CHAPTER 10

The Math of Finance

The price then that the borrower has to pay for the loan of capital, and which he regards as interest, is from the point of view of the lender more properly to be regarded as profits: for it includes insurance against risks which are often very heavy, and earnings of arrangement for the task, which is often very arduous, of keeping those risks as small as possible. Variations in the nature of these risks and of the task of management will of course occasion corresponding variations in the gross interest—so called that is paid of the use of money.

—Alfred Marshall, *Principles of Economics*: *Volume 2* (London: MacMillan & Co., 1890), p. 623

Investment decisions made by financial managers, to acquire capital assets such as plant and equipment, and asset managers, to acquire securities such as stocks and bonds, require the valuation of investments and the determination of yields on investments. The concept that must be understood to determine the value of an investment, the yield on an investment, and the cost of funds is the time value of money. This simple mathematical concept allows financial and asset managers to translate future cash flows to a value in the present, translate a value today into a value at some future point in time, and calculate the yield on an investment. The time-value-of-money mathematics allows an evaluation and comparison of investments and financing arrangements and is the subject of this chapter.

WHY THE TIME VALUE OF MONEY?

The notion that money has a time value is one of the most basic concepts in investment analysis. Making decisions today regarding future cash flows requires understanding that the value of money does not remain the same throughout time.

A dollar today is worth less than a dollar at some future for two reasons:

Reason 1: Cash flows occurring at different times have different values relative to any one point in time.

One dollar one year from now is not as valuable as one dollar today. After all, you can invest a dollar today and earn interest so that the value it grows to next year is greater than the one dollar today. This means we have to take into account the *time value of money* to quantify the relation between cash flows at different points in time.

Reason 2: Cash flows are uncertain.

Expected cash flows may not materialize. Uncertainty stems from the nature of forecasts of the timing and the amount of cash flows. We do not know for certain when, whether, or how much cash flows will be in the future. This uncertainty regarding future cash flows must somehow be taken into account in assessing the value of an investment.

Translating a current value into its equivalent future value is *compounding*. Translating a future cash flow or value into its equivalent value in a prior period is *discounting*. In this chapter, we outline the basic mathematical techniques of compounding and discounting.

Suppose someone wants to borrow \$100 today and promises to pay back the amount borrowed in one month. Would the repayment of only the \$100 be fair? Probably not. There are two things to consider. First, if the lender didn't lend the \$100, what could he or she have done with it? Second, is there a chance that the borrower may not pay back the loan? So, when considering lending money, we must consider the opportunity cost (that is, what could have been earned or enjoyed), as well as the uncertainty associated with getting the money back as promised.

Let's say that someone is willing to lend the money, but that they require repayment of the \$100 plus some compensation for the opportunity cost and any uncertainty the loan will be repaid as promised. Then:

- the amount of the loan, the \$100, is the principal; and
- the compensation required for allowing someone else to use the \$100 is the interest.

Looking at this same situation from the perspective of time and value, the amount that you are willing to lend today is the loan's present value. The amount that you require to be paid at the end of the loan period is the loan's future value. Therefore, the future period's value is comprised of two parts:

Amount paid at the end of the loan		Principal		Interest
↓		↓		↓
Future value	=	Present value	+	Interest
or, using notation,				
FV	=	PV	+	$(i \times PV)$

If you would know the value of money, go and try to borrow some.
—Benjamin Franklin

The interest is compensation for the use of funds for the period of the loan. It consists of:

1. compensation for the length of time the money is borrowed; and
2. compensation for the risk that the amount borrowed will not be repaid exactly as set forth in the loan agreement.

CALCULATING THE FUTURE VALUE

Suppose you deposit \$1,000 into a savings account at the Safe Savings Bank and you are promised 5% interest per period. At the end of one period, you would have \$1,050. This \$1,050 consists of the return of your principal amount of the investment (the \$1,000) and the interest or return on your investment (the \$50). Let's label these values:

- \$1,000 is the value today, the present value, PV.
- \$1,050 is the value at the end of one period, the future value, FV.
- 5% is the rate interest is earned in one period, the interest rate, i.

To get to the future value from the present value:

$$\begin{aligned}
FV &= PV + \text{Interest} \\
FV &= PV + PV \times i \\
FV &= PV \times (1 + i) \\
\$1{,}050 &= \$1{,}000 \times (1.05)
\end{aligned}$$

If the \$50 interest is withdrawn at the end of the period, the principal is left to earn interest at the 5% rate. Whenever you do this, you earn *simple interest*. It is simple because it repeats itself in exactly the same way from one period to the next as long as you take out the interest at the end of each period and the principal remains the same.

Time is money.

—Benjamin Franklin

If, on the other hand, both the principal and the interest are left on deposit at the Safe Savings Bank, the balance earns interest on the previously paid interest, referred to as *compound interest*. Earning interest on interest is called compounding because the balance at any time is a combination of the principal, interest on principal, and *interest on accumulated interest* (or simply, *interest on interest*).

If you compound interest for one more period in our example, the original \$1,000 grows to \$1,052.50:

$$
\begin{aligned}
FV &= \text{Principal} &&+ \text{First period interest} &&+ \text{Second period interest}\\
&= PV &&+ PV \times i &&+ [PV(1+i)] \times i\\
&= \$1{,}000.00 &&+ (\$1{,}000.00 \times 0.05) &&+ (\$1{,}050.00 \times 0.05)\\
&= \$1{,}000.00 &&+ 50.00 &&+ 52.50\\
&= \$1{,}052.50
\end{aligned}
$$

The present value of the investment is \$1,000, theinterest earned over two years is \$52.50, and the future value of the investment after two years is \$1,052.50. If this were simple interest, the future value would be \$1,050. Therefore, the interest on interest—the results of compounding—is \$2.50.

We can use some shorthand to represent the FV at the end of two periods:

$$\text{FV} = \text{PV}(1 + i)^2$$

The balance in the account two years from now, \$1,052.50, is comprised of three parts:

- The principal, \$1,000.
- Interest on principal: \$50 in the first period plus \$50 in the second period.
- Interest on interest: 5% of the first period's interest, or $0.05 \times \$50 = \$2.50$.

To determine the future value with compound interest for more than two periods, we follow along the same lines:

$$\text{FV} = \text{PV}(1 + i)^N \tag{10.1}$$

The value of N is the number of compounding periods, where a compounding period is the unit of time after which interest is paid at the rate i. A period may be any length of time: a minute, a day, a month, or a year. The important thing is to be consistent through the calculations. The term "$(1 + i)^N$" is the *compound factor*, and it is the rate of exchange between present dollars and future dollars, n compounding periods into the future.

> The entire essence of America is the hope to first make money—then make money with money—then make lots of money with lots of money.
>
> —Paul Erdman

Equation (10.1) is the foundation of financial mathematics. It relates a value at one point in time to a value at another point in time, considering the compounding of interest.

We show the relation between present and future values for a principal of \$1,000 and interest of 5% per period through 10 compounding periods in Exhibit 10.1. For example, the value of \$1,000, earning interest at 5% per period, is \$1,628.89, which is 10 periods into the future:

$$FV = \$1{,}000(1 + 0.05)^{10} = \$1{,}000(1.62898) = \$1{,}628.89$$

After ten years, there will be \$1,628.89 in the account, consisting of:

- The principal, \$1,000;
- Interest on the principal of \$1,000: \$50 per period for 10 periods or \$500; and
- Interest on interest totaling \$128.89.

If you left the money in the bank, after 50 years you would have:

$$FV = \$1{,}000(1 + 0.05)^{50} = \$11{,}467.40$$

If this were simple interest instead of compound interest, the balance after 50 years would be: \$1,000 + [50 × \$1,000 × 0.05] = \$3,500. In other words, the \$11,467.40 – 3,500 = \$7,967.40. This is the power of compounding.

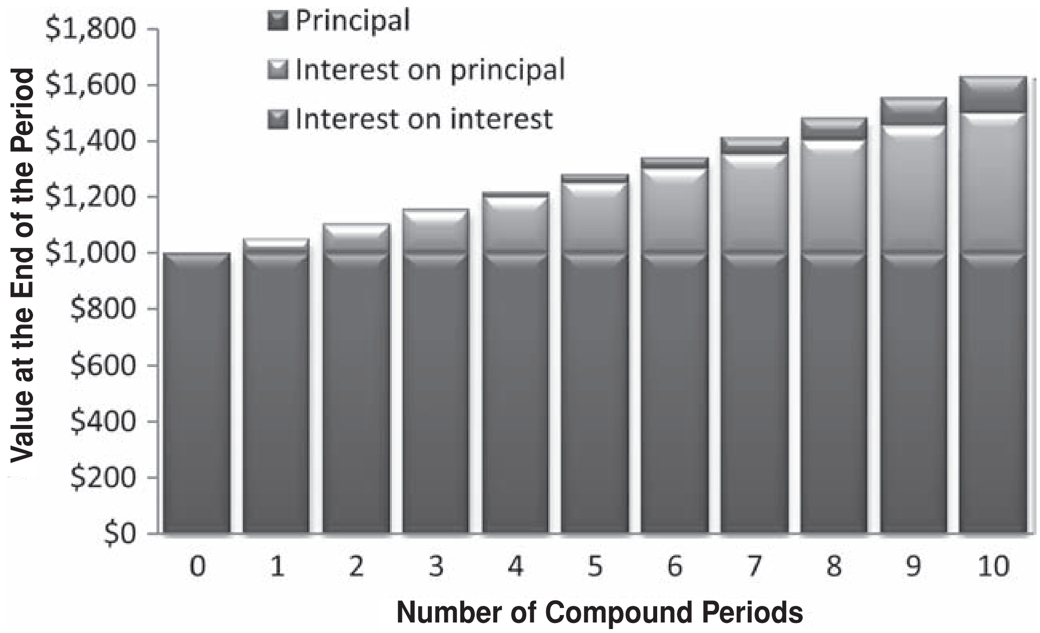

EXHIBIT 10.1 The Future Value of $1,000 Invested for 10 Years in an Account That Pays 10% Compounded Interest per Year

We can use financial calculators, scientific calculators with financial functions, or spreadsheets to solve most any financial problem. Consider the problem of calculating the future value of $1,000 at 5% for 10 years:

Hewlett-Packard 10B	Texas Instruments 83/84	Microsoft Excel
1000 +/− PV	N = 10	=FV(.05,10,0,−1000)
10 N	I% = 5	
5 I/YR	PV = −1000	
PV	*Place cursor at* FV = *and then* SOLVE	

A few notes about entering the data into the calculator or spreadsheet:

1. You need to change the sign of the present value to negative, reflecting the investment (negative cash flow).
2. You enter interest rates as whole values form when using the financial functions within a calculator, but enter these in decimal form if using the math functions of a calculation or the financial functions of a spreadsheet.

3. If you are using the financial function of a scientific calculator, you need to first enter this function. In the case of the Texas Instruments 83 or 84 calculator, for example, this is done through APPS >Finance>TVM Solver.
4. If you are using a spreadsheet function, you must enter a 0 in place of an unused argument.[1]

EXAMPLE 10.1: GUARANTEED INVESTMENT CONTRACTS

A common investment product of a life insurance company is a guaranteed investment contract (GIC). With this investment, an insurance company guarantees a specified interest rate for a period of years. Suppose that the life insurance company agrees to pay 6% annually for a five-year GIC and the amount invested by the policyholder is $10 million.

The amount of the liability (that is, the amount this life insurance company has agreed to pay the GIC policyholder) is the future value of $10 million when invested at 6% interest for five years:

$$PV = \$10{,}000{,}000,\ i = 6\%,\ \text{and}\ N = 5,$$

so that the future value is

$$FV = \$10{,}000{,}000(1 + 0.06)^5 = \$13{,}382{,}256$$

TRY IT! FUTURE VALUE

If you deposit $100 in a saving account that pays 2% interest per year, compounded annually, how much will you have in the account at the end of

a. five years?
b. 10 years?
c. 20 years?

[1]For example, the FV function has the following arguments: interest rate, number of periods, payment, and present value. Because this last problem does not involve any periodic payments, we used a zero for that argument.

Growth Rates and Returns

We can express the change in the value of the savings balance as a growth rate. A *growth rate* is the rate at which a value appreciates (a positive growth) or depreciates (a negative growth) over time. Our \$1,000 grew at a rate of 5% per year over the 10-year period to \$1,628.89. The average annual growth rate of our investment of \$1,000 is 5%—the value of the savings account balance increased 5% per year.

We could also express the appreciation in our savings balance in terms of a return. A *return* is the income on an investment, generally stated as a change in the value of the investment over each period divided by the amount at the investment at the beginning of the period. We could also say that our investment of \$1,000 provides an average annual return of 5% per year. The average annual return is not calculated by taking the change in value over the entire 10-year period (\$1,629.89 − \$1,000) and dividing it by \$1,000. This would produce an *arithmetic average return* of 62.889% over the 10-year period, or 6.2889% per year. But the arithmetic average ignores the process of compounding, so this is not the correct annual return.

The correct way of calculating the average annual return is to use a *geometric average return*:

$$\text{Geometric average return} = \sqrt[N]{\frac{FV}{PV}} - 1 \tag{10.2}$$

which is a rearrangement of equation (10.1). Using the values from the example,

$$\text{Geometric average return} = \sqrt[10]{\frac{\$1,628.89}{\$1,000.00}} - 1 = 5\%$$

Therefore, the annual return on the investment as the *compound average annual return* or the *true return*—is 5% per year.

Hewlett-Packard 10B	**Texas Instruments 83/84**	**Microsoft Excel**
1000 +/– PV	N = 10	=RATE(10,0,−1000,1628.89)
10 N	PV = −1000	
1628.89 FV	FV = 1628.89	
I/YR	*Place cursor at* I% = *and then* SOLVE	

TRY IT! GROWTH RATES

Suppose you invest $2,000 today and you double your money after five years. What is the annual growth rate on your investment?

Compounding More Than Once per Year

An investment may pay interest more than one time per year. For example, interest may be paid semiannually, quarterly, monthly, weekly, or daily, even though the stated rate is quoted on an annual basis. If the interest is stated as, say, 4% per year, compounded semiannually, the nominal rate—often referred to as the *annual percentage rate* (APR)—is 4%.

Suppose we invest $10,000 in an account that pays interest stated at a rate of 4% per year, with interest compounded quarterly. How much will we have after five years if we do not make any withdrawals? We can approach problems when compounding is more frequent than once per year using two different methods:

Method 1: Convert the information into compounding periods and solve

The inputs:

$PV = \$10{,}000$

$N = 5 \times 4 = 20$

$i = 4\% \div 4 = 1\%$

Solve for FV:

$FV = \$10{,}000\ (1 + 0.01)^{20} = \$12{,}201.90$

Method 2: Convert the APR into an effective annual rate and solve

The inputs:

$PV = \$10{,}000$

$N = 5$

$i = (1 + 0.01)^4 - 1 = 4.0604\%$

Solve for FV:

$FV = \$10{,}000\ (1 + 0.040601)^5 = \$12{,}201.90$

Both methods will get you to the correct answer. In Method 1, you need to adjust both the number of periods and the rate. In Method 2, you need to first calculate the effective annual rate, in this case 4.0601%, before calculating the future value.

Compounding frequency	Compound period	Rate per compound period	Number of compound periods in 10 years	Future value
Annual	1 year	8%	10	$215.89
Semiannual	6 months	4%	20	$219.11
Quarterly	3 months	2%	40	$220.80
Monthly	1 month	0.67%	120	$221.96

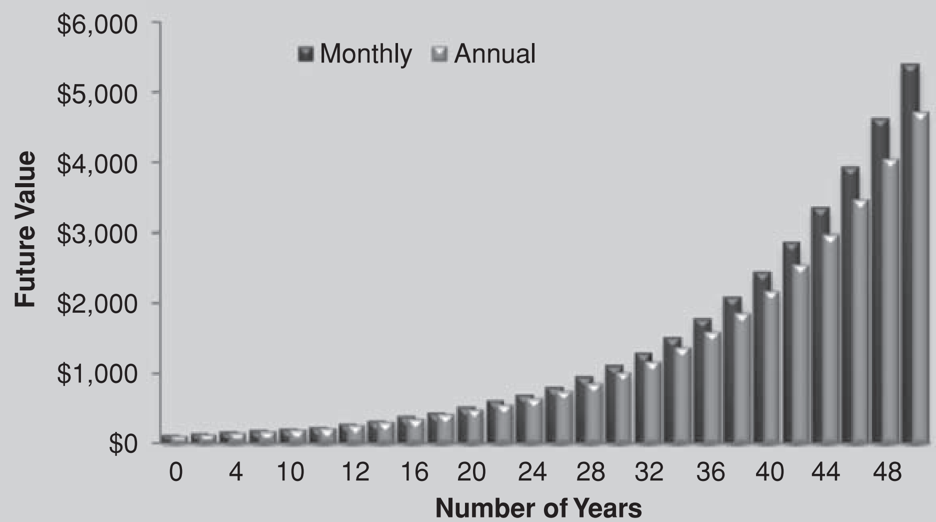

EXHIBIT 10.2 Value of $100 Invested in the Account That Pays 8% Interest per Year for 10 Years for Different Frequencies of Compounding

The frequency of compounding matters. To see how this works, let's use an example of a deposit of $100 in an account that pays interest at a rate of 8% per year, with interest compounded for different compounding frequencies. How much is in the account after, say, 10 years depends on the compounding frequency, as we show in Exhibit 10.2. At the end of ten years, the difference in the future values between annual and monthly compounding is a little more than $6. After 50 years, the difference is $5,388 – 4,690 = $698.

EXAMPLE 10.2: QUARTERLY COMPOUNDING

Suppose we invest $200,000 in an investment that pays 4% interest per year, compounded quarterly. What will be the future value of this investment at the end of 10 years?

Solution:

The given information is:

$$i = 4\%/4 = 1\% \text{ and } N = 10 \times 4 = 40 \text{ quarters.}$$

$$\text{Therefore, } FV = \$200{,}000(1 + 0.01)^{40} = \$297{,}772.75$$

TRY IT! MORE GROWTH RATES

Complete the following table, calculating the annual growth rate for each investment.

Present Value	Future Value	Number of Years	Growth Rate
\$1	\$3	6	
\$1,000	\$2,000	9	
\$500	\$600	7	
\$1	\$1.50	4	

Continuous Compounding

The extreme frequency of compounding is *continuous compounding*—interest is compounded instantaneously. The factor for compounding continuously for one year is e_{APR}, where e is 2.71828 . . . , the base of the natural logarithm. And the factor for compounding continuously for two years is $e_{\text{APR}} \times e^{\text{APR}}$ or $e^{2\text{APR}}$. The future value of an amount that is compounded continuously for N years is

$$FV = PVe^{N(\text{APR})} \tag{10.3}$$

where APR is the annual percentage rate and $e^{N(\text{APR})}$ is the compound factor.

If $1,000 is deposited in an account for five years, with interest of 12% per year, compounded continuously,

$$\begin{aligned} FV &= \$1{,}000\, e^{5(0.12)} \\ &= \$1{,}000(e^{0.60}) \\ &= \$1{,}000 \times 1.82212 \\ &= \$1{,}822.12 \end{aligned}$$

Comparing this future value with that if interest is compounded annually at 12% per year for five years, $\$1{,}000\ (1 + 0.12)^5 = \$1{,}762.34$, we see the effects of this extreme frequency of compounding.

> This process of growing proportionately, at every instant, to the magnitude at that instant, some people call a logarithmic rate of growing. Unit logarithmic rate of growth is that rate which in unit time will cause 1 to grow to 2.718281.
>
> It might also be called the organic rate of growing: because it is characteristic of organic growth (in certain circumstances) that the increment of the organism in a given time is proportional to the magnitude of the organism itself.
>
> —Silvanus P. Thompson, *Calculus Made Easy* (London: MacMillan and Co. Limited, 1914), p. 140

TRY IT! FREQUENCY OF COMPOUNDING

If you deposit $100 in a saving account today that pays 2% interest per year, how much will you have in the account at the end of 10 years if interest is compounded:

a. annually?
b. quarterly?
c. continuously?

Multiple Rates

In our discussion thus far, we have assumed that the investment will earn the same periodic interest rate, *i*. We can extend the calculation of a future value to allow for different interest rates or growth rates for different periods.

Suppose an investment of \$10,000 pays 5% during the first year and 4% during the second year. At the end of the first period, the value of the investment is \$10,000 (1 + 0.05), or \$10,500. During the second period, this \$10,500 earns interest at 4%. Therefore, the future value of this \$10,000 at the end of the second period is

$$FV = \$10{,}000\,(1 + 0.05)\,(1 + 0.4) = \$10{,}920$$

We can write this more generally as:

$$FV = PV(1 + i_1)(1 + i_2)(1 + i_3)\ldots(1 + i_N) \tag{10.4}$$

where i_N is the interest rate for period N.

EXAMPLE 10.3: DIFFERENT INTEREST RATES FOR DIFFERENT PERIODS

Consider a \$50,000 investment in a one-year bank *certificate of deposit* (CD) today and rolled over annually for the next two years into one-year CDs. The future value of the \$50,000 investment will depend on the one-year CD rate each time the funds are rolled over. Assume that the one-year CD rate today is 5% and that it is expected that the one-year CD rate one year from now will be 6%, and the one-year CD rate two years from now will be 6.5%.

a. What is the future value of this investment at the end of three years?
b. What is the average annual return on your CD investment?

Solution

a. $FV = \$50{,}000(1 + 0.05)(1 + 0.06)(1 + 0.065) = \$59{,}267.25$

b. $i = \sqrt[3]{\dfrac{\$59,267.25}{\$50,000}} - 1 = 5.8315\%$

CALCULATING A PRESENT VALUE

Now that we understand how to compute future values, let's work the process in reverse. Suppose that for borrowing a specific amount of money today, the Trustworthy Company promises to pay lenders \$5,000 two years from today. How much should the lenders be willing to lend Trustworthy in exchange for this promise? This dilemma is different than calculating a future value. Here we are given the future value and have to calculate the present value. But we can use the same basic idea from the future value problems to solve present value problems.

If you can earn 5% on other investments that have the same amount of uncertainty as the \$5,000 Trustworthy promises to pay, then:

- The future value, $FV = \$5{,}000$.
- The number of compounding periods, $N = 2$.
- The interest rate, $i = 5\%$.

We also know the basic relation between the present and future values:

$$FV = PV(1 + i)^N$$

Substituting the known values into this equation:

$$\$5{,}000 = PV(1 + 0.05)^2$$

To determine how much you are willing to lend now, PV, to get \$5,000 one year from now, FV, requires solving this equation for the unknown present value:

$$FV = PV(1 + i)^N$$
$$\$5{,}000 = PV(1 + 0.05)^2$$

Therefore, you would be willing to lend \$4,535.15 to receive \$5,000 one year from today if your opportunity cost is 5%. We can check our work by reworking the problem from the reverse perspective. Suppose you invested \$4,535.15 for two years and it earned 5% per year. What is the value of this investment at the end of the year?

We know: $PV = \$4{,}535.15.25$, $N = 5\%$ or 0.05, and $i = 2$. Therefore, the future value is \$5,000:

$$FV = PV(1 + i)^N = \$4{,}535.15(1 + 0.05)^2 = \$5{,}000.00$$

Compounding translates a value in one point in time into a value at some future point in time. The opposite process translates future values into present values: Discounting translates a value back in time. From the basic valuation equation,

$$FV = PV(1 + i)^N$$

we divide both sides by $(1 + i)^N$ and exchange sides to get the present value,

$$PV = \frac{FV}{(1 + i)^N} = FV\left(\frac{1}{1 + i}\right)^N = FV\left[\frac{1}{(1 + i)^N}\right] \qquad (10.5)$$

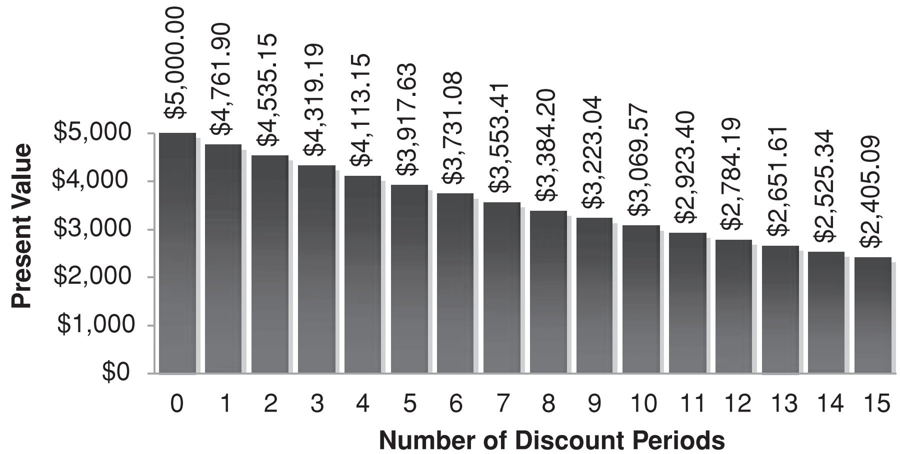

EXHIBIT 10.3 Present Value of \$5,000 for 0 to 15 Periods, at a Discount Rate of 5% per Period

In the right-most form, the term in square brackets is referred to as the *discount factor* since it is used to translate a future value to its equivalent present value. We can restate our problem as:

$$PV = \frac{\$5{,}000}{(1+0.05)^2} = \$5{,}000\left[\frac{1}{(1+0.05)^2}\right] = \$5{,}000 \times 0.90703$$
$$= \$4{,}535.15,$$

where the discount factor is 0.90703. We provide the present value of \$5,000 for discount periods ranging from 0 to 15 in Exhibit 10.3.

We can also calculate this present using a calculator or a spreadsheet. Consider the present value of the \$5,000 at 5% for ten years:

Hewlett-Packard 10B	Texas Instruments 83/84	Microsoft Excel
5000 FV	N = 10	=PV(.05,10,0,5000)
10 N	FV = 5000	
5 I/YR	I% = 5	
PV	Place cursor at I% = and then SOLVE	

If the frequency of compounding is greater than once a year, we make adjustments to the rate per period and the number of periods as we did in compounding. For example, if the future value five years from today is \$100,000 and the interest is 6% per year, compounded semiannually,

$i = 6\% \div 2 = 3\%$, $N = 5 \times 2 = 10$, and the present value is \$134,392:

$$PV = \$100{,}000(1 + 0.03)^{10} = \$100{,}000 \times 1.34392 = \$134{,}392$$

TRY IT! PRESENT VALUE

You are presented with an investment that promises \$1,000 in ten years. If you consider the appropriate discount rate to be 6%, based on what you can earn on similar risk investments, what would you be willing to pay for this investment today?

EXAMPLE 10.4: MEETING A SAVINGS GOAL

Suppose that the goal is to have \$75,000 in an account by the end of four years. And suppose that interest on this account is paid at a rate of 5% per year, compounded semiannually. How much must be deposited in the account today to reach this goal?

Solution

We are given $FV = \$75{,}000$, $i = 5\% \times 2 = 2.5\%$ per six months, and $N = 4 \times 2 = 8$ six-month periods. Therefore, the amount of the required deposit is:

$$PV = \frac{\$75{,}000}{(1 + 0.025)^8} = \$61{,}555.99$$

DETERMINING THE UNKNOWN INTEREST RATE

As we saw earlier in our discussion of growth rates, we can rearrange the basic equation to solve for i:

$$i = \sqrt[N]{\frac{FV}{PV}} - 1$$

which is the same as:

$$i = (FV/PV)^{1/N} - 1$$

As an example, suppose that the value of an investment today is \$2,000 and the expected value of the investment in five years \$3,000. What is the annual rate of appreciation in value of this investment over the five-year period?

$$i = \sqrt[5]{\frac{\$3{,}000}{\$2{,}000}} - 1 = 8.447\%$$

There are many applications in finance where it is necessary to determine the rate of change in values over a period of time. If values are increasing over time, we refer to the rate of change as the growth rate. To make comparisons easier, we usually specify the growth rate as a rate per year.

EXAMPLE 10.5: INTEREST RATES

Consider the growth rate of dividends for General Electric. General Electric pays dividends each year. In 1996, for example, General Electric paid dividends of \$0.317 per share of its common stock, whereas in 2006 the company paid \$1.03 in dividends per share in 2006.

Solution

This represents a growth rate of 12.507%:

$$i = \sqrt[10]{\frac{\$1.03}{\$0.317}} - 1 = 12.507\%$$

THE TIME VALUE OF A SERIES OF CASH FLOWS

Applications in finance may require determining the present or future value of a series of cash flows rather than simply a single cash flow. The principles of determining the future value or present value of a series of cash flows are the same as for a single cash flow, yet the math becomes a bit more cumbersome.

Suppose that the following deposits are made in a Thrifty Savings and Loan account paying 5% interest, compounded annually:

Period	End of Period Cash Flow
0	\$1,000
1	\$2,000
2	\$1,500

What is the balance in the savings account at the end of the second year if there are no withdrawals and interest is paid annually?

Let's simplify any problem like this by referring to today as the end of period 0, and identifying the end of the first and each successive period as 1, 2, 3, and so on. Represent each end-of-period cash flow as *CF* with a subscript specifying the period to which it corresponds. Thus, CF_0 is a cash flow today, CF_{10} is a cash flow at the end of period 10, and CF_{25} is a cash flow at the end of period 25, and so on. In our example, CF_0 is \$1,000, CF_1 is \$2,000, and CF_2 is \$1,500.

Representing the information in our example using cash flow and period notation:

$$FV = CF_0(1+i)^2 + CF_1(1+i)^1 + CF_2(1+i)^0$$

It is important to get the compounding correct. For example, there is no compounding of the cash flow that occurs at the end of the second period to arrive at a future value at the end of the second period. Hence, the factor is $(1 + i)^0 = 1$.

We can represent these cash flows in a time line in Exhibit 10.4 to help graphically depict and sort out each cash flow in a series. From this example, you can see that the future value of the entire series is the sum of each of the

EXHIBIT 10.4 Time Line for the Future Value of a Series of Uneven Cash Flows Deposited to Earn 5% Compound Interest per Period

0		1		2
\$1,000.00		\$2,000.00		\$1,500.00
		↳	\$2,000 (1 + 0.05) =	2,100.00
↳	$\$1,000.00\,(1 + 0.05)^2 =$			1,102.50
				\$4,702.50

compounded cash flows comprising the series. In much the same way, we can determine the future value of a series comprising any number of cash flows. And if we need to, we can determine the future value of a number of cash flows before the end of the series.

To determine the present value of a series of future cash flows, each cash flow is discounted back to the present, where we designate the beginning of the first period, today, as 0. As an example, consider the Thrifty Savings & Loan problem from a different angle. Instead of calculating what the deposits and the interest on these deposits will be worth in the future, let's calculate the present value of the deposits. The present value is what these future deposits are worth today.

Suppose you are promised the following cash flows:

Period	Cash Flow	End of Period Cash Flow
0	CF_0	$1,000
1	CF_1	$2,000
2	CF_2	$1,500

What is the present value of these cash flows—that is, at the end of period 0—if the discount rate is 5%? We would use the same method that we used in the previous problem—just backwards. We show this in Exhibit 10.5. As you can see in this exhibit, we don't discount the cash flow that occurs today. We discount the first period's cash flow one period, and discount the second period's cash flow two periods.

EXHIBIT 10.5 Time Line for the Present Value of a Series of Uneven Cash Flows Deposited to Earn 5% Compounded Interest per Period

0		1		2
$1,000.00		$2,000.00		$1,500.00
1,904.76	$\frac{\$2,000}{(1+0.05)}$			
1,360.54			$\frac{\$1,500}{(1+0.05)^2}$	
$4,265.30				

You may also notice a relation between the future value that we calculated in Exhibit 10.4 and the present value that we calculated in Exhibit 10.5, with both examples using the same set of cash flows and same interest rate—just going in different directions:

$$\$4{,}265.30\,(1 + 0.05)^2 = \$4{,}702.50$$

Gettin' Fancy

We can represent the future value of a series of cash flows as:

$$FV = \sum_{t=0}^{N} CF_{\mathrm{t}}(1 + i)^{N-t} \tag{10.6}$$

This, simply, means that the future value of a series of cash flows is the sum of the future value of each cash flow, where each of the future value considers the amount of the cash flow and the number of compounding period. Therefore, if there are 10 periods, the cash flow from occurring at the end of the sixth period, CF_6, would have interest compounded $N - t = 10 - 6 = 4$ periods, and the cash flow occurring at the end of the tenth period would not have any compounding.

And, likewise, we can represent the present value of a series using summation notation as:

$$PV = \sum_{t=0}^{N} \frac{CF_t}{(1 + i)^t} \tag{10.7}$$

with a similar explanation. For example, the cash flow occurring at the end of the fifth period is discounted five periods at the discount rate of i.

Multiple Rates

In our illustrations thus far, we have used one interest rate to compute the present value of all cash flows in a series. However, there is no reason that one interest rate must be used. For example, suppose that the cash flow is the same as used earlier: \$1,000 today, \$2,000 at the end of period 1, and \$1,500 at the end of period 2. Now, instead of assuming that a 5% interest rate can be earned if a sum is invested today until the end of period 1 and the end of period 2, it is assumed that an amount invested today for one period can earn 5% but an amount invested today for two periods can earn 6%.

In this case, the calculation of the present value of the cash flow at the end of period 1 (the \$2,000) is obtained in the same way as before: computing the present value using an interest rate of 5%. However, we

EXHIBIT 10.6 Time Line for the Present Value of a Series of Uneven Cash Flows Deposited to Earn 5% Compounded Interest per Period

0		1		2
\$1,000.00		\$2,000.00		\$1,500.00
1,904.76	$\frac{\$2,000}{(1+0.05)}$			
1,334.99			$\frac{\$1,500}{(1+0.06)^2}$	
\$4,239.75				

must calculate the present value for the cash flow at the end of period 2 (the \$1,500) using an interest rate of 6%. We depict the present value calculation in Exhibit 10.6. As expected, the present value of the cash flows is less than a 5% interest rate is assumed to be earned for two periods (\$4,239.75 versus \$4,265.39).

Although in many illustrations and applications throughout this book we will assume a single interest rate for determining the present value of a series of cash flows, in many real-world applications multiple interest rates are used. This is because in real-world financial markets the interest rate that can be earned depends on the amount of time the investment is expected to be outstanding. Typically, there is a positive relationship between interest rates and the length of time the investment must be held. The relationship between interest rates on investments and the length of time the investment must be held is called the yield curve.

The formula for the present value of a series of cash flows when there is a different interest rate is a simple modification of the single interest rate case. In the formula, i is replaced by i with a subscript to denote the period, i_t. That is,

$$PV = \sum_{t=0}^{N} \frac{CF_t}{(1 + i_t)^t}$$

ANNUITIES

There are valuation problems that require us to evaluate a series of level cash flows—each cash flow is the same amount as the others—received at regular intervals. Let's suppose you expect to deposit \$2,000 at the end of

EXHIBIT 10.7 Time Line for a Series of Even Cash Flows Deposited to Earn 5% Interest per Period

A: Future Value

0	1	2	3	4
	$2,000.00	$2,000.00	$2,000.00	$2,000.00
				2,100.00
				2,205.00
				2,315.25
				$8,620.25

B: Present Value

0	1	2	3	4
	$2,000.00	$2,000.00	$2,000.00	$2,000.00
$1,904.76				
1,814.06				
1,727.68				
1,645.40				
$7,091.90				

each of the next four years in an account earning 8% compounded interest. How much will you have available at the end of the fourth year?

As we just did for the future value of a series of uneven cash flows, we can calculate the future value (as of the end of the fourth year) of each $2,000 deposit, compounding interest at 5%, as we show in Exhibit 10.7. The future value of this series is $8,620.25. Modifying the future value of a series equation to reflect that all of the cash flows are the same,

$$FV = \sum_{t=0}^{N} CF(1+i)^{N-t} = CF \sum_{t=0}^{N} (1+i)^{N-t} \tag{10.8}$$

A series of cash flows of equal amount, occurring at even intervals is referred to as an *annuity*. Determining the value of an annuity, whether compounding or discounting, is simpler than valuing uneven cash flows.

Consider the same series of $2,000 for four periods, but calculate the present value of the series. We show this calculation in Panel B of Exhibit 10.7. The present value of this series is $7,091.90.

EXAMPLE 10.6: FUTURE VALUE OF AN ANNUITY

Suppose you wish to determine the future value of a series of deposits of $1,000, deposited each year in the No Fault Vault Bank for five years, with the first deposit made at the end of the first year. If the NFV Bank pays 5% interest on the balance in the account at the end of each year and no withdrawals are made, what is the balance in the account at the end of the five years?

Solution

In equation form,

$$FV = \$1{,}000 \sum_{t=1}^{5} (1 + 0.05)^{N-t} = \$1{,}000(5.5263) = \$5{,}525.63$$

Summing the individual future values:

Cash Flow	Amount	Future Value
CF_1	$1,000	$1,215.51
CF_2	$1,000	1,157.63
CF_3	$1,000	1,102.50
CF_4	$1,000	1,050.00
CF_5	$1,000	1.000.00
	Total	$5,525.63

Calculator and spreadsheet inputs:

Periodic payment = PMT = 1,000

i = 5% (input as 5 for calculator, 0.05 for spreadsheet)

N = 5

Solve for FV

As we did with the future valueof an even series, we can simplify the equation for the present value of a series of level cash flows beginning after one period as:

$$PV = \sum_{t=0}^{N} \frac{CF}{(1+i)^t} = CF \sum_{t=0}^{N} \frac{1}{(1+i)^t} \tag{10.9}$$

EXHIBIT 10.8 Time Line for a Series of Even Cash Flows Deposited to Earn 4% Interest per Period

A: Future Value of the Ordinary Annuity

0	1	2	3	4
	$500.00	$500.00	$500.00	$500.00
				520.00
				540.80
				562.43
				$2,123.23

B: Future Value of the Annuity Due

0	1	2	3	4
$500.00	$500.00	$500.00	$500.00	
				$520.00
				540.80
				562.43
				584.93
				$2,208.16

Another way of looking at this is that the present value of an annuity is equal to the amount of one cash flow multiplied by the sum of the discount factors.

If the cash flows occur at the end of each period (that is, the first cash flow occurs one period from today), we refer to this as an *ordinary annuity*. The two examples that we provide in Exhibit 10.8 are both ordinary annuities.

EXAMPLE 10.7: PRESENT VALUE OF AN ANNUITY

Consider a five-payment annuity, with payments of $500 at the end of each of the next five years.

a. If the appropriate discount rate is 4%, what is the present value of this annuity?

b. If the appropriate discount rate is 5%, what is the present value of this annuity?

Solution

a. Given: $PMT = \$500; i = 4\%; N = 5$. Solve for PV. $PV = \$2{,}225.91$
b. Given: $PMT = \$500, i = 4\%, N = 5$. Solve for PV. $PV = \$2{,}164.74$

Note: The higher the discount rate, the lower the present value of the annuity.

Equations (10.8) and (10.9) are the valuation—future and present value—formulas for an ordinary annuity. An ordinary annuity is therefore a special form of annuity, where the first cash flow occurs at the end of the first period.

This annuity short-cut is built into financial calculators and spreadsheet functions. For example, in the case of the present value of the four-payment ordinary annuity of $2,000 at 5%:

Hewlett-Packard 10B	Texas Instruments 83/84	Microsoft Excel
2000 PMT	N = 4	=PV(.05,4,2000,0)
4 N	I% = 5	
5 I/YR	PMT = 2000	
PV	FV = 0	
	Place cursor at PV = *and then* SOLVE	

Valuing a Perpetuity

There are some circumstances where cash flows are expected to continue forever. For example, a corporation may promise to pay dividends on preferred stock forever, or, a company may issue a bond that pays interest every six months, forever. How do you value these cash flow streams? Recall that when we calculated the present value of an annuity, we took the amount of one cash flow and multiplied it by the sum of the discount factors that corresponded to the interest rate and number of payments. But what if the number of payments extends forever—into infinity?

A series of cash flows that occur at regular intervals, forever, is a *perpetuity*. Valuing a perpetual cash flow stream is just like valuing an ordinary annuity, but the N is replaced by ∞:

$$PV = CF \sum_{t=1}^{\infty} \left(\frac{1}{1+i} \right)^t$$

As the number of discounting periods approaches infinity, the summation approaches $1/i$, so:

$$PV = \frac{CF}{i} \tag{10.10}$$

Suppose you are considering an investment that promises to pay \$100 each period forever, and the interest rate you can earn on alternative investments of similar risk is 5% per period. What are you willing to pay today for this investment?

$$PV = \frac{\$100}{0.05} = \$2{,}000$$

Therefore, you would be willing to pay \$2,000 today for this investment to receive, in return, the promise of \$100 each period forever.

EXAMPLE 10.8: PERPETUITY

Suppose that you are given the opportunity to purchase an investment for \$5,000 that promises to pay \$50 at the end of every period forever. What is the periodic interest per period—the return—associated with this investment?

Solution

We know that the present value is $PV = \$5{,}000$ and the periodic, perpetual payment is $CF = \$50$. Inserting these values into the formula for the present value of a perpetuity,

$$\$5{,}000 = \frac{\$50}{i}$$

Solving for i, CF = \$50, $i = 0.01$ or 1%. Therefore, an investment of \$5,000 that generates \$50 per period provides 1% compounded interest per period.

Valuing an Annuity Due

In the ordinary annuity cash flow analysis, we assume that cash flows occur at the end of each period. However, there is another fairly common cash flow pattern in which level cash flows occur at regular intervals, but the first cash flow occurs immediately. This pattern of cash flows is called an *annuity due*. For example, if you win the Mega Millions grand prize, you will receive your winnings in 20 installments (after taxes, of course). The 20 installments are paid out annually, beginning immediately. The lottery winnings are therefore an annuity due.

Like the cash flows we have considered thus far, the future value of an annuity due can be determined by calculating the future value of each cash flow and summing them. And, the present value of an annuity due is determined in the same way as a present value of any stream of cash flows.

Let's consider first an example of the future value of an annuity due, comparing the values of an ordinary annuity and an annuity due, each comprising four cash flows of \$500, compounded at the interest rate of 4% per period. We show the calculation of the future value of both the ordinary annuity and the annuity due at the end of three periods in Exhibit 10.8. You will notice that the future value of the annuity due is $1 + i$ multiplied by the future value of the ordinary annuity. This is because each cash flow earns interest for one more periods in the case of the annuity due.

The present value of the annuity due is calculated in a similar manner, adjusting the ordinary annuity formula for the different number of discount periods. Because the cash flows in the annuity due situation are each discounted one less period than the corresponding cash flows in the ordinary annuity, the present value of the annuity due is greater than the present value of the ordinary annuity for an equivalent amount and number of cash flows. We show this in Exhibit 10.9 for the same four-payment, \$500 annuity, but this time we compare the present value of the ordinary annuity with the present value of the annuity due.

You will notice that there is one more period of discounting for each cash flow in the ordinary annuity, as compared to the annuity due. Therefore, the present value of the annuity due is equal to the present value of the ordinary annuity multiplied by $1 + i$; that is, \$1,814.95 $(1 + 0.04) =$ \$1,887.55.

Calculating the value of an annuity due using a calculator or a spreadsheet is similar to that of the ordinary annuity, but with one small difference. With calculators, you need to change the mode to the "due" or "begin" mode. For example, when calculating the present value of the four-payment, \$500 annuity with the HP10B calculator,

EXHIBIT 10.9 Time Line for a Series of Even Cash Flows Deposited to Earn 4% Interest per Period

A: Present Value of the Ordinary Annuity

0	1	2	3	4
	$500.00	$500.00	$500.00	$500.00
$480.77				
462.28				
444.50				
427.40				
$1,814.95				

B: Present Value of the Annuity Due

0	1	2	3	4
$500.00	$500.00	$500.00	$500.00	
480.77				
462.28				
444.50				
$1,887.55				

Ordinary Annuity	**Annuity Due**
PMT = 500	PMT = 500
i = 4	i = 4
N = 4	N = 4
END mode	BEG mode

Using spreadsheets, the only difference is the last argument in the function (0 or nothing for an ordinary annuity, 1 for an annuity due):

Ordinary Annuity	**Annuity Due**
=PV(0.04,4,500,0,0)	=PV(0.04,4,500,0,1)

Valuing a Deferred Annuity

A *deferred annuity* has a stream of cash flows of equal amounts at regular periods starting at some time after the end of the first period. When we calculated the present value of an annuity, we brought a series of cash flows back to the beginning of the first period—or, equivalently the end of the period 0. With a deferred annuity, we determine the present value of the ordinary annuity and then discount this present value to an earlier period.

Suppose you want to deposit an amount today in an account such that you can withdraw \$100 per year for three years, with the first withdrawal occurring three years from today. We diagram this set of cash flows in Panel A of Exhibit 10.9.

We can solve this problem in two steps:

Step 1: Solve for the present value of the withdrawals.

Step 2: Discount this present value to the present.

The first step requires determining the present value of a three-cash-flow ordinary annuity of \$100. This calculation provides the present value as of the end of the second year (one period prior to the first withdrawal), using an ordinary annuity. Based on this calculation (present value of an ordinary annuity, $N = 3$, $i = 5\%$, $PMT = \$100$), you need \$272.32 in the account at the end of the second period in order to satisfy the three withdrawals. We show this in Panel B of Exhibit 10.9.[2]

The next step is to determine how much you need to deposit today to meet the savings goal of \$272.32 at the end of the second year. The \$272.32 is the future value, $N = 2$, and $i = 5\%$. Therefore, you need to deposit \$247.01 today so that you will have \$272.32 in two years, so that you can then begin to make withdrawals starting at the end of the third year. We show this in Panel C of Exhibit 10.9.

We can check our work by looking at the balance in the account at the end of each period, as we show in Panel D of Exhibit 10.9. If we have performed the calculations correctly, we should end up with a zero balance at the time of the last \$100 withdrawal. Remember, the funds left in the account earn 5%. Therefore, for example, in the third period, you begin with \$272.33 in the account. The account balance earns 5% or \$13.62 of interest during the third year. This brings the balance in the account to

[2]We could have also solved this problem using an annuity due in the first step, which would mean that we would discount the value from the first step three periods instead of two.

$\$272.33 + 13.62 = \$285.95$. Once we remove the \$100, the balance at the end of the third year is \$185.95.

EXAMPLE 10.9: DEFERRED ANNUITY

Suppose you want to retire and be able to withdraw \$40,000 per year each year for twenty years after your retirement. If you plan to stop deposits in your retirement account ten years prior to retirement, what is the balance that you must have in your retirement account ten years before you retire if you can earn 4% per year on your retirement investments?

Solution

Balance in the account one year before retirement is the present value of an ordinary annuity with:

$PMT = \$40{,}000$
$N = 20$
$i = 4\%$
Solve for PV. $PV_{\text{one year before retirement}} = \$543{,}613.05$

Balance needed ten years before retirement:

$$PV_{\text{10 years before retirement}} = PV_{\text{one year before retirement}} \div (1 + 0.04)^9 = \$381{,}935.32$$

Deferred annuity problems can become more complex, such as determining a set of payments needed for some future goal. However, all deferred annuity problems can be solved easily by breaking down the problem into steps.[3]

LOAN AMORTIZATION

If an amount is loaned and then repaid in installments, we say that the loan is amortized. Therefore, *loan amortization* is the process of calculating

[3]Unfortunately, there are no calculator functions or spreadsheet functions that perform deferred annuity calculations specifically, because there are so many variations possible on how these are designed.

the loan payments that amortize the loaned amount. We can determine the amount of the loan payments once we know the frequency of payments, the interest rate, and the number of payments.

Consider a loan of \$100,000. If the loan is repaid in four annual installments (at the end of each year) and the interest rate is 6% per year. The first thing we need to do is to calculate the amount of each payment. In other words, we need to solve for *CF:*

$$\$100{,}000 = \sum_{t=1}^{4} \frac{CF}{(1+0.06)^t}$$

We want to solve for the loan payment, that is, the amount of the annuity. The calculator and spreadsheet inputs for this calculation are:

$$\begin{aligned} PV &= 100{,}000 \\ i &= 6\% \\ N &= 4 \end{aligned}$$

and then solve for *PMT*. This is the *CF*, the loan payment.

The loan payment is \$28,859.15. We can calculate the amount of interest and principal repayment associated with each loan payment using a loan amortization schedule, as we show in Panel A of Exhibit 10.10.

The loan payments are determined such that after the last payment is made there is no loan balance outstanding. Thus, the loan is referred to as a *fully amortizing loan*. You can see this in Panel B of Exhibit 10.3. Even though the loan payment each year is the same, the proportion of interest and principal differs with each payment: the interest is 5% of the principal amount of the loan that remains at the beginning of the period, whereas the principal repaid with each payment is the difference between the payment and the interest. As the payments are made, the remainder is applied to repayment of the principal. This is the scheduled principal repayment or the *amortization*. As the principal remaining on the loan declines, less interest is paid with each payment.

Loan amortization works the same whether this is a mortgage loan to purchase a home, a term loan, or any other loan such as an automobile loan in which the interest paid is determined on the basis of the remaining amount of the loan. You can modify the calculation of the loan amortization to suit different principal repayments, such as additional lump-sum payments, known as *balloon payments*. See Exhibit 10.11.

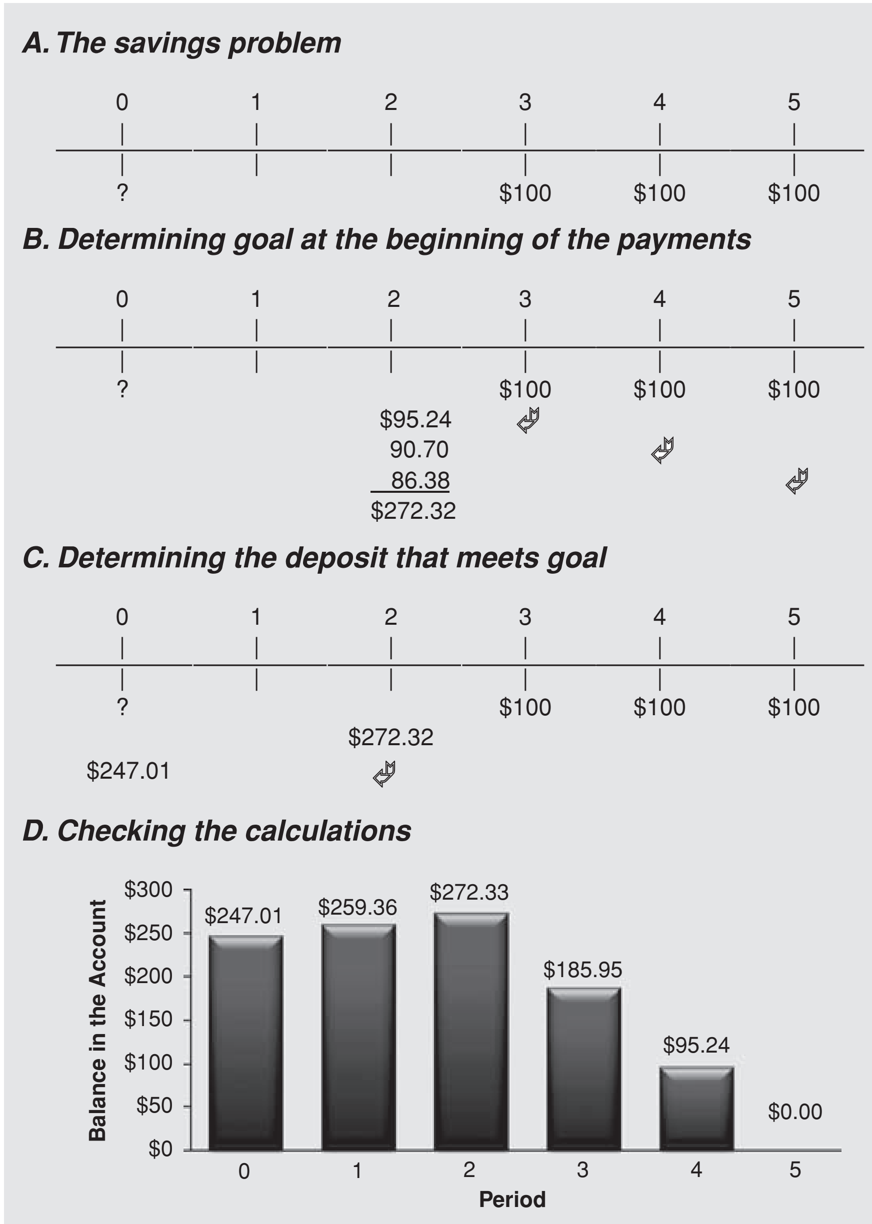

EXHIBIT 10.10 Deferred Annuity Time Lines for a Three-Period, $100 Annuity with the First Cash Flow Deferred Three Periods

INTEREST RATES AND YIELDS

Calculating the present or future value of a lump-sum or set of cash flows requires information on the timing of cash flows and the compound or discount rate. However, there are many applications in which we are presented with values and cash flows, and wish to calculate the yield or implied

A. Amortization of the loan

Year	Beginning balance of the loan outstanding	Payment	Interest = 6% × beginning balance of the loan	Principal repaid with payment = payment – interest	Remaining principal = beginning balance – principal repaid
1	\$100,000.00	\$28,859.15	\$6,000.00	\$22,859.15	\$77,140.85
2	\$77,140.85	\$28,859.15	\$4,628.45	\$24,230.70	\$52,910.15
3	\$52,910.15	\$28,859.15	\$3,174.61	\$25,684.54	\$27,225.61
4	\$27,225.61	\$28,859.15	\$1,633.54	\$27,225.61	\$0.00

B. Payoff of loan

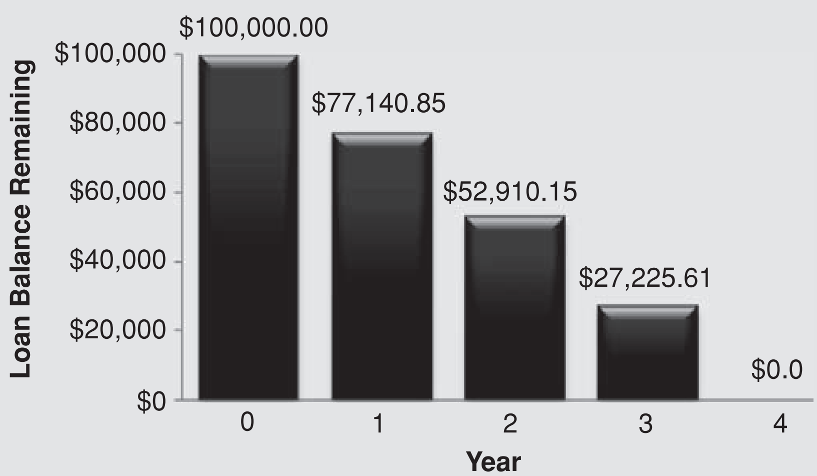

EXHIBIT 10.11 Loan Amortization of a Four-Year \$100,000 Loan, with an Interest Rate of 6%

interest rate associated with these values and cash flows. By calculating the yield or implied interest rate, we can then compare investment or financing opportunities.

Annual Percentage Rate vs. Effective Annual Rate

A common problem in finance is comparing alternative financing or investment opportunities when the interest rates are specified in a way that makes it difficult to compare terms. The Truth in Savings Act of 1991 requires institutions to provide the annual percentage yield for savings accounts. As a result of this law, consumers can compare the yields on different savings arrangements. But this law does not apply beyond savings accounts. One investment may pay 10% interest compounded semiannually, whereas another investment may pay 9% interest compounded daily. One financing arrangement may require interest compounding quarterly, whereas another may require interest compounding monthly.

Want to compare investments or financing with different frequencies of compounding? We must first translate the stated interest rates into a common basis. There are two ways to convert interest rates stated over different time intervals so that they have a common basis: the annual percentage rate and the effective annual interest rate.

One obvious way to represent rates stated in various time intervals on a common basis is to express them in the same unit of time—so we annualize them. The annualized rate is the product of the stated rate of interest per compound period and the number of compounding periods in a year. Let i be the rate of interest per period and n be the number of compounding periods in a year. The annualized rate, which is as we indicated earlier in this chapter also referred to as the nominal interest rate or the annual percentage rate (APR), is

$$APR = i \times n \tag{10.11}$$

Another way of converting stated interest rates to a common basis is the effective rate of interest. The *effective annual rate* (*EAR*) is the true economic return for a given time period because it takes into account the compounding of interest. We also refer to this rate as the *effective rate of interest*. The formula is

$$EAR = (1 + i)n - 1 \tag{10.12}$$

Let's look how the *EAR* is affected by the compounding. Suppose that the Safe Savings and Loan promises to pay 2% interest on accounts, compounded annually. Because interest is paid once, at the end of the year, the effective annual return, *EAR*, is 2%. If the 2% interest is paid on a semiannual basis—1% every six months—the effective annual return is larger than 2% since interest is earned on the 1% interest earned at the end of the first six months. In this case, to calculate the *EAR*, the interest rate per compounding period—six months—is 0.01 (that is, $0.02 \div 2$) and the number of compounding periods in an annual period is 2:

$$EAR = (1 + 0.01)^2 - 1 = 1.0201 - 1 = 0.0201 \text{ or } 2.01\%$$

In the case of continuous compounding, the EAR is simply:

$$EAR_{\text{continuous compounding}} = e^{APR} - 1 \tag{10.13}$$

Extending this example to the case of quarterly compounding and continuous compounding with a nominal interest rate of 2%, we first calculate the interest rate per period, i, and the number of compounding periods in a year, n:

Frequency of Compounding	Calculation	Effective Annual Rate
Annual	$(1 + 0.02)^1 - 1$	2.00%
Semiannual	$(1 + 0.01)^2 - 1$	2.01%
Quarterly	$(1 + 0.005)^4 - 1$	2.02%
Continuous	$e^{0.02} - 1$	2.02%

Figuring out the effective annual rate is useful when comparing interest rates for different investments. It doesn't make sense to compare the APRs for different investments having a different frequency of compounding within a year. But since many investments have returns stated in terms of APRs, we need to understand how to work with them.

To illustrate how to calculate effective annual rates, consider the rates offered by two banks, Bank A and Bank B. Bank A offers 4.2% compounded semiannually and Bank B other offers 4.158% compounded continuously. We can compare these rates using the *EAR*s. Which bank offers the highest interest rate? The effective annual rate for Bank A is $(1 + 0.021)^2 - 1 =$ 4.2441%. The effective annual rate for Bank B is $e^{0.04158} - 1 = 4.2457\%$. Therefore, Bank B offers a slightly higher interest rate.

Yields on Investments

Suppose an investment opportunity requires an investor to put up $10,000 million and offers cash inflows of $4,000 after one year and $7,000 after two years. The return on this investment, or *yield*, is the interest rate that equates the present values of the $4,000 and $7,000 cash inflows to equal the present value of the $1 million cash outflow. This yield is also referred to as the *internal rate of return* (*IRR*) and is calculated as the rate that solves the following:

$$\$10{,}000 = \frac{\$4{,}000}{(1 + IRR)^1} + \frac{\$7{,}000}{(1 + IRR)^2}$$

Unfortunately, there is no direct mathematical solution (that is, closed-form solution) for the *IRR*, but rather we must use an iterative procedure. Fortunately, financial calculators and financial software ease our burden in this calculation. The *IRR* that solves this equation is 6.023%. In other words, if you invest $10,000 today and receive $4,000 in one year and $7,000 in two years, the return on your investment is 6.023%.

Another way of looking at this same yield is to consider that an investment's *IRR* is the interest rate that makes the present value of all expected future cash flows—both the cash outflows for the investment and the subsequent inflows—equal to zero. We can represent the IRR as the rate that solves

$$\$0 = \sum_{t=0}^{N} \frac{CF_t}{(1 + IRR)^t}$$

We can use a calculator or a spreadsheet to solve for *IRR*. To do this, however, we must enter the series of cash flows in a manner that can be used with the appropriate function. Consider the problem with the present value of $10,000 and cash flows of $4,000 and $7,000. The financial routines require that the cash flows be entered in chronological order, and then the IRR function be used with these cash flows.[4]

Hewlett-Packard 10B	Texas Instruments 83/84	Microsoft Excel	
			A
10000 +/– CFj	{4000,7000} →	1	– 10000
4000 CFj	STO L1	2	4000
7000 CFj	IRR(– 10000,L1)	3	7000
IRR		4	= IRR(A1:A3)

EXAMPLE 10.10: CALCULATING A YIELD

Suppose an investment of $1 million produces no cash flow in the first year but cash flows of $200,000, $300,000, and $900,000 two, three, and four years from now, respectively. What is the return on this investment?

[4]If there is no cash flow for a given period, both the calculators and the spreadsheets require you to enter a zero in place of that cash flow; failing to do so will result in an incorrect *IRR*.

Solution

The *IRR* for this investment is the interest rate that solves:

$$\$1{,}000{,}000 = \frac{\$200{,}000}{(1+IRR)^2} + \frac{\$300{,}000}{(1+IRR)^3} + \frac{\$900{,}000}{(1+IRR)^4}$$

The return is 10.172%.

We can use this approach to calculate the yield on any type of investment, as long as we know the cash flows—both positive and negative—and the timing of these flows. Consider the case of the yield to maturity on a bond. Most bonds pay interest semiannually—that is, every six months. Therefore, when calculating the yield on a bond, we must consider the timing of the cash flows to be such that the discount period is six months.

TRY IT! THE YIELD ON AN INVESTMENT

Suppose you invest \$1,000 today in an investment that promises you \$1,000 in two years and \$10,000 in three years. What is the *IRR* on this investment?

EXAMPLE 10.11: CALCULATING THE YIELD ON A BOND

Consider a bond that has a current price of 90; that is, if the par value of the bond is \$1,000, the bond's price is 90% of \$1,000 or \$900. And suppose that this bond has five years remaining to maturity and an 8% coupon rate. With five years remaining to maturity, the bond has 10 six-month periods remaining.

(*continued*)

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Solution

With a coupon rate of 8%, this means that the cash flows for interest is $40 every six months. For a given bond, we therefore have the following information:

Present value = $900

Number of periods to maturity = 10

Cash flow every six months = $40

Additional cash flow at maturity = $1,000

The six-month yield, r_d, is the discount rate that solves:

$$\$900 = \left[\sum_{t=1}^{10} \frac{\$40}{(1 + r_d)^t}\right] + \frac{\$1{,}000}{(1 + r_d)^{10}}$$

Using a calculator or spreadsheet, we calculate the six-month yield as 5.315% [PV = $900; N = '10; PMT = $40; FV = $1,000]. Bond yields are generally stated on the basis of an annualized yield, referred to as the *yield to maturity* on a bond-equivalent basis. This measure is analogous to the *APR* with semiannual compounding. Therefore, yield to maturity is 10.63%.

THE BOTTOM LINE

- The time value of money is one of the foundation concepts and tools in financial and investment management.
- Using compound interest, we can estimate a value of in the future; using discounting, we can translate a future value into a value today—a present value.
- It is important to consider the type of interest—compounding vs. simple—and the frequency of compounding in determining a present value of a future value.
- The time value of money mathematics can be used to determine the present value or future value of a lump-sum amount or of a series of cash flows, the growth rate of values, the number of periods of interest to meet a goal, or to simply amortize a loan.

- Given the cost of an investment and its cash flows, we can calculate the yield or implied interest rate. By calculating the yield or implied interest rate, we can then compare investment or financing opportunities. The yield or internal rate of return on an investment is the interest rate at which the present value of the cash flows equals the initial investment outlay.

SOLUTIONS TO TRY IT! PROBLEMS

Future Value

a. $FV = \$100\ (1+ 0.02)^5 = \$110.41$
b. $FV = \$100\ (1+ 0.02)^{10} = \$121.90$
c. $FV = \$100\ (1+ 0.02)^{20} = \$148.59$

Growth Rates

$PV = \$2{,}000$; $FV = \$4{,}000$; $N = 5$ Solve for i. $i = 14.87\%$

More Growth Rates

Present Value	Future Value	Number of Periods	Growth Rate
\$1	\$3	6	20.094%
\$1000	\$2000	9	8.006%
\$500	\$600	7	2.639%
\$1	\$1.50	4	10.668%

Frequency of Compounding

a. $\$100\ (1 + 0.02)^{10} = \$121.899$
b. $\$100\ (1 + 0.005)^{40} = \$122.079$
c. $\$100\ e^{0.2} = \$122.140$

Present Value

$PV = \$1{,}000 \div (1 + 0.06)10 = \$558.39$

The Yield on an Investment

Cash flows are −\$10,000, \$0, \$1,000 and \$10,000. The yield is 3.332%

QUESTIONS

1. What is the relationship between compounding and discounting of a lump-sum?
2. Complete the following: "The larger the interest rate, the ________ (larger/smaller) the future value of a value today."
3. Holding everything else the same, what is the effect of using a higher discount rate to discount a future value to the present?
4. If you invest the same amount in each of three accounts today, which account produces the highest future value if the annual percentage rate is the same? Account A: annual compounding, Account B: quarterly compounding, Account C: continuous compounding.
5. What distinguishes an ordinary annuity from an annuity due?
6. What distinguishes an ordinary annuity from a deferred annuity?
7. If a cash flow is the same amount each period, *ad infinitum*, how do we value the present value of this series of cash flows?
8. Which is most appropriate to use in describing the annual growth of the value of an investment: the arithmetic average growth rate or the geometric average growth rate? Why?
9. How can we break down the valuation of a deferred annuity into manageable parts for computation purposes?
10. Which has the highest present value if the payments and number of payments are identical, an ordinary annuity or an annuity due?
11. If you are offered two investments, one that pays 5% simple interest per year and one that pays 5% compound interest per year, which would you choose? Why?
12. Consider a borrowing arrangement in which the annual percentage rate (APR) is 8%.
    a. Under what conditions does the effective annual rate of interest (EAR) differ from the APR of 8%?
    b. As the frequency of compounding increases within the annual period, what happens to the relationship between the EAR and the APR?
13. Suppose you deposit $1,000 in an account with an APR of 4%, with compounding quarterly.
    a. After 10 years, what is the balance in the account if you make no withdrawals?
    b. After 10 years, how much interest on interest did you earn?
14. Suppose you are promised $10,000 five years from today. If the appropriate discount rate is 6%, what is this $10,000 worth to you today?

15. Suppose you buy a car today and finance \$10,000 of its cost at an APR of 3%, with payments made monthly.

a. If you finance the car for 24 months, what is the amount of your monthly car payment?

b. If you finance the car for 36 months, what is the amount of your monthly car payment?