look more like a line. A line has been drawn in to emphasize the trend in the data.

Finding a Formula for the Curve

We say that the data in the third column of Table 5.9 have been *transformed*. A calculator or computer gives a regression line for the transformed data:[13]

$$y = 5.52 - 0.011c.$$

[12]Data from Recording Industry Association of America, Inc., 1998.
[13]The values obtained by a computer or another calculator may vary slightly from the ones given.

Table 5.9 *Values of $y = \ln l$ and c.*

c, CDs	l, LPs	$y = \ln l$
0	244	5.50
0.8	210	5.35
5.8	205	5.32
23	167	5.12
53	125	4.83
102	107	4.67
150	72	4.28
207	35	3.56
287	12	2.48
333	4.8	1.57
408	2.3	0.83
495	1.2	0.18

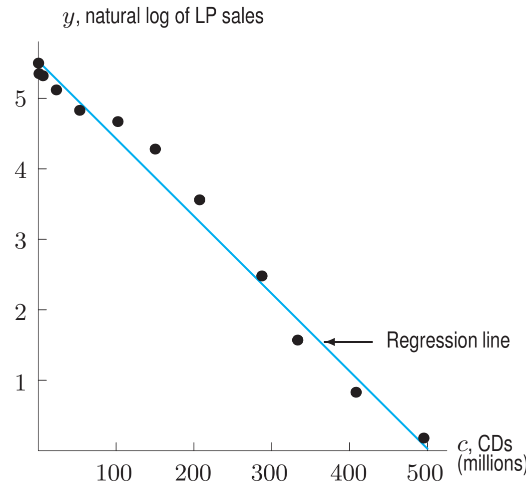

Figure 5.25: The y-axis of this graph gives the natural log of LP sales

Notice that this equation gives y in terms of c. To transform the equation back to our original variables, l and c, we substitute $\ln l$ for y, giving

$$\ln l = 5.52 - 0.011c.$$

We solve for l by raising e to both sides:

$$\begin{aligned} e^{\ln l} &= e^{5.52-0.011c} \\ &= (e^{5.52})(e^{-0.011c}). \qquad \text{Using an exponent rule} \end{aligned}$$

Since $e^{\ln l} = l$ and $e^{5.52} \approx 250$, we have

$$l = 250e^{-0.011c}.$$

This is the equation of the curve in Figure 5.24.

Fitting an Exponential Function to Data

In general, to fit an exponential formula, $N = ae^{kt}$, to a set of data of the form (t, N), we use three steps. First, we transform the data by taking the natural log of both sides and making the substitution $y = \ln N$. This leads to the equation

$$\begin{aligned} y = \ln N &= \ln\left(ae^{kt}\right) \\ &= \ln a + \ln e^{kt} \\ &= \ln a + kt. \end{aligned}$$

Setting $b = \ln a$ gives a linear equation with k as the slope and b as the y-intercept:

$$y = b + kt.$$

Secondly, we can now use linear regression on the variables t and y. (Remember that $y = \ln N$.) Finally, as step three, we transform the linear regression equation back into our original variables by substituting $\ln N$ for y and solving for N.

Exercises and Problems for Section 5.4

Skill Refresher

In Exercises S1–S6, write the numbers in scientific notation.

S1. One million four hundred fifty-five thousand

S2. Four hundred twenty three billion

S3. 64.7×10^3

S4. 12,310,000

S5. 0.00036

S6. 0.00471

In Exercises S7–S10, without a calculator, determine between which two powers of ten the following numbers lie.

S7. 12,500

S8. 0.000881

S9. $\frac{1}{3}$

S10. $3{,}850 \cdot 10^8$

Exercises

In Exercises 1–4, you wish to graph the quantities on a standard piece of paper. On which should you use a logarithmic scale? On which a linear scale? Why?

1. The wealth of 20 different people, one of whom is a multi-billionaire.

2. The number of diamonds owned by 20 people, one of whom is a multi-billionaire.

3. The number of meals per week eaten in restaurants for a random sample of 20 people worldwide.

4. The number of tuberculosis bacteria in 20 different people, some never exposed to the disease, some slightly exposed, some with mild cases, and some dying of it.

5. **(a)** Use a calculator to fill in the following tables (round to 4 decimal digits).

n	1	2	3	4	5	6	7	8	9
$\log n$									

n	10	20	30	40	50	60	70	80	90
$\log n$									

(b) Using the results of part (a), plot the integer points 2 through 9 and the multiples of 10 from 20 to 90 on the log scaled axis shown in Figure 5.26.

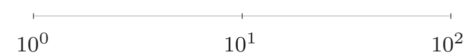

Figure 5.26

For the tables in Exercises 6–8,

(a) Use linear regression to find a linear function $y = b+mx$ that fits the data. Record the correlation coefficient.

(b) Use linear regression on the values x and $\ln y$ to fit a function of the form $\ln y = b + mx$. Record the correlation coefficient. Convert to an exponential function $y = ae^{kx}$.

(c) Compare the correlation coefficients. Graph the data and the two functions to assess which function fits best.

6.

x	y
30	70
85	120
122	145
157	175
255	250
312	300

7.

x	y
8	23
17	150
23	496
26	860
32	2720
37	8051

8.

x	y
3.2	35
4.7	100
5.1	100
5.5	150
6.8	200
7.6	300

Problems

9. The signing of the Declaration of Independence is marked on the log scale in Figure 5.20 on page 208. To two decimal places, what is its position?

10. **(a)** Draw a line segment about 5 inches long. On it, choose an appropriate linear scale and mark points that represent the integral powers of two from zero to the sixth power. What is true about the location of the points as the exponents get larger?

(b) Draw a second line segment. Repeat the process in (a) but this time use a logarithmic scale so that the units are now powers of ten. What do you notice about the location of these points?

11. Figure 5.27 shows the prices of seven different items, with the scale markings representing the logarithm of the price in dollars. Give the approximate price of each item,and explain why a log scale was necessary.

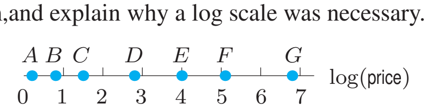

Figure 5.27

12. Microfinance refers to financial services, such as loans, offered to people with very low incomes. Table 5.10 shows the number of microborrowers in 2006.[14]

(a) Plot the data (in millions of borrowers) on a linear scale.
(b) Plot the data on a logarithmic scale.
(c) Which scale is more appropriate? Why?

Table 5.10

Region	Borrowers (millions)
A: Africa	8.4
B: Asia	112.7
C: Eastern Europe and Central Asia	3.4
D: Latin America and the Caribbean	6.8
E: Middle East and North Africa	1.7
F: North America and Western Europe	0.05

13. The usual distances for track (running) events are 100 meters, 200 meters, 400 meters, 800 meters, 1500 meters, 3000 meters, 5000 meters, and 10,000 meters.

(a) Plot the length of each track event on a linear scale.
(b) Plot the length of each track event on a logarithmic scale.
(c) Which scale, (a) or (b), is more useful to the runner?
(d) On each figure identify the point corresponding to 50 meters.

14. Table 5.11 shows the numbers of deaths in 2006 due to various causes in the US.[15]

(a) Explain why a log scale is necessary to plot the data from Table 5.11.
(b) Find the log of each value given.
(c) Plot the data using a log scale. Label each point with the related cause.

Table 5.11

Cause	Deaths
Scarlet fever	2
Whooping cough	9
Asthma	3613
HIV	12,113
Kidney diseases	46,095
Accidents	121,599
Malignant neoplasms	559,888
Cardiovascular disease	823,746
All causes	2,426,264

15. Table 5.12 shows the dollar value of some items in 2004. Plot and label these values on a log scale.

Table 5.12

Item	Dollar value	Item	Dollar value
Pack of gum	0.50	New house	264,400
Movie ticket	9.00	Lottery winnings	100 million
New computer	1200	Bill Gates' worth	46.6 bn
Year at college	27,500	National debt	7,500 bn
Luxury car	60,400	US GDP	11,700 bn

16. Table 5.13 shows the sizes of various organisms. Plot and label these values on a log scale.

Table 5.13

Animal	Size (cm)	Animal	Size (cm)
Virus	0.0000005	Domestic cat	60
Bacterium	0.0002	Wolf (with tail)	200
Human cell	0.002	Thresher shark	600
Ant	0.8	Giant squid	2200
Hummingbird	12	Sequoia	7500

17. **(a)** Plot the data in Table 5.14.
(b) What kind of function might the data from part (a) represent?
(c) Now plot $\log y$ versus x instead of y versus x. What do you notice?

Table 5.14

x	0.2	1.3	2.1	2.8	3.4	4.5
y	5.7	12.3	21.4	34.8	52.8	113.1

18. **(a)** Complete the Table 5.15 with values of $y = 3^x$.
(b) Complete Table 5.16 with values for $y = \log(3^x)$. What kind of function is this?
(c) Complete tables for $f(x) = 2 \cdot 5^x$ and $g(x) = \log(2 \cdot 5^x)$. What kinds of functions are these?
(d) What seems to be true about a function which is the logarithm of an exponential function? Is this true in general?

Table 5.15

x	0	1	2	3	4	5
$y = 3^x$						

Table 5.16

x	0	1	2	3	4	5
$y = \log(3^x)$						

[14]Based on data in Daley-Harris, *State of the Microsummit Campaign Report 2007*, p. 22.
[15]www.cdc.gov/nchs/data/nvsr/nvsr58/nvsr58_01.pdf, accessed May 30, 2010.

19. Repeat part (b) and (c) of Problem 18 using the natural log function. Is your answer to part (d) the same?

20. Table 5.17 shows newspapers' share of the expenditure of national advertisers. Using the method of Problem 22, fit an exponential function of the form $y = ae^{kx}$ to the data, where y is percent share and x is the number of years since 1950.

Table 5.17 *Newspapers' share of advertising*

	1950	1960	1970	1980	1990	1992
x	0	10	20	30	40	42
y	16.0	10.8	8.0	6.7	5.8	5.0

21. To study how recognition memory decreases with time, the following experiment was conducted. The subject read a list of 20 words slowly aloud, and later, at different time intervals, was shown a list of 40 words containing the 20 words that he or she had read. The percentage, P, of words recognized was recorded as a function of t, the time elapsed in minutes. Table 5.18 shows the averages for 5 different subjects.[16] This is modeled by $P = a \ln t + b$.

(a) Find $\ln t$ for each value of t, and then use regression on a calculator or computer to estimate a and b.
(b) Graph the data points and regression line on a coordinate system of P against $\ln t$.
(c) When does this model predict that the subjects will recognize no words? All words?
(d) Graph the data points and curve $P = a \ln t + b$ on a coordinate system with P against t, with $0 \le t \le 10{,}500$.

Table 5.18 *Percentage of words recognized*

t, min	5	15	30	60	120	240
P%	73.0	61.7	58.3	55.7	50.3	46.7
t, min	480	720	1440	2880	5760	10,080
P%	40.3	38.3	29.0	24.0	18.7	10.3

22. Table 5.19 shows the value, y, of US imports from China with x in years since 2002.

(a) Find a formula for a linear function $y = b+mx$ that approximates the data.
(b) Find $\ln y$ for each y value, and use the x and $\ln y$ values to find a formula for a linear function $\ln y = b + mx$ that approximates the data.
(c) Use the equation in part (b) to find an exponential function of the form $y = ae^{kx}$ that fits the data.

Table 5.19 *Value of US imports from China in millions of dollars*

	2002	2003	2004	2005	2006	2007	2008
x	0	1	2	3	4	5	6
y	125,193	152,436	196,682	243,470	287,774	321,443	337,773

23. Table 5.20 gives the length ℓ (in cm) and weight w (in gm) of 16 different fish known as threadfin bream (*Nemipterus marginatus*) found in the South China Sea.[17]

(a) Let $W = \ln w$ and $L = \ln \ell$. For these sixteen data points, plot W on the vertical axis and L on the horizontal axis. Describe the resulting scatterplot.
(b) Fitting a line to the scatterplot you drew in part (a), find a possible formula for W in terms of L.
(c) Based on your formula for part (b), find a possible formula for w in terms of ℓ.
(d) Comment on your formula, keeping in mind what you know about units as well the typical relationship between weight, volume, and length.

Table 5.20 *Length and weight of fish*

Type	1	2	3	4	5	6	7	8
ℓ	8.1	9.1	10.2	11.9	12.2	13.8	14.8	15.7
w	6.3	9.6	11.6	18.5	26.2	36.1	40.1	47.3
Type	9	10	11	12	13	14	15	16
ℓ	16.6	17.7	18.7	19.0	20.6	21.9	22.9	23.5
w	65.6	69.4	76.4	82.5	106.6	119.8	169.2	173.3

24. A light, flashing regularly, consists of cycles, each cycle having a dark phase and a light phase. The frequency of this light is measured in cycles per second. As the frequency is increased, the eye initially perceives a series of flashes of light, then a coarse flicker, a fine flicker, and ultimately a steady light. The frequency at which the flickering disappears is called the fusion frequency.[18] Table 5.21 shows the results of an experiment[19] in which the fusion frequency F was measured as a function of the light intensity I. It is modeled by $F = a \ln I + b$.

(a) Find $\ln I$ for each value of I, and then use linear regression on a calculator or computer to estimate a and b in the equation $F = a \ln I + b$.
(b) Plot F against $\ln I$, showing the data points and the line.

[16] Adapted from D. Lewis, *Quantitative Methods in Psychology* (New York: McGraw-Hill, 1960).
[17] Data taken from *Introduction to Tropical Fish Stock Assessment* by Per Sparre, Danish Institute for Fisheries Research, and Siebren C. Venema, FAO Fisheries Department, available at http://www.fao.org/docrep/W5449E/w5449e00.htm. This source cites the following original reference: Pauly, D., 1983. Some simple methods for the assessment of tropical fish stocks.
[18] R. S. Woodworth, *Experimental Psychology* (New York: Holt and Company, 1948).
[19] D. Lewis, *Quantitative Methods in Psychology* (New York: McGraw-Hill, 1960).

(c) Plot F against I, showing the data points and the curve and give its equation.

(d) The units of I are arbitrary, that is, not given. If the units of I were changed, which of the constants a and b would be affected, and in what way?

Table 5.21 *Fusion frequency, F, as a function of the light intensity, I*

I	0.8	1.9	4.4	10.0	21.4	48.4	92.5	218.7	437.3	980.0
F	8.0	12.1	15.2	18.5	21.7	25.3	28.3	31.9	35.2	38.2

CHAPTER SUMMARY

- **Logarithms**
  Common log: $y = \log x$ means $10^y = x$.
  $\log 10 = 1, \log 1 = 0.$
  Natural log: $y = \ln x$ means $e^y = x$.
  $\ln e = 1, \ln 1 = 0.$
- **Properties of Logs**

$$\log(ab) = \log a + \log b \qquad \ln(ab) = \ln a + \ln b.$$
$$\log(a/b) = \log a - \log b \qquad \ln(a/b) = \ln a - \ln b.$$
$$\log(b^t) = t\log b \qquad \ln(b^t) = t\ln b.$$
$$\log(10^x) = 10^{\log x} = x \qquad \ln(e^x) = e^{\ln x} = x.$$

- **Converting Between Base b and Base e**
  If $Q = ab^t$ and $Q = ae^{kt}$, then $k = \ln b$.
- **Solving Equations Using Logs**
  Solve equations such as $ab^t = c$ and $ae^{kt} = c$ using logs.
  Not all exponential equations can be solved with logs, e.g. $2^t = 3 + t$.
- **Logarithmic Functions**
  Graph; domain; range; concavity; asymptotes.
- **Applications of Logarithms**
  Doubling time; half life;
  Chemical acidity; orders of magnitude; decibels.
- **Logarithmic Scales**
  Plotting data; log-log scales. Linearizing data and fitting curves to data using logs.
- **Limits and Limits from the Right and from the Left**

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER FIVE

Exercises

In Exercises 1–2, convert to the form $Q = ab^t$.

1. $Q = 7e^{-10t}$

2. $Q = 5e^t$

In Exercises 3–6, convert to the form $Q = ae^{kt}$.

3. $Q = 4 \cdot 7^t$

4. $Q = 2 \cdot 3^t$

5. $Q = 4 \cdot 8^{1.3t}$

6. $Q = 973 \cdot 6^{2.1t}$

Solve the equations in Exercises 7–22 exactly if possible.

7. $1.04^t = 3$

8. $e^{0.15t} = 25$

9. $3(1.081)^t = 14$

10. $40e^{-0.2t} = 12$

11. $5(1.014)^{3t} = 12$

12. $5(1.15)^t = 8(1.07)^t$

13. $5(1.031)^x = 8$

14. $4(1.171)^x = 7(1.088)^x$

15. $3\log(2x + 6) = 6$

16. $1.7(2.1)^{3x} = 2(4.5)^x$

17. $3^{4\log x} = 5$

18. $100^{2x+3} = \sqrt[3]{10{,}000}$

19. $13e^{0.081t} = 25e^{0.032t}$

20. $87e^{0.066t} = 3t + 7$

21. $\dfrac{\log x^2 + \log x^3}{\log(100x)} = 3$

22. $\log x + \log(x - 1) = \log 2$

In Exercises 23–25, simplify fully.

23. $\log\left(100^{x+1}\right)$

24. $\ln\left(e \cdot e^{2+M}\right)$

25. $\ln(A + B) - \ln(A^{-1} + B^{-1})$

In Exercises 26–31, state the domain of the function and identify any vertical asymptote of its graph. You need not graph the function.

26. $y = \ln(x + 8)$

27. $y = \log(x - 20)$

28. $y = \log(12 - x)$

29. $y = \ln(300 - x)$

30. $y = \ln\left(x - e^2\right)$

31. $y = \log(x + 15)$

In Exercises 32–37, say where you would mark the given animal lifespan on an inch scale, where 0 inches represents $10^0 = 1$ year and 5 inches represents $10^5 = 100{,}000$ years. Give your answer to the nearest tenth of an inch.[20]

32. A colony of quaking aspen in Utah is estimated to be 80,000 years old.

33. A bristlecone pine in California named Methuselah is estimated by ring count to be 4838 years old.

34. A specimen of antarctic sponge is estimated to be 1550 years old.

35. The Puget Sound saltwater clam called a geoduck can live for 160 years.

36. The oldest recorded dog lived 29 years.

37. The common house mouse can live 4 years in captivity.

Problems

38. Suppose that $x = \log A$ and that $y = \log B$. Write the following expressions in terms of x and y.

(a) $\log(AB)$ **(b)** $\log(A^3 \cdot \sqrt{B})$
(c) $\log(A - B)$ **(d)** $\dfrac{\log A}{\log B}$
(e) $\log \dfrac{A}{B}$ **(f)** AB

39. Let $p = \ln m$ and $q = \ln n$. Write the following expressions in terms of p and/or q without using logs.

(a) $\ln(nm^4)$ **(b)** $\ln\left(\dfrac{1}{n}\right)$
(c) $\dfrac{\ln m}{\ln n}$ **(d)** $\ln(n^3)$

40. Let $x = 10^U$ and $y = 10^V$. Write the following expressions in terms of U and/or V without using logs.

(a) $\log xy$ **(b)** $\log\left(\dfrac{x}{y}\right)$
(c) $\log x^3$ **(d)** $\log\left(\dfrac{1}{y}\right)$

41. Solve the following equations. Give approximate solutions if exact ones can't be found.

(a) $e^{x+3} = 8$ **(b)** $4(1.12^x) = 5$
(c) $e^{-0.13x} = 4$ **(d)** $\log(x - 5) = 2$
(e) $2\ln(3x) + 5 = 8$ **(f)** $\ln x - \ln(x-1) = 1/2$
(g) $e^x = 3x + 5$ **(h)** $3^x = x^3$
(i) $\ln x = -x^2$

42. Solve for x exactly.

(a) $\dfrac{3^x}{5^{x-1}} = 2^{x-1}$
(b) $-3 + e^{x+1} = 2 + e^{x-2}$
(c) $\ln(2x - 2) - \ln(x - 1) = \ln x$
(d) $9^x - 7 \cdot 3^x = -6$
(e) $\ln\left(\dfrac{e^{4x} + 3}{e}\right) = 1$
(f) $\dfrac{\ln(8x) - 2\ln(2x)}{\ln x} = 1$

In Problems 43–46, , the Richter scale ratings for two earthquakes are M_1 and M_2, with $M_2 > M_1$. If the earthquakes have seismic waves of sizes W_1 and W_2, respectively, then

$$M_2 - M_1 = \log\left(\frac{W_2}{W_1}\right).$$

How many times greater than the smaller one are the seismic waves for the larger one?

43. $M_1 = 4.2$ and $M_2 = 6.4$

44. $M_1 = 5.3$ and $M_2 = 5.8$

45. $M_1 = 4.4$ and $M_2 = 5.6$

46. $M_1 = 5.7$ and $M_2 = 8.1$

47. With t in years, the formulas for dollar balances of two bank accounts are:

$$f(t) = 1100(1.05)^t \quad \text{and} \quad g(t) = 1500e^{0.05t}.$$

(a) Describe in words the bank account modeled by f.
(b) Describe the account modeled by g. State the effective annual rate.
(c) What continuous interest rate has the same effective growth rate as f?

48. **(a)** Let $B = 5000(1.06)^t$ give the balance of a bank account after t years. If the formula for B is written $B = 5000e^{kt}$, estimate the value of k correct to four decimal places. What is the financial meaning of k?
(b) The balance of a bank account after t years is given by the formula $B = 7500e^{0.072t}$. If the formula for B is written $B = 7500b^t$, find b exactly, and give the value of b correct to four decimal places. What is the financial meaning of b?

49. The number of bacteria present in a culture after t hours is given by the formula $N = 1000e^{0.69t}$.

(a) How many bacteria will there be after $1/2$ hour?
(b) How long before there are $1{,}000{,}000$ bacteria?
(c) What is the doubling time?

[20]From http://en.wikipedia.org/wiki/Maximum_life_span and http://en.wikipedia.org/wiki/List_of_long-living_organisms, accessed June, 2010.

50. In 2010, the population of the country Erehwon was 50 million people and increasing by 2.9% every year. The population of the country Ecalpon, on other hand, was 45 million people and increasing by 3.2% every year.

(a) For each country, write a formula expressing the population as a function of time t, where t is the number of years since 2010.
(b) Find the value(s) of t, if any, when the two countries have the same population.
(c) When is the population of Ecalpon double that of Erehwon?

51. The price $P(t) = 5(2)^{t/7}$ of a good is rising due to inflation, where t is time in years.

(a) What is the doubling time?
(b) What is the annual inflation rate?

52. Let $P = 15(1.04)^t$ give the population (in thousands) of a town, with t in years.

(a) Describe the population growth in words.
(b) If the formula for P is written $P = 15(b)^{12t}$, find b exactly. What is the meaning of b in the context of the population?
(c) If the formula for P is written $P = 15(2)^{t/c}$, find the value of c correct to 2 decimals. What is the meaning of c in this context?

In Problems 53–55, use $v(t) = 20e^{0.2t}$ and $w(t) = 12e^{0.22t}$.

53. Solve $v(t) = 30$ exactly.

54. Solve $3v(2t) = 2w(3t)$ exactly.

55. Find the doubling time of w.

56. A calculator confirms that $5 \approx 10^{0.7}$. Show how to use this fact to approximate the value of $\log 25$.

57. **(a)** What are the domain and range of $f(x) = 10^x$? What is the asymptote of $f(x) = 10^x$?
(b) What does your answer to part (a) tell you about the domain, range, and asymptotes of $g(x) = \log x$?

58. What is the domain of $y = \ln(x^2 - x - 6)$?

59. **(a)** Plot the data given by Table 5.22. What kind of function might fit this data well?
(b) Using the substitution $z = \ln x$, transform the data in Table 5.22, and compile your results into a new table. Plot the transformed data as $\ln x$ versus y.
(c) What kind of function gives a good fit to the plot you made in part (b)? Find a formula for y in terms of z that fits the data well.
(d) Using the formula from part (c), find a formula for y in terms of x that gives a good fit to the data in Table 5.22.
(e) What does your formula from part (d) tell you about x as a function of y (as opposed to y as a function of x)?

Table 5.22

x	0.21	0.55	1.31	3.22	5.15	12.48
y	-11	-2	6.5	16	20.5	29

60. Radioactive carbon-14 decays according to the function $Q(t) = Q_0e^{-0.000121t}$ where t is time in years, $Q(t)$ is the quantity remaining at time t, and Q_0 is the amount of present at time $t = 0$. Estimate the age of a skull if 23% of the original quantity of carbon-14 remains.

61. Suppose 2 mg of a drug is injected into a person's bloodstream. As the drug is metabolized, the quantity diminishes at the continuous rate of 4% per hour.

(a) Find a formula for $Q(t)$, the quantity of the drug remaining in the body after t hours.
(b) By what percent does the drug level decrease during any given hour?
(c) The person must receive an additional 2 mg of the drug whenever its level has diminished to 0.25 mg. When must the person receive the second injection?
(d) When must the person receive the third injection?

62. A rubber ball is dropped onto a hard surface from a height of 6 feet, and it bounces up and down. At each bounce it rises to 90% of the height from which it fell.

(a) Find a formula for $h(n)$, the height reached by the ball on bounce n.
(b) How high will the ball bounce on the 12^{th} bounce?
(c) How many bounces before the ball rises no higher than an inch?

63. Oil leaks from a tank. At hour $t = 0$ there are 250 gallons of oil in the tank. Each hour after that, 4% of the oil leaks out.

(a) What percent of the original 250 gallons has leaked out after 10 hours? Why is it less than $10 \cdot 4\% = 40\%$?
(b) If $Q(t) = Q_0e^{kt}$ is the quantity of oil remaining after t hours, find the value of k. What does k tell you about the leaking oil?

64. Before the advent of computers, logarithms were calculated by hand. Various tricks were used to evaluate different logs. One such trick exploits the fact that $2^{10} \approx 1000$. (It actually equals 1024.) Using this fact and the log properties, show that

(a) $\log 2 \approx 0.3$ **(b)** $\log 7 \approx 0.85$

65. A googol is the number 1 followed by 100 zeros, or 10^{100}. A googolplex is the number 1 followed by a googol zeros, or 10^{googol}. Evaluate:

(a) $\sqrt{\log(\text{googol})}$ **(b)** $\log\sqrt{\text{googol}}$

(c) $\sqrt{\log(\text{googolplex})}$

66. Since $e = 2.718\ldots$ we know that $2 < e < 3$, which means that $2^2 < e^2 < 3^2$. Without using a calculator, explain why

(a) $1 < \ln 3 < 2$ **(b)** $1 < \ln 4 < 2$

67. Simplify the expression $\sqrt{1000^{\frac{1}{12}\cdot\log k}}$. Your answer should be exact and should not involve exponents or logs, though it may involve radicals.

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–37 true or false? Give an explanation for your answer.

1. The log of 2000 is less than 3.

2. The inverse of $y = \ln x$ is $y = e^x$.

3. If $2^x = 1024$ then $x = 10$.

4. If a quantity grew to 4 times its original amount in 8 hours then its doubling time is one-half hour.

5. If the function $y = ab^t$ is converted to the form $y = ae^{kt}$, k is always equal to $\ln b$.

6. If x is a positive number, $\log x$ is the exponent of 10 that gives x.

7. If $10^y = x$ then $\log x = y$.

8. The quantity 10^{-k} is a negative number when k is positive.

9. For any n, we have $\log(10^n) = n$.

10. If $n > 0$, then $10^{\log n} = n$.

11. If a and b are positive, $\log\left(\dfrac{a}{b}\right) = \dfrac{\log a}{\log b}$.

12. If a and b are positive, $\ln(a + b) = \ln a + \ln b$.

13. For any value a, $\log a = \ln a$.

14. For any value x, $\ln(e^{2x}) = 2x$.

15. The function $y = \log x$ has an asymptote at $y = 0$.

16. The graph of the function $y = \log x$ is concave down.

17. The reflected graph of $y = \log x$ across the line $y = x$ is the graph of $y = 10^x$.

18. If $y = \log\sqrt{x}$ then $y = \frac{1}{2}\log x$.

19. The function $y = \log(b^t)$is always equal to $y = (\log b)^t$.

20. The values of $\ln e$ and $\log 10$ are both 1.

21. If $7.32 = e^t$ then $t = \dfrac{7.32}{e}$.

22. If $50(0.345)^t = 4$, then $t = \dfrac{\log(4/50)}{\log 0.345}$.

23. If $ab^t = n$, then $t = \dfrac{\log(n/a)}{\log b}$.

24. The doubling time of a quantity $Q = Q_0e^{kt}$ is the time it takes for any t-value to double.

25. The half-life of a quantity is the time it takes for the quantity to be reduced by half.

26. If the half-life of a substance is 5 hours then there will be $\frac{1}{4}$ of the substance in 25 hours.

27. If $y = 6(3)^t$, then $y = 6e^{(\ln 3)t}$.

28. If a population doubles in size every 20 years, its annual continuous growth rate is 20%.

29. If $Q = Q_0e^{kt}$, then $t = \dfrac{\ln(Q/Q_0)}{k}$.

30. Log scales provide a way to graph quantities that have vastly different magnitudes.

31. An elephant weighs about 8000 pounds. Its weight, plotted on a log scale, would be a little before 4.

32. A virus has a cell size of about 0.0000005 cm. On a log scale, its size would be plotted at about 7.

33. The closest star outside our solar system is about 26,395,630,000,000 miles away from Earth. This distance, plotted on a log scale, would occur a little after 26.

34. In a graph made using a log-log scale, consecutive powers of 10 are equally spaced on the horizontal axis and on the vertical axis.

35. One million and one billion differ by one order of magnitude.

36. After fitting a data set with both an exponential function, $y = Ae^{kx}$, and a power function, $y = Bx^n$, we must have $B = A$.

37. Given the points on a cubic curve, $(1, 1)$, $(2, 8)$, $(3, 27)$ and $(4, 64)$ it is not possible to fit an exponential function to this data.

SKILLS REFRESHER FOR CHAPTER 5: LOGARITHMS

We list the definitions and properties of the common and natural logarithms.

Properties of Logarithms If $M, N > 0$:

- Logarithm of a product: $\log MN = \log M + \log N$ $\quad \ln MN = \ln M + \ln N$
- Logarithm of a quotient: $\log M/N = \log M - \log N$ $\quad \ln M/N = \ln M - \ln N$
- Logarithm of a power: $\log M^P = P \log M$ $\quad \ln M^P = P \ln M$
- Logarithm of 1: $\log 1 = 0$ $\quad \ln 1 = 0$
- Logarithm of the base: $\log 10 = 1$ $\quad \ln e = 1$

Be aware of the following two common errors,

$$\log(M + N) \neq (\log M)(\log N)$$

and

$$\log(M - N) \neq \frac{\log M}{\log N}.$$

Relationships Between Logarithms and Exponents

- $\log N = x$ if and only if $10^x = N$ $\quad \ln N = x$ if and only if $e^x = N$
- $\log 10^x = x$ $\quad \ln e^x = x$
- $10^{\log x} = x$, for $x > 0$ $\quad e^{\ln x} = x$, for $x > 0$

Example 1 Evaluate without a calculator:

(a) $\log 10{,}000$ (b) $\ln 1$

Solution

(a) Common logarithms are powers of 10. The power of 10 needed to get 10,000 is 4, so $\log 10{,}000 = 4$.

(b) Natural logarithms are powers of e. The power of e needed to get 1 is zero, so $\ln 1 = 0$.

Example 2 Write the equation in exponential form:

(a) $\log x = -3$ (b) $\ln x = \sqrt{2}$

Solution

(a) By definition $\log x = -3$ means $10^{-3} = x$.

(b) By definition $\ln x = \sqrt{2}$ means $e^{\sqrt{2}} = x$.

Example 3 Write the equation in logarithmic form:

(a) $10^x = 1000$ (b) $10^{-2} = 0.01$ (c) $e^{-1} = 0.368$

Solution

(a) By definition $10^x = 1000$ means $\log 1000 = x$.

(b) By definition $10^{-2} = 0.01$ means $\log 0.01 = -2$.

(c) By definition $e^{-1} = 0.368$ means $\ln 0.368 = -1$.

Example 4 Write the expression using sums and/or differences of logarithmic expressions that do not contain the logarithms of products, quotients or powers.

(a) $\log(10x)$ (b) $\ln\left(\dfrac{e^2}{\sqrt{x}}\right)$

Solution (a)

$$\begin{aligned}\log(10x) &= \log 10 + \log x && \text{Logarithm of a product}\\ &= 1 + \log x. && \text{Logarithm of the base}\end{aligned}$$

(b)

$$\begin{aligned}\ln\left(\frac{e^2}{\sqrt{x}}\right) &= \ln e^2 - \ln\sqrt{x} && \text{Logarithm of a quotient}\\ &= \ln e^2 - \ln x^{1/2}\\ &= 2\ln e - \frac{1}{2}\ln x && \text{Logarithm of a power}\\ &= 2\cdot 1 - \frac{1}{2}\ln x && \text{Logarithm of the base}\\ &= 2 - \frac{1}{2}\ln x.\end{aligned}$$

Example 5 Write the expression as a single logarithm.

(a) $\ln x - 2\ln y$ (b) $3(\log x + \frac{4}{3}\log y)$

Solution (a)

$$\begin{aligned}\ln x - 2\ln y &= \ln x - \ln y^2 && \text{Logarithm of a power}\\ &= \ln\left(\frac{x}{y^2}\right). && \text{Logarithm of a quotient}\end{aligned}$$

(b)

$$\begin{aligned}3\left(\log x + \frac{4}{3}\log y\right) &= 3\log x + 4\log y\\ &= \log x^3 + \log y^4 && \text{Logarithm of a power}\\ &= \log(x^3y^4). && \text{Logarithm of a product}\end{aligned}$$

Example 6 Express in terms of x without logarithms.

(a) $e^{3\ln x}$ (b) $\log 10^{2x}$

Solution (a)

$$\begin{aligned}e^{3\ln x} &= e^{\ln x^3} && \text{Logarithm of a power}\\ &= x^3. && \text{Logarithm of the base}\end{aligned}$$

(b) $\log 10^{2x} = 2x.$ Logarithm of the base

Example 7 Solve the equation for x.

(a) $12e^x = 5$ (b) $2^{-3x} = 17$

Solution (a)

$$\begin{aligned} 12e^x &= 5 \\ e^x &= \frac{5}{12} && \text{Dividing by 12} \\ \ln e^x &= \ln\left(\frac{5}{12}\right) && \text{Taking ln of both sides} \\ x &= \ln\left(\frac{5}{12}\right). && \text{Logarithm of the base} \end{aligned}$$

(b)

$$\begin{aligned} 2^{-3x} &= 17 \\ \log 2^{-3x} &= \log 17 && \text{Taking logs of both sides} \\ -3x \log 2 &= \log 17 && \text{Logarithm of a power} \\ -3x &= \frac{\log 17}{\log 2} && \text{Dividing by log 2} \\ x &= -\frac{\log 17}{3\log 2}. && \text{Dividing by -3} \end{aligned}$$

Example 8 Solve the equation for x.

(a) $2(\log(2x+50)) - 4 = 0$ (b) $\ln(x+2) = 3$

Solution (a)

$$\begin{aligned} 2(\log(2x+50)) - 4 &= 0 \\ 2(\log(2x+50)) &= 4 \\ \log(2x+50) &= 2 \\ 10^{\log(2x+50)} &= 10^2 && \text{Converting to exponential form} \\ 2x+50 &= 10^2 \\ 2x+50 &= 100 \\ 2x &= 50 \\ x &= 25. \end{aligned}$$

(b)

$$\begin{aligned} \ln(x+2) &= 3 \\ e^{\ln(x+2)} &= e^3 && \text{Raise } e \text{ to each side} \\ x+2 &= e^3 \\ x &= e^3 - 2. \end{aligned}$$

Exercises to Tools for Chapter 5

For Exercises 1–8, evaluate without a calculator.

1. $\log(\log 10)$

2. $\ln(\ln e)$

3. $2\ln e^4$

4. $\ln\left(\frac{1}{e^5}\right)$

5. $\frac{\log 1}{\log 10^5}$

6. $e^{\ln 3} - \ln e$

7. $\sqrt{\log 10{,}000}$

8. $10^{\log 7}$

For Exercises 9–12, rewrite the exponential equation in equivalent logarithmic form.

9. $10^{-4} = 0.0001$

10. $10^{0.477} = 3$

11. $e^{-2} = 0.135$

12. $e^{2x} = 7$

For Exercises 13–15, rewrite the logarithmic equation in equivalent exponential form.

13. $\log 0.01 = -2$

14. $\ln x = -1$

15. $\ln 4 = x^2$

For Exercises 16–24, if possible, write the expression using sums and/or differences of logarithmic expressions that do not contain the logarithms of products, quotients, or powers.

16. $\log 2x$

17. $\frac{\ln x}{2}$

18. $\log\left(\frac{x}{5}\right)$

19. $\log\left(\frac{x^2+1}{x^3}\right)$

20. $\ln\sqrt{\frac{x-1}{x+1}}$

21. $\log(x^2+y^2)$

22. $\log(x^2-y^2)$

23. $(\log x)(\log y)$

24. $\frac{\ln x^2}{\ln(x+2)}$

For Exercises 25–31, rewrite the expression as a single logarithm.

25. $\log 12 + \log x$

26. $\ln x^2 - \ln(x+10)$

27. $\frac{1}{2}\log x + 4\log y$

28. $\log 3 + 2\log\sqrt{x}$

29. $3\left(\log(x+1) + \frac{2}{3}\log(x+4)\right)$

30. $\ln x + \ln\left(\frac{y}{2}(x+4)\right) + \ln z^{-1}$

31. $2\log(9-x^2) - (\log(3+x) + \log(3-x))$

For Exercises 32–39, simplify the expression if possible.

32. $2\ln e^{\sqrt{x}}$

33. $\log(A^2+B^2)$

34. $\log 10x - \log x$

35. $2\ln x^{-2} + \ln x^4$

36. $\ln\sqrt{x^2+16}$

37. $\log 100^{2z}$

38. $\frac{\ln e}{\ln e^2}$

39. $\ln\frac{1}{e^x+1}$

For Exercises 40–45, solve for x using logarithms.

40. $12^x = 7$

41. $3\cdot 5^x = 9$

42. $4\cdot 13^{3x} = 17$

43. $e^{-5x} = 9$

44. $12^{5x} = 3\cdot 15^{2x}$

45. $19^{6x} = 77\cdot 7^{4x}$

In Exercises 46–49, solve for x.

46. $3\log(4x+9) - 6 = 2$

47. $4\log(9x+17) - 5 = 1$

48. $\ln(3x+4) = 5$

49. $2\ln(6x-1) + 5 = 7$