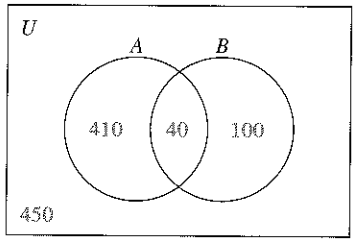
15-1 Venn Diagrams

Objective *To use Venn diagrams to illustrate intersections and unions of sets and to use the inclusion-exclusion principle to solve counting problems involving intersections and unions of sets.*

In this chapter we investigate the theory of counting, more formally known as **combinatorics**. Although counting may seem to be a very simple activity—one that merely involves the pairing of the natural numbers 1, 2, 3, . . . with the objects in some set—the task can be challenging when the objects are numerous. Shortcuts in counting are therefore an important part of combinatorics.

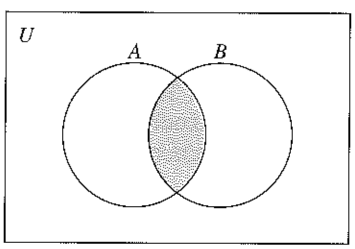
Before we can get started counting, we must know precisely what we are counting. We can use sets to separate the objects to be counted from all others, and we can use a *Venn diagram* to illustrate those sets.



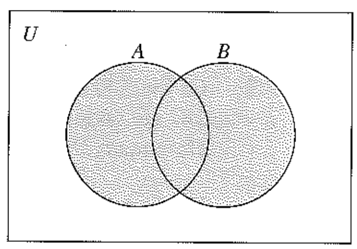
When drawing a Venn diagram, we begin with a rectangle, which represents a *universal set U*. Inside the rectangle we draw circles to represent *subsets* of the universal set. For example, consider the Venn diagram at the right. Here we have a universal set of 1000 typical Americans. The set A represents those who have a certain protein called antigen A on the surface of their red blood cells. Likewise, the set B represents those who have antigen B. Of our 1000 typical Americans, a total of 450 are in set A and a total of 140 are in set B.

Notice that sets A and B overlap. This represents the fact that 40 of our 1000 typical Americans have *both* antigen A and antigen B on the surface of their red blood cells. We call the set of elements that any two sets A and B have in common the **intersection** of A and B, which we denote by $A \cap B$. It is the shaded region in the Venn diagram at the left below.

The set of people who have *either* antigen A or antigen B represents the **union** of sets A and B, which we denote by $A \cup B$. It is the shaded region in the Venn diagram at the right below.



$A \cap B$



$A \cup B$

◀ How many license plates can be made using three letters followed by three digits? In this chapter you will learn how to find the answer—17,576,000.

Teaching Notes, p. 564B

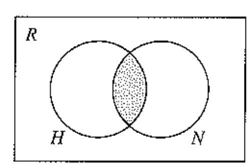
Warm-Up Exercises

For Exercises 1-5, consider the integers from 1 to 10 inclusive.

1. List all the integers that are:
   a. prime. 2, 3, 5, 7
   b. odd. 1, 3, 5, 7, 9
   c. factors of 6. 1, 2, 3, 6
2. List each integer that is both prime and odd. 3, 5, and 7
3. List each integer that is prime, odd, and a factor of 6. 3
4. List each integer that is *not* prime, *not* odd, and *not* a factor of 6. 4, 8, 10
5. List each integer that is both odd and a factor of 6, but is not prime. 1

Motivating the Section

A consumer-advocate group in Wisconsin wants to direct a mailing to homeowners in the state who heat their homes with natural gas. The shaded region in the diagram below indicates the portion of the state's population to whom the mailing will be sent. In the diagram, R represents residents of Wisconsin, H represents homeowners in Wisconsin, and N represents people in Wisconsin whose homes or apartments are heated by natural gas.



Communication Note

When discussing the Venn diagrams on the preceding page, be sure students understand that sets A and B are defined in terms of *antigens*, not *blood types*. Thus, there is a total of 450 people who have antigen A, and of these, 410 have "type A" blood. Ask students to make a similar statement regarding antigen B and "type B" blood. (A total of 140 people have antigen B, and of these, 100 have "type B" blood.)

Making Connections

Venn diagrams can also be used to illustrate logical statements (see Lewis Caroll's *Symbolic Logic*). Ask students to draw a Venn diagram to illustrate each of the following statements:

1. All A's are B's. (Circle for A is completely inside circle for B.)
2. No A's are B's. (Circle for A is completely outside circle for B.)
3. Some A's are B's. (Circle for A partially overlaps circle for B.)

Assessment Note

In discussing the inclusion-exclusion principle, ask students for $n(A \cup B)$ if A and B are disjoint sets, that is, if $A \cap B = \emptyset$. ($n(A \cup B) = n(A) + n(B)$ since $n(A \cap B) = 0$.)

The set of all elements *not* in a set A is called the **complement** of A and is denoted $\overline{A}$. In the union diagram on the right at the bottom of the preceding page, the unshaded region inside the rectangle represents $\overline{A \cup B}$, the complement of $A \cup B$. This complement consists of people who have *neither* antigen A nor antigen B on the surface of their red blood cells. (Such people are said to have "type O" blood. Moreover, those with antigen A but not antigen B have "type A" blood, those with antigen B but not antigen A have "type B" blood, and those with both antigen A and antigen B have "type AB" blood.)

Look again at the first Venn diagram shown on the preceding page. If we let $n(A \cup B)$ designate the number of elements in the union of sets A and B, we see that $n(A \cup B) = 550$. Notice that

$$n(A \cup B) \neq n(A) + n(B)$$

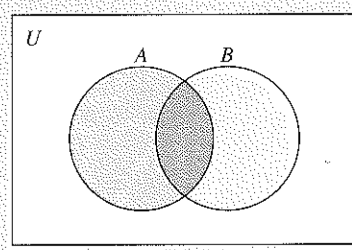
(that is, $550 \neq 450 + 140$), because the number of people in $A \cap B$ is counted twice when $n(A)$ and $n(B)$ are added. To compensate, we must subtract $n(A \cap B)$ once from the sum:

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 550 &= 450 + 140 - 40 \end{aligned}$$

This gives us the following counting principle.

The Inclusion-Exclusion Principle

For any sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.



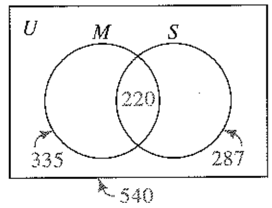
Example 1 Of the 540 seniors at Central High School, 335 are taking mathematics, 287 are taking science, and 220 are taking both mathematics and science. How many are taking neither mathematics nor science?

Solution Let U = the universal set of seniors at Central High School,
M = the subset of seniors taking mathematics,
and S = the subset of seniors taking science.

These sets and the number of people in each are shown in the Venn diagram at the top of the next page.

Using the inclusion-exclusion principle, we have:

$$n(M \cup S) = n(M) + n(S) - n(M \cap S)$$
$$= 335 + 287 - 220$$
$$= 402$$



Thus, the number of seniors *not* taking mathematics or science is:

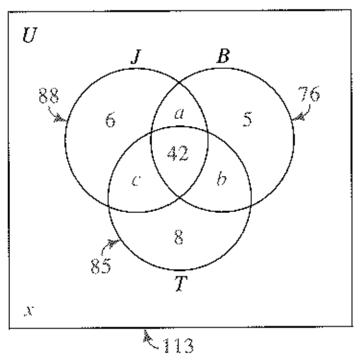
$$n(\overline{M \cup S}) = 540 - 402 = 138$$

Example 2 In a survey, 113 business executives were asked if they regularly read the *Wall Street Journal, Business Week* magazine, and *Time* magazine. The results of the survey are as follows:

88 read the *Journal.*	6 read only the *Journal.*
76 read *Business Week.*	5 read only *Business Week.*
85 read *Time.*	8 read only *Time.*
42 read all three.	

How many read none of the three publications?

Solution Letting J, B, and T represent the sets of *Journal, Business Week,* and *Time* readers, we draw a Venn diagram and place the numbers given above in the diagram, as shown at the right. (Be sure you understand the placement of each number in the diagram.) To find the numbers for the remaining regions of the diagram, we let a represent the number of executives who read just the *Journal* and *Business Week*, b represent the number who read just *Business Week* and *Time*, and c represent the number who read just the *Journal* and *Time*. We can then write the following system of equations:



$$\begin{aligned} 88 &= 6 + 42 + a + c, &\quad \text{or} \quad 40 &= a + c \\ 76 &= 5 + 42 + a + b, &\quad \text{or} \quad 29 &= a + b \\ 85 &= 8 + 42 + b + c, &\quad \text{or} \quad 35 &= b + c \end{aligned}$$

Solving this system gives $a = 17$, $b = 12$, and $c = 23$. The number x, representing those who read none of the three publications, can be found by adding the numbers within the three circles and subtracting the sum from 113. This gives $x = 0$, which means that every one of the business executives surveyed reads at least one of the three publications.

Example Note

Students may need help reading the numbers in a Venn diagram: A number by itself inside a region that has *not* been subdivided indicates how many objects are in that region only (for example, the "42" in Example 2 refers to $n(J \cap B \cap T)$); a number with an arrow pointing to a region that *has* been subdivided indicates the total number of objects in all of the subdivisions of the region (for example, the "88" in Example 2 refers to $n(J)$).

Suggested Assignments

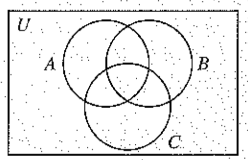
Discrete Math
568/1, 7, 9, 13–19 odd

Supplementary Materials

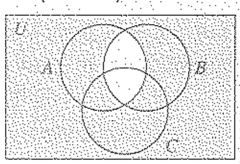
Alternative Assessment, 46

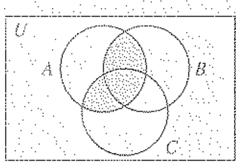
Additional Examples

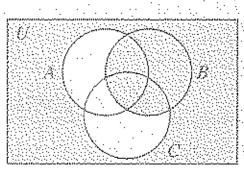
1. For each of the following, copy the Venn diagram and shade the specified set.



a. $\overline{A \cap B}$
b. $A \cap (B \cup C)$
c. $B \cup (\overline{A} \cap \overline{C})$

a.


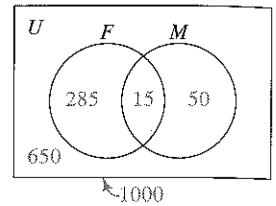
b.


c.


In Example 2 we found that the set $\overline{J \cup B \cup T}$ has no elements in it. Such a set is called the **empty set** and is denoted by the Greek letter ϕ (phi). Thus, $\overline{J \cup B \cup T} = \phi$.

CLASS EXERCISES

A certain small college has 1000 students. Let F = the set of college freshmen, and let M = the set of music majors. These sets are shown in the Venn diagram at the right. Describe each of the following sets in words and tell how many members it has.



1. $F \cap M$ 15 **2.** $F \cup M$ 350
3. $\overline{F}$ 700 **4.** $\overline{M}$ 935
5. $\overline{F} \cap M$ 50 **6.** $F \cap \overline{M}$ 285
7. $\overline{F \cup M}$ 650 **8.** $F \cup \overline{M}$ 950

9. *Reading* Refer to the first Venn diagram on page 565. People who have "type A" blood belong to the set $A \cap \overline{B}$. To what set do people with each of the following blood types belong?
a. B $B \cap \overline{A}$ **b.** AB $A \cap B$ **c.** O $\overline{A \cup B}$ or $\overline{A} \cap \overline{B}$

10. If A is any subset of a universal set U, complete the following.
a. $A \cup \phi =$ _?_ A **b.** $A \cap \phi =$ _?_ ϕ **c.** $A \cup \overline{A} =$ _?_ U
d. $A \cap \overline{A} =$ _?_ ϕ **e.** $A \cup U =$ _?_ U **f.** $A \cap U =$ _?_ A

WRITTEN EXERCISES

For Exercises 1–4, draw a Venn diagram and shade the region representing the set given in part (a). Then draw a separate Venn diagram and shade the region representing the set given in part (b).

A

1. a. $P \cap Q$ **b.** $P \cup Q$
2. a. $\overline{P} \cap Q$ **b.** $P \cap \overline{Q}$
3. a. $\overline{P \cup Q}$ **b.** $\overline{P} \cap \overline{Q}$
4. a. $\overline{P \cap Q}$ **b.** $\overline{P} \cup \overline{Q}$

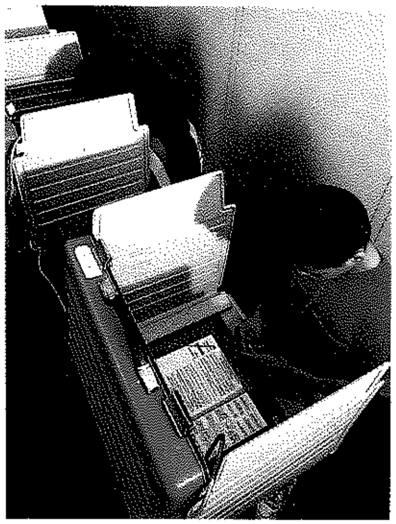
Let U = the universal set of all teachers in your school. Let the subsets of mathematics teachers, biology teachers, physics teachers, and chemistry teachers be represented by M, B, P, and C, respectively. Describe in words each of the following sets, and name a teacher belonging to each set if such a teacher exists in your school.

5. a. $M \cup P$ **b.** $M \cap \overline{P}$
6. a. $P \cap C$ **b.** $\overline{P \cup C}$
7. a. $B \cup (P \cap C)$ **b.** $(B \cup P) \cap (B \cap C)$
8. a. $(B \cup C) \cap M$ **b.** $(B \cap M) \cup (C \cap M)$

9. a. $\overline{M \cup P \cup C}$ **b.** $\overline{M} \cap \overline{P} \cap \overline{C}$

10. a. $\overline{M} \cap \overline{B} \cap \overline{P} \cap \overline{C}$ **b.** $\overline{M \cup B \cup P \cup C}$

In Exercises 11–14, draw a Venn diagram to illustrate each situation described. Then use the diagram to answer the question asked.



11. In an election-day survey of 100 voters leaving the polls, 52 said they voted for Proposition 1, and 38 said they voted for Proposition 2. If 18 said they voted for both, how many voted for neither? 28

12. Although the weather was perfect for the beach party, 17 of the 30 people attending got a sunburn and 25 people were bitten by mosquitoes. If 12 people were both bitten and sunburned, how many had neither affliction? 0

13. In a survey of 48 high school students, 20 liked classical music and 16 liked bluegrass music. Twenty students said they didn't like either. How many liked classical but not bluegrass? 12

14. Of the 52 teachers at Roosevelt High School, 27 said they like to go sailing, 25 said they like to go fishing, and 12 said they don't enjoy either recreation. How many enjoy fishing but not sailing? 13

B **15.** **Astronomy** Consider the sets defined below. (You may need to consult an encyclopedia to determine the elements of each set.)

Let U = the universal set of planets in our solar system,
S = the subset of planets smaller than Earth,
and F = the subset of planets farther from the Sun than Earth.

a. Draw a Venn diagram with overlapping circles representing S and F inside a rectangular region U. Inside each of the four regions of your diagram, list the planets described by that region.

b. The smallest set to which the planet Venus belongs is $S \cap \overline{F}$. What is the smallest set to which the planet Uranus belongs? $\overline{S} \cap F$

16. **Geography** Consider the sets defined below.

Let U = the universal set of states in the United States,
P = the subset of states bordering the Pacific Ocean,
and M = the subset of states bordering Mexico.

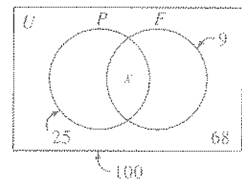
a. Name all the states in the set $\overline{P} \cap M$. Arizona, New Mexico, Texas

b. What is the only state in the set $P \cap M$? California

c. How many states are in the set $\overline{P \cup M}$? 42

Additional Examples cont.

2. Of the first 100 positive integers, 25 are prime, 9 are factors of 100, and 68 are neither prime nor a factor of 100. How many are:
a. both prime and a factor of 100?
b. a factor of 100 but not prime?



$n(P \cup F) =$
$n(P) + n(F) -$
$n(P \cap F) =$
$25 + 9 - x$;
$(34 - x) + 68 = 100$;
$x = 2$

a. Two numbers (2 and 5) are both prime and factors of 100.
b. Seven (9 − 2) numbers are factors of 100 but not prime.

Additional Answers Written Exercises

15. a. $\overline{S} \cap \overline{F}$ = {Earth}
$S \cap \overline{F}$ = {Mercury, Venus}
$S \cap F$ = {Mars, Pluto}
$\overline{S} \cap F$ = {Jupiter, Saturn, Uranus, Neptune}

ana, Guyana, Suriname, Uruguay, Venezuela}

$A \cap P \cap \overline{R} =$ {Colombia, Chile}

$\overline{A} \cap P \cap \overline{R} =$ {Ecuador}

$A \cap \overline{P} \cap R =$ {Brazil}

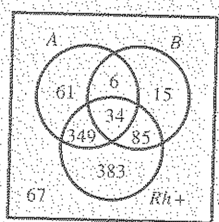
$A \cap P \cap R = \emptyset$

$\overline{A} \cap P \cap R =$ {Peru}

$\overline{A} \cap \overline{P} \cap R = \emptyset$

$\overline{A} \cap \overline{P} \cap \overline{R} =$ {Bolivia, Paraguay}

18.



Application

Many of the properties developed in the C-level exercises are used in the design of integrated logic circuits for computers.

Communication Note

Have a student do research and make a report to the class on Augustus De Morgan's contributions to mathematics and logic.

17. Geography Consider the sets defined below. (You may need to consult an encyclopedia or atlas to determine the elements of each set.)

Let U = the universal set of 13 countries in South America,
A = the subset of South American countries bordering the Atlantic Ocean or the Caribbean Sea,
P = the subset of South American countries bordering the Pacific Ocean,
and R = the subset of countries through which the Amazon River runs.

a. Name two countries that belong to $\overline{A} \cap \overline{P} \cap \overline{R}$. Bolivia, Paraguay

b. Draw a Venn diagram with overlapping circles representing A, P, and R all inside a rectangular region U. Inside each of the eight regions of your diagram, list the South American countries described by that region.

18. Medical Science In addition to antigens A and B, a red blood cell may have a protein called the Rhesus factor on its surface. If the protein is present, the blood is said to be "Rh positive"; if not, the blood is "Rh negative." About 85% of the American population is "Rh positive." Using this information, redraw the first Venn diagram on page 565 and introduce a third set, $Rh+$, that overlaps sets A and B. Then indicate the approximate number of people who belong to each of the eight regions of the new Venn diagram.

19. In a parking lot containing 85 cars, there are 45 cars with automatic transmissions, 43 cars with rear-wheel drive, and 46 cars with four-cylinder engines. Of the cars with automatic transmissions, 26 also have rear-wheel drive. Of the cars with rear-wheel drive, 29 also have four-cylinder engines. Of the cars with four-cylinder engines, 27 also have automatic transmissions. There are 21 cars with all three features.

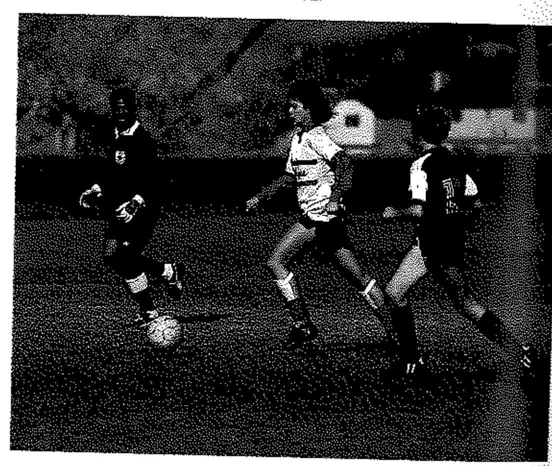
a. How many cars do not have automatic transmissions and rear-wheel drive but do have four-cylinder engines? 11

b. How many cars do not have any of the three features? 12

20. Of the 415 girls at Gorham High School last year, 100 played fall sports, 98 played winter sports, and 96 played spring sports. Twenty-two girls played sports all three seasons while 40 played only in fall, 47 only in winter, and 33 only in spring.

a. How many girls played fall and winter sports but not a spring sport? 13

b. How many girls did not play sports in any of the three seasons? 219



21. For a universal set U, what is $\overline{U}$? ϕ

22. If A is a subset of a universal set U, find $\overline{\overline{A}}$. A

C 23. When we say that multiplication is distributive over addition, we mean that for all real numbers a, b, and c,

$$a(b + c) = ab + ac.$$

Similarly, when we say that intersection is distributive over union, we mean that for all sets A, B, and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Show that this equation is true by drawing two three-circle Venn diagrams. In one diagram shade the region corresponding to $A \cap (B \cup C)$, and in the other shade the region corresponding to $(A \cap B) \cup (A \cap C)$.

24. Use the method of Exercise 23 to determine whether union distributes over intersection. Yes, union distributes over intersection.

25. a. One of De Morgan's laws states that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. Use the method of Exercise 23 to verify this law.

b. Another of De Morgan's laws writes $\overline{A \cap B}$ in terms of $\overline{A}$ and $\overline{B}$. Use Venn diagrams to determine what this law is. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

c. Use parts (a) and (b) to write $\overline{A \cap (B \cup C)}$ in terms of $\overline{A}$, $\overline{B}$, and $\overline{C}$. $\overline{A} \cup (\overline{B} \cap \overline{C})$

26. Simplify each of the following by using the properties of intersection, union, and complementation from Exercises 23–25.

a. $A \cap (\overline{A} \cup B)$ $A \cap B$

b. $A \cup (\overline{A \cup B})$ $A \cup \overline{B}$

27. Extend the inclusion-exclusion principle to three sets. (*Hint*: Even though $n(A \cup B \cup C) \neq n(A) + n(B) + n(C)$, you can alter the right side of this inequality to obtain equality.) $n(A \cup B \cup C) =$
$n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

15-2 The Multiplication, Addition, and Complement Principles

Objective *To use the multiplication, addition, and complement principles to solve counting problems.*

In Section 15-1 we saw that Venn diagrams and the inclusion-exclusion principle are helpful in certain counting problems. In this section we consider three other counting principles.

The Multiplication Principle

If an action can be performed in n_1 ways, and for each of these ways another action can be performed in n_2 ways, then the two actions can be performed together in $n_1 n_2$ ways.

Teaching Notes, p. 564B

Warm-Up Exercises

Seven sculptures were included in an art display. Alex, Beth, Chen-Fa, and Donna were the North High School artists, and José, Kim, and Lois were the South High School artists.

1. In how many ways can a best-of-show prize be awarded? 7

2. a. List the different ways in which exactly one student from each school could be awarded a prize. AJ, AK, AL, BJ, BK, BL, CJ, CK, CL, DJ, DK, DL

b. Describe a way to determine the number of possibilities found in Ex. 2. Multiply the number of students from North H.S., 4, by the number from South H.S., 3.

3. If the three sculptures created by the South High School students are lined up, list all the possible arrangements. JKL, JLK, KJL, KLJ, LJK, LKJ

Motivating the Section

Ask students if they can predict which offers more choices for license plates:

1. a plate with three different letters of the alphabet in any order
2. a plate with four different nonzero digits in any order

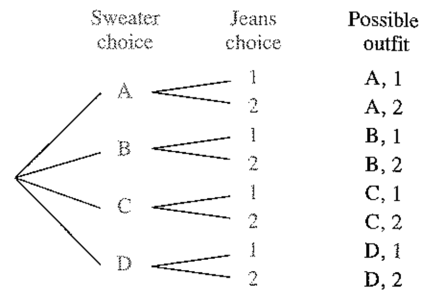
(The plate with three different letters offers 15,600 choices; the plate with four different digits offers 3024 choices.)

Example 1 If you have 4 sweaters and 2 pairs of jeans, how many different sweater-and-jeans outfits can you make?

Solution There are two actions to perform:

(1) choosing a sweater and (2) choosing a pair of jeans

Since there are 4 sweaters from which to choose and 2 pairs of jeans from which to choose, there are $4 \times 2 = 8$ ways to choose a sweater-and-jeans outfit. If we call the sweaters A, B, C, and D, and the pairs of jeans 1 and 2, the so-called *tree diagram* at the right shows the 8 possible outfits.



The multiplication principle can be extended to three or more actions that are performed together, as the next two examples show.

Example 2 In how many ways can 8 people line up in a cafeteria line?

Solution The diagram below represents the eight places in the cafeteria line.

$\square\ \square\ \square\ \square\ \square\ \square\ \square\ \square$

The first place in line can be filled by any of the 8 people. Then the second place in line can be filled by any of the remaining 7 people, the third place by any of the remaining 6 people, and so on. The diagram below illustrates this reasoning.

$\boxed{8}\ \boxed{7}\ \boxed{6}\ \boxed{5}\ \boxed{4}\ \boxed{3}\ \boxed{2}\ \boxed{1}$

By the multiplication principle, the answer is:

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$$

In Example 2, the product $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can be abbreviated as $8!$ (read "8 factorial"). The definition of $n!$, where n is a nonnegative integer, is:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$
$$0! = 1$$

Notice in Example 2 that no person can occupy more than one space in the cafeteria line. Thus, the spaces of the cafeteria line are filled *without repetition*. This is in contrast with the next example, where repetition does occur.

Students ... the use of the multiplication principle in Example 2, for they argue that the selection of the first person in line influences the choices available for the next position. For example, if Tom is first in line, he can't be second; while if Jill is first, then Tom *can* be second. Be sure students understand that the multiplication principle relies on each choice leaving the same *number* of choices for the next step, not necessarily the same specific choices.

Mathematical Note

You may wish to introduce an alternate notation for $n!$:

$$\text{For } n \geq 1,\ n! = \prod_{i=1}^{n} i$$

Note that Π (pi) stands for "product" just as Σ (sigma) stands for "sum."

Using Technology

Many calculators have a factorial key. Students may gain an appreciation for the rate at which $n!$ grows by finding various factorials on their calculators. Ask them:

1. What is the largest value of n for which the calculator can evaluate $n!$?
2. What is the largest value of n for which the calculator displays the *exact* value of $n!$?
3. Using the answer for question 2, calculate $\frac{(n+1)!}{n+1}$. Is the answer exact?

Example 3 How many license plates can be made using 2 letters followed by 3 digits?

Solution The diagram below represents the five spaces of a license plate.

$$\boxed{}\quad\boxed{}\quad\boxed{}\quad\boxed{}\quad\boxed{}$$

Each of the first two spaces can be filled with any of the 26 letters of the alphabet. Each of the last three spaces can be filled with any of the 10 digits. The diagram below illustrates this reasoning.

$$\boxed{26}\quad\boxed{26}\quad\boxed{10}\quad\boxed{10}\quad\boxed{10}$$

Thus, by the multiplication principle, the number of license plates is:

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676{,}000$$

In Example 3, suppose that each of the five spaces of a license plate can be filled with *either* a letter *or* a digit. For any given space, then, there are 26 ways of choosing a letter, 10 ways of choosing a digit, and $26 + 10 = 36$ ways of filling the space. Note that we *add* the number of ways of performing the actions of choosing a letter and choosing a digit because the actions are **mutually exclusive**, that is, they cannot be performed together. This leads us to the following principle.

The Addition Principle

If two actions are mutually exclusive, and the first can be done in n_1 ways and the second in n_2 ways, then one action *or* the other can be done in $n_1 + n_2$ ways.

Example 4 In Morse code, the letters of the alphabet are represented by sequences of dots (•) and dashes (–). For example, • – represents the letter A and – – • represents the letter G. Show that sequences of no more than 4 symbols (dots and dashes) are needed to represent all of the alphabet.

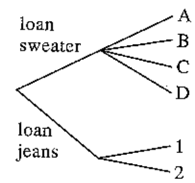
Solution We can think of a 1-symbol sequence as a box to be filled with either a dot or a dash; a 2-symbol sequence as two boxes to be filled, each with a dot or a dash; and so on. Using the multiplication principle and then the addition principle, we have:

$$\begin{aligned}
n_1 &= \text{number of 1-symbol sequences} = 2 && = 2\\
n_2 &= \text{number of 2-symbol sequences} = 2 \times 2 && = 4\\
n_3 &= \text{number of 3-symbol sequences} = 2 \times 2 \times 2 && = 8\\
n_4 &= \text{number of 4-symbol sequences} = 2 \times 2 \times 2 \times 2 && = \underline{16}\\
&\text{Total number of sequences with no more than 4 symbols} && = 30
\end{aligned}$$

Thus, the 30 possible sequences are obviously enough to accommodate the 26 letters of the alphabet.

Communication Note

Emphasize that multiplication is used for situations in which you select from set *A and* from set *B*; addition is used when you select from set *A or* from set *B*. A nice contrast between the addition and multiplication principles can be made by returning to Example 1 and assuming that the student agrees to loan a sibling either a sweater *or* a pair of jeans. A tree diagram for this situation might be:



Note that 6 different loans can be made.

Example Note

Point out that Example 4 uses both the multiplication and addition principles. You might ask a student to obtain the complete Morse code and give a brief report on its history.

Additional Examples

1. Classify as true or false: $(3 \cdot 4)! = 3! \cdot 4!$
   $(3 \cdot 4)! = 12! = 12 \cdot 11 \cdot 10 \cdot \ldots \cdot 3 \cdot 2 \cdot 1$
   $3! \cdot 4! = 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
   Thus, $(3 \cdot 4)! \neq 3! \cdot 4!$

2. Given the digits 0, 1, 2, 3, 4, and 5, how many four-digit numbers can be formed if:
   a. digits can be repeated?
   b. digits cannot be repeated?
   c. digits cannot be repeated and the number must be even?
   In each case, the first digit cannot be zero.
   a. $\boxed{5} \cdot \boxed{6} \cdot \boxed{6} \cdot \boxed{6} =$ 1080 numbers
   b. Choices for first digit: 1, 2, 3, 4, or 5. Choices for second digit: 0 and the 4 digits remaining after the first choice.
   $\boxed{5} \cdot \boxed{5} \cdot \boxed{4} \cdot \boxed{3} =$ 300 numbers
   c. The last digit must be 0, 2, or 4. When the last digit is 0, we have $\boxed{5} \cdot \boxed{4} \cdot \boxed{3} \cdot \boxed{1} =$ 60 choices. When the last digit is 2, we have $\boxed{4} \cdot \boxed{4} \cdot \boxed{3} \cdot \boxed{1} =$ 48 choices. When the last digit is 4, we have $\boxed{4} \cdot \boxed{4} \cdot \boxed{3} \cdot \boxed{1} =$ 48 choices. By the addition principle, there are $60 + 48 + 48 = 156$ even 4-digit numbers without repetition of digits.

When counting the number of elements in a set, you may find it easier to count the number in the complement of the set and then use the following principle, which is illustrated in Example 5.

The Complement Principle

If A is a subset of a universal set U, then:

$$n(A) = n(U) - n(\overline{A})$$

Example 5 Find the number of 4-digit numbers containing at least one digit 5.

Solution Let U = the universal set of all 4-digit numbers,
A = the set of 4-digit numbers containing at least one digit 5,
and $\overline{A}$ = the set of 4-digit numbers containing no 5's.

Then:
$$\begin{aligned} n(A) &= n(U) - n(\overline{A}) \\ &= 9 \cdot 10 \cdot 10 \cdot 10 - 8 \cdot 9 \cdot 9 \cdot 9 \\ &= 9000 - 5832 = 3168 \end{aligned}$$

CLASS EXERCISES

1. Evaluate: **a.** 2! 2 **b.** 3! 6 **c.** 4! 24
2. If 9! = 362,880, what is 10!? 3,628,800
3. ***Reading*** In Example 3, suppose the first of the three digits in a license plate cannot be 0. How many such license plates are possible? 608,400
4. **a.** If a girl has 6 different skirts and 10 different blouses, how many different skirt-and-blouse outfits are possible? 60
   b. If she also has 3 different sweaters, how many skirt-blouse-and-sweater outfits are possible? 180
5. A boy has 2 sports coats and 4 sweaters.
   a. How many coat-and-sweater outfits can he wear? 8
   b. Suppose he decides to wear either a sports coat or a sweater, but not both. How many choices does he have? 6
6. If 10 runners compete in a race, in how many different ways can prizes be awarded for first, second, and third places? 720
7. The "home row" of a standard typewriter gives one arrangement of the letters A, S, D, F, G, H, J, K, L.
   a. How many *other* arrangements of these letters are possible? 362,879
   b. If *any* 9 letters of the alphabet could be placed on the "home row" of a typewriter, how many arrangements of the letters would be possible? See below.
8. ***Reading*** Give the conclusion of the complement principle in words rather than symbols. Then use a Venn diagram to illustrate this principle.

7. b. 1,133,836,704,000

WRITTEN EXERCISES

A 1. Evaluate: **a.** $5!$ 120 **b.** $6!$ 720 **c.** $7!$ 5040 **d.** $0!$ 1

2. Evaluate: **a.** $\frac{10!}{9!}$ 10 **b.** $\frac{20!}{18!}$ 380 **c.** $\frac{n!}{(n-1)!}$ n **d.** $\frac{(n+1)!}{(n-1)!}$ $n^2 + n$

3. In how many different orders can you arrange 5 books on a shelf? 120

4. In how many different orders can 9 people stand in a line? 362,880

5. In how many different ways can you answer 10 true-false questions? 1024

6. In how many different ways can you answer 10 multiple-choice questions if each question has 5 choices? $5^{10} = 9{,}765{,}625$

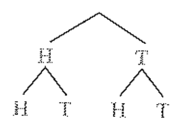
7. Many radio stations have 4-letter call signs beginning with K. How many such call signs are possible if letters **(a)** can be repeated? **(b)** cannot be repeated? See below.

8. How many 3-digit numbers can be formed using the digits 4, 5, 6, 7, 8 if the digits **(a)** can be repeated? **(b)** cannot be repeated? 125; 60

9. In how many ways can 4 people be seated in a row of 12 chairs? 11,880

10. In how many ways can 4 different prizes be given to any 4 of 10 people if no person receives more than 1 prize? 5040

11. The tree diagram shows the four possible outcomes when a coin is tossed twice. If H and T represent "heads" and "tails," respectively, then the four outcomes are HH, HT, TH, and TT.



a. Make a tree diagram showing the outcomes if a coin is tossed 3 times.

b. The solid lines of the tree diagram are called *branches* and the elements of the bottom row (H, T, H, and T) are called *leaves*. How many branches and leaves are in your diagram for part (a)? 14 branches, 8 leaves

c. How many branches and leaves would there be in a tree diagram showing the toss of a coin 10 times? 2046 branches, 1024 leaves

12. Four cards numbered 1 through 4 are shuffled and 3 different cards are chosen one at a time. Make a tree diagram showing the various possible outcomes.

B 13. **Sports** A high school coach must decide on the batting order for a baseball team of 9 players.

a. The coach has how many different batting orders from which to choose? 362,880

b. How many different batting orders are possible if the pitcher bats last? 40,320

c. How many different batting orders are possible if the pitcher bats last and the team's best hitter bats third? 5040

7. **a.** 17,576 **b.** 13,800

Assessment Note

The first letter of the call signs for radio stations throughout the United States is limited to K and W. (Generally speaking, K is used west of the Mississippi River, and W is used east.) Ask students how many four-letter call signs are available nationwide. $(2 \times 26 \times 26 \times 26 = 35{,}152)$

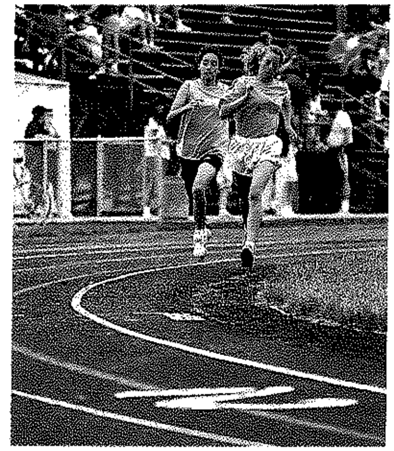
Suggested Assignments

Discrete Math
Day 1: 575/1–15 odd
Day 2: 576/17–31 odd

Supplementary Materials

Alternative Assessment, 46–47
Student Resource Guide, 140–142

14. Sports A track coach must choose a 4-person 400 m relay team and a 4-person 800 m relay team from a squad of 7 sprinters, any of whom can run on either team. If the fastest sprinter runs last in both races, in how many ways can the coach form the two teams if each of the 6 remaining sprinters runs only once and each different order is counted as a different team? 720



15. How many numbers consisting of 1, 2, or 3 digits (without repetitions) can be formed using the digits 1, 2, 3, 4, 5, 6? 156

16. If you have 5 signal flags and can send messages by hoisting one or more flags on a flagpole, how many messages can you send? 325

17. In some states license plates consist of 3 letters followed by 2 or 3 digits (for example, RRK-54 or ABC-055). How many such possibilities are there for those plates with 2 digits? for those with 3 digits? In all, how many license plates are possible? 1,757,600; 17,576,000; 19,333,600

18. How many possibilities are there for a license plate with 2 letters and 3 or 4 nonzero digits? 4,928,040

19. **a.** How many 3-digit numbers contain no 7's? 648
    b. How many 3-digit numbers contain at least one 7? 252

20. **a.** How many 4-digit numbers contain no 8's or 9's? 3584
    b. How many 4-digit numbers contain at least one 8 or 9? 5416

21. How many numbers from 5000 to 6999 contain at least one 3? 542

22. Many license plates in the U.S. consist of 3 letters followed by a 3-digit number from 100 to 999. How many of these contain at least one of the vowels A, E, I, O, and U? 7,483,500

23. Telephone numbers in the U.S. and Canada have 10 digits as follows:

3-digit area code number:	first digit is *not* 0 or 1; second digit *must be* 0 or 1
3-digit exchange number:	first and second digits are *not* 0 or 1
4-digit line number:	*not* all zeros

a. How many possible area codes are there? 160
b. The area code for Chicago is 312. Within this area code how many exchange numbers are possible? 640
c. One of the exchange numbers for Chicago is 472. Within this exchange, how many line numbers are possible? 9999
d. How many 7-digit phone numbers are possible in the 312 area code? 6,399,360
e. How many 10-digit phone numbers are possible in the U.S. and Canada? 1,023,897,600

31,996,800

24. Suppose a state has 5 telephone area codes. Refer to Exercise 23 and tell how many phone numbers there could be in the state without adding any more area codes.

25. a. How many 9-letter "words" can be formed using the letters of the word FISHERMAN? (*Note*: We allow any arrangement of letters, such as "HAMERSNIF," to count as a "word." We also assume each letter is used exactly once.) 362,880

b. How many 9-letter "words" begin and end with a vowel? (*Hint*: There are 3 choices for the first letter and 2 for the last letter, as shown below.) 30,240

$\boxed{3}\ \boxed{?}\ \boxed{?}\ \boxed{?}\ \boxed{?}\ \boxed{?}\ \boxed{?}\ \boxed{?}\ \boxed{2}$

26. Suppose the letters of VERMONT are used to form "words."

a. How many 7-letter "words" can be formed? 5040

b. How many 6-letter "words" can be formed? How does your answer here compare with your answer in part (a)? 5040; =

c. How many 5-letter "words" begin with a vowel and end with a consonant? 600

27. A school has 677 students. Explain why at least two students must have the same pair of initials. $26^2 = 676 < 677$

28. a. Ohio State University has about 40,000 students. Explain why at least two students must have the same first, middle, and last initials. $26^3 < 40{,}000$

b. What is the minimum number of students needed to be *certain* that at least two students have the same three initials? 17,577

29. ***Writing*** Exercises 27 and 28 involve the *pigeonhole principle*, which states:

If you are putting x pigeons in y pigeonholes, and $x > y$, then __?__.

Write a paragraph in which you complete the statement of the pigeonhole principle and explain its use in Exercises 27 and 28. Be sure to tell what corresponds to pigeons and what corresponds to pigeonholes in those exercises.

30. ***Research*** In a computer, a *bit* stores the smallest unit of information. Bits are usually arranged in groups of 8, called *bytes*, in order to store larger pieces of information. Find out what a bit is and from this determine how many different pieces of information can be stored in a byte. Then find out how "byte-sized" pieces of information are part of the ASCII (pronounced "ask' ee") code used by many computers for converting text characters to numbers.

31. Use your calculator to evaluate $\log_{10} 9!$ Then, without using your calculator, find the value of $\log_{10} 10!$ Check your answer with a calculator. See below.

32. Use the properties of logarithms to evaluate $\log_{10} 100! - \log_{10} 99!$ 2

33. a. Show that $10 \cdot 9 \cdot 8 \cdot 7 = \frac{10!}{6!}$. $\frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7$

b. For $1 \le r \le n$, show that:

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

C 34. In how many zeros does the number 100! end? 24

31. $\log_{10} 9! \approx 5.56$; $\log_{10} 10! \approx 6.56$

Exercise Note

Emphasize that the exercises in this section and in following sections consider a "word" to be any arrangement of letters, even meaningless arrangements such as *XZGJ*.

Additional Answers Written Exercises

29. If you are putting x pigeons in y pigeonholes, and $x > y$, then at least one pigeonhole will contain more than one pigeon. In Exercises 27 and 28, the students correspond to pigeons and the initials correspond to pigeonholes. Because our alphabet has 26 letters, there are 26^2 sets of initials with 2 letters and 26^3 sets of initials with 3 letters. If a given number of people is greater than the number of sets of possible initials, then there will be people who have the same initials.

30. We can think of a bit as storing either a 0 or a 1. A byte is composed of 8 bits and thus can store 2^8 or 256 numbers in binary code. The ASCII code associates with each keyboard character a number between 0 and 255. Thus, when a key is pressed on the keyboard, the computer stores not the character itself but a number corresponding to that character. For example, the ASCII code for the character H is 72, or 01001000 in binary code.

Teaching Notes, p. 564C

Warm-Up Exercises

Evaluate each expression.

1. $7!$ 5040 **2.** $\frac{7!}{4!}$ 210

3. $\frac{7!}{4!3!}$ 35

Tell whether the order in which the following choices are made is important or *not.*

4. Four students are chosen from a class of 87 students to be student council representatives. not imp.

5. Four digits are chosen for a bank-card identification code. imp.

Motivating the Section

Ideas from combinatorics have long been used in encoding and decoding information. One recent application arose with the appearance of "cash stations," where people can make machine-assisted withdrawals from or deposits to their accounts. Suppose an automatic teller machine requires a customer to enter a 4-digit code in order to initiate a transaction. How many different 4-digit codes can be generated using the digits 0 through 9? (10,000)

Communication Note

Students will need help in reading the symbols for permutations and combinations out loud: ${}_nP_r$ is usually read as "the number of permutations of n things taken r at a time."

15-3 Permutations and Combinations

Objective *To solve problems involving permutations and combinations.*

In some situations involving choices, the order in which the choices are made is important, whereas in others it is not. Suppose that a club with 12 members wishes to choose a president, a vice president, and a treasurer. In this case, the order of the choices is important: for instance, the order A, B, and C for president, vice president, and treasurer, respectively, is different from the order B, A, and C for the 3 offices. The number of ways of filling the 3 offices is:

$$12 \cdot 11 \cdot 10 = 1320$$

Suppose, on the other hand, that the club merely wants to choose a governing council of 3. In this case, the order of selection is *not* important, since a selection of A, B, and C is the same as a selection of B, A, and C. Now for each governing council of 3 that can be chosen (for example, A, B, C), there are 3! different slates of officers (ABC, ACB, BAC, BCA, CAB, CBA). Thus:

$$n(\text{governing councils}) \times 3! = n(\text{slates of officers})$$

$$n(\text{governing councils}) = \frac{n(\text{slates of officers})}{3!}$$

$$= \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220$$

In the first situation, where the order of selection is important, each selection is called a **permutation** of 3 people from a set of 12; the number of such selections is denoted ${}_{12}P_3$. In the second situation, where the order of selection is not important, each selection is called a **combination** of 3 people from a set of 12; the number of such selections is denoted ${}_{12}C_3$.

In general, for $0 \le r \le n$, the symbols ${}_nP_r$ and ${}_nC_r$ denote the number of permutations and the number of combinations, respectively, of r things chosen from n things. (Sometimes $P(n, r)$ and $C(n, r)$ are used instead of ${}_nP_r$ and ${}_nC_r$ in order to emphasize that permutations and combinations are *functions* of n and r.)

	Formula	Example
Permutation (order important)	${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$ $= \frac{n!}{(n-r)!}$ $\left\{\text{Formula derived in Exercise 33 on page 577}\right.$	${}_{12}P_3 = 12 \cdot 11 \cdot 10$ $= \frac{12!}{(12-3)!}$ $= 1320$
Combination (order not important)	${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}$ $= \frac{n!}{(n-r)!\,r!}$ $\left\{\text{Formula derived in Exercise 28 on page 582}\right.$	${}_{12}C_3 = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3}$ $= \frac{12!}{(12-3)!\,3!}$ $= 220$

Example 1 A company advertises two job openings, one for a copywriter and one for an artist. If 10 people who are qualified for either position apply, in how many ways can the openings be filled?

Solution Since the jobs are different, the order of selecting people matters: X as copywriter and Y as artist is different from Y as copywriter and X as artist. Thus, the solution is found by counting permutations:

$$_{10}P_2 = 10 \cdot 9 = 90$$

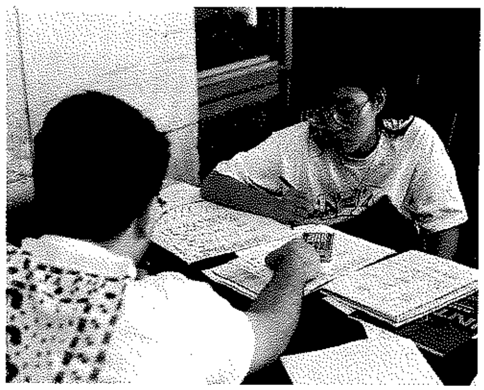
Example 2 A company advertises two job openings for computer programmers, both with the same salary and job description. In how many ways can the openings be filled if 10 people apply?

Solution Since the two jobs are identical, the order of selecting people is not important: Choosing X and Y for the positions is the same as choosing Y and X. Thus, the solution is found by counting combinations:

$$_{10}C_2 = \frac{10 \cdot 9}{1 \cdot 2} = 45$$

If you have a calculator with permutation and combination keys (or just a factorial key), you can use it to help solve counting problems that involve large numbers, as in the next example.

Example 3 As shown below, a standard deck of playing cards consists of 52 cards, with 13 cards in each of four *suits* (clubs, spades, diamonds, and hearts). Clubs and spades are black cards, and diamonds and hearts are red cards. Also, jacks, queens, and kings are called *face cards.*



Clubs (♣): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king
Spades (♠): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king
Diamonds (♦): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king
Hearts (♥): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

How many ways are there to deal 13 cards from a standard deck if the order in which the cards are dealt is **(a)** important? **(b)** not important?

Solution **a.** $_{52}P_{13} = \frac{52!}{39!} \approx 3.95 \times 10^{21}$ **b.** $_{52}C_{13} = \frac{52!}{39!13!} \approx 6.35 \times 10^{11}$

Example Note

Point out that the answer to Example 1 is twice that of Example 2. Ask what would happen if there were 3 jobs instead of 2. (The answer to Example 1 would be 3!, or 6, times the answer to Example 2.)

Additional Examples

1. For a certain raffle, 845 tickets are sold.
   a. In how many ways can four \$50 gift certificates be awarded?
   b. In how many ways can a \$100, a \$50, a \$20, and a \$10 gift certificate be awarded?

   a. Since the awards are the same, order is unimportant. $_{845}C_4 = \frac{845!}{841!4!} = \frac{845 \cdot 844 \cdot 843 \cdot 842}{4 \cdot 3 \cdot 2 \cdot 1} \approx 2.11 \times 10^{10}$

   b. Since the awards are different, the order is important. $_{845}P_4 = \frac{845!}{841!} = 845 \cdot 844 \cdot 843 \cdot 842 \approx 5.06 \times 10^{11}$

2. At the Red Lion Diner, an omelet can be ordered plain or with any or all of the following fillings: cheese, onions, peppers. How many different kinds of omelets are possible?

   A customer can choose 0, 1, 2, or 3 fillings. $_3C_0 + {_3C_1} + {_3C_2} + {_3C_3} = \frac{3!}{3!0!} + \frac{3!}{2!1!} + \frac{3!}{1!2!} + \frac{3!}{0!3!} = 1 + 3 + 3 + 1 = 8$

Additional Answers Class Exercises

8. a. ABC, ABD, ACB, ACD, ADB, ADC, BAC, BAD, BCA, BCD, BDA, BDC, CAB, CAD, CBA, CBD, CDA, CDB, DAB, DAC, DBA, DBC, DCA, DCB

Suggested Assignments

Discrete Math
Day 1: 580/1–15 odd
Day 2: 581/17–23, 25, 29

Supplementary Materials

Alternative Assessment, 47
Activities Book, 41–43

CLASS EXERCISES

In Exercises 1–4, find the value of each expression.

1. a. ${}_5P_2$ 20 **b.** ${}_5C_2$ 10 **2. a.** ${}_6P_3$ 120 **b.** ${}_6C_3$ 20

3. a. ${}_{10}P_3$ 720 **b.** ${}_{10}C_3$ 120 **4. a.** ${}_4P_4$ 24 **b.** ${}_4C_4$ 1

5. In how many ways can a club with 10 members choose a president, a vice president, and a treasurer? 720

6. In how many ways can a club with 10 members choose a 3-person governing council? 120

7. A lock on a safe has a dial with 50 numbers on it. To open it, you must turn the dial left, then right, and then left to 3 different numbers. Such a lock is usually called a combination lock, but a more accurate name would be permutation lock. Explain. Order of the 3 numbers is important.

8. a. Four people (A, B, C, and D) apply for three jobs (clerk, secretary, and receptionist). If each person is qualified for each job, make a list of the ways the jobs can be filled. For example, ABC means that A is clerk, B is secretary, and C is receptionist. This is different from BAC and CBA.

b. Your list should contain ${}_4P_3$ entries. How many entries involve persons A, B, and C? How many entries involve persons A, C, and D? 6; 6

c. If A, B, C, and D apply for three identical job openings as a clerk, make a list of the number of ways the openings can be filled. ABC, ABD, ACD, BCD

WRITTEN EXERCISES

A

1. a. In how many ways can a club with 20 members choose a president and a vice president? 380

b. In how many ways can the club choose a 2-person governing council? 190

2. a. In how many ways can a club with 13 members choose 4 different officers? 17,160

b. In how many ways can the club choose a 4-person governing council? 715

3. a. In how many ways can a host-couple choose 4 couples to invite for dinner from a group of 10 couples? 210

b. Ten students each submit a woodworking project in an industrial arts competition. There are to be first-, second-, and third-place prizes plus an honorable mention. In how many ways can these awards be made? 5040

4. A teacher has a collection of 20 true-false questions and wishes to choose 5 of them for a quiz. How many quizzes can be made if the order of the questions is considered **(a)** important? **(b)** unimportant? 1,860,480; 15,504

5. Each of the 200 students attending a school dance has a ticket with a number for a door prize. If 3 different numbers are selected, how many ways are there to award the prizes, given that the 3 prizes are **(a)** identical? **(b)** different? See below.

6. Suppose you bought 4 books and gave one to each of 4 friends. In how many ways can the books be given if they are **(a)** all different? **(b)** all identical? 24; 1

 5. a. 1,313,400 b. 7,880,400

7. Eight people apply for 3 job positions. In how many different ways can the 3 positions be filled if the positions are **(a)** all different? **(b)** all the same? 336; 56

8. How many different ways are there to deal a hand of 5 cards from a standard deck of 52 cards if the order in which the cards are dealt is **(a)** important? **(b)** not important? 311,875,200; 2,598,960

9. In how many ways can 6 hockey players be chosen from a group of 12 if the playing positions are **(a)** considered? **(b)** not considered? 665,280; 924

10. Of the 12 players on a school's basketball team, the coach must choose 5 players to be in the starting lineup. In how many ways can this be done if the playing positions are **(a)** considered? **(b)** not considered? 95,040; 792

11. **a.** A hiker would like to invite 7 friends to go on a trip but has room for only 4 of them. In how many ways can they be chosen? 35
    b. If there were room for only 3 friends, in how many ways could they be chosen? How is your answer related to the answer for part (a)? Why? 35

12. **a.** In how many ways can you choose 3 letters from the word LOGARITHM if the order of letters is unimportant? 84
    b. In how many ways can you choose 6 letters from LOGARITHM if the order of letters is unimportant? Compare with part (a). 84; =

13. Show that $_{100}C_2 = {}_{100}C_{98}$ in the following ways: **(a)** by using the formula for $_nC_r$; **(b)** by explaining how choosing 2 out of 100 is related to choosing 98 out of 100.

14. **a.** Show that $_{11}C_3 = {}_{11}C_8$.
    b. Study part (a) and Exercise 13. Then make a generalization and prove it.

15. **a.** Evaluate $_5C_0$ to find how many ways you can select no objects from a group of 5. 1
    b. Evaluate $_5C_5$ to find how many ways you can select 5 objects from a group of 5. 1

16. How does the formula for $_nC_n$ suggest the definition $0! = 1$?

17. A certain chain of ice cream stores sells 28 different flavors, and a customer can order a single-, double-, or triple-scoop cone. Suppose on a multiple-scoop cone that the order of the flavors is important and that the flavors can be repeated. How many possible cones are there? 22,764

18. Refer to Exercise 17 and find the number of double-scoop ice cream cones that are possible if repetition of flavors is allowed but the order of flavors is unimportant. 406

Additional Answers Written Exercises

11. b. The answer is the same as that in part (a). Choosing 3 out of 7 to go is the same as choosing 4 out of 7 to stay.

13. a. $_{100}C_2 = \frac{100!}{98!2!}$; $_{100}C_{98} = \frac{100!}{2!98!}$. Therefore, $_{100}C_2 = {}_{100}C_{98}$.

14. a. $_{11}C_3 = \frac{11!}{8!3!}$ and $_{11}C_8 = \frac{11!}{3!8!}$; therefore, $_{11}C_3 = {}_{11}C_8$.
    b. $_nC_r = {}_nC_{n-r}$. $_nC_r = \frac{n!}{(n-r)!r!}$ and $_nC_{n-r} = \frac{n!}{[n-(n-r)]!(n-r)!} = \frac{n!}{r!(n-r)!}$; thus, $_nC_r = {}_nC_{n-r}$.

16. $_nC_n = \frac{n!}{0!n!} = \frac{1}{0!}$. There is only one way to select n objects from a group of n; therefore, $\frac{1}{0!} = 1$ and $0! = 1$.

Students ... with Exercise 20 might ... about using a decision algorithm (see the teaching notes for Section 15–2). Here the first decision might be: In what order will the *subjects* be placed?

Additional Answers Written Exercises

23. b. There will be fewer line segments if three or more points are collinear.

27. The first and last entry of each row is 1. Each entry in between is the sum of the two entries immediately above it.

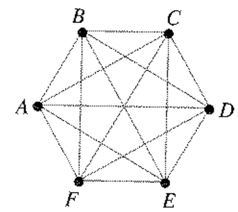
21. a. 792 b. 658,008 c. 1,940,952

19. Three couples go to the movies and sit together in a row of six seats. In how many ways can these people arrange themselves if each couple sits together? 48

20. A mathematics teacher uses 4 algebra books, 2 geometry books, and 3 precalculus books for reference. In how many ways can the teacher arrange the books on a shelf if books covering the same subject matter are kept together? 1728

21. From a standard deck of 52 cards, 5 cards are dealt and the order of the cards is unimportant. In how many ways can you receive **(a)** all face cards? **(b)** no face cards? **(c)** at least one face card? See above.

22. Answer part (c) of Exercise 21 if the order of receiving the 5 cards is important. 232,914,240



23. a. How many line segments can be drawn joining the six points A, B, C, D, E, and F shown? 15

b. If the six points were positioned differently, the answer to part (a) could be different. Explain.

24. A convex polygon has n vertices. Find the number of diagonals by calculating the total number of ways of connecting two vertices and then subtracting the number of sides. (You may want to compare your answer with that for Exercise 25 on page 482 where difference equations were used.) $\frac{n(n-3)}{2}$

25. Solve for n: ${}_nC_2 = 45$ 10

26. Solve for n: ${}_nC_2 = {}_{n-1}P_2$ 4

27. ***Investigation*** Give the values of the combinations shown in the triangular array below. What patterns can you discover in this array?

$$\begin{array}{c} {}_1C_0 \quad {}_1C_1 \\ {}_2C_0 \quad {}_2C_1 \quad {}_2C_2 \\ {}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3 \\ {}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4 \\ {}_5C_0 \quad {}_5C_1 \quad {}_5C_2 \quad {}_5C_3 \quad {}_5C_4 \quad {}_5C_5 \end{array}$$

$$\begin{array}{c} 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

28. For $1 \le r \le n$, show that $\frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdot \cdots r} = \frac{n!}{(n-r)!r!}$.

29. If the 52 cards of a standard deck are dealt to four people, 13 cards at a time, the first person can receive ${}_{52}C_{13}$ possible hands. Then the second person can receive 13 of the remaining 39 cards in ${}_{39}C_{13}$ possible ways. The third person can receive 13 of the remaining 26 cards in ${}_{26}C_{13}$ ways, and the fourth person can receive 13 of the remaining 13 cards in ${}_{13}C_{13}$ ways. Thus, the total number of ways of distributing the 52 cards into four 13-card hands is:

$${}_{52}C_{13} \cdot {}_{39}C_{13} \cdot {}_{26}C_{13} \cdot {}_{13}C_{13}$$

Without using a calculator, show that the above product simplifies to $\frac{52!}{(13!)^4}$. Then use a calculator to find this number. 5.36×10^{28}

//// COMPUTER EXERCISE

Write a program that outputs the values of ${}_nP_r$ and ${}_nC_r$ for input values of n and r where $0 \le r \le n$.

15-4 Permutations with Repetition; Circular Permutations

Objective *To solve counting problems that involve permutations with repetition and circular permutations.*

Activity

a. Write down all possible arrangements of the letters of the word MOP.
b. Write down all *distinguishable* arrangements of the letters of the word MOM.
c. What accounts for the fact that part (b) gives fewer permutations than part (a)?

As the preceding activity shows, fewer permutations result when some of the objects being rearranged are the same. For example, consider the two 6-letter words MEXICO and CANADA. Since the letters in the word MEXICO are all different, there are $6! = 720$ possible arrangements of the letters. On the other hand, the word CANADA has fewer distinguishable arrangements of its letters because of the 3 identical A's. To see why, suppose we distinguish the A's by color, as follows:

CANADA

Since the six letters are now all different, there are 6! arrangements of the letters, including:

NADACA	NADACA	NADACA
NADACA	NADACA	NADACA

As soon as the 3 different A's are changed back to the same A, however, the 6 different arrangements above become the *same* arrangement NADACA. Because the 3 different A's have 3! different arrangements, we have:

$$n(\text{arrangements of CANADA}) = \frac{n(\text{arrangements of CANADA})}{3!}$$
$$= \frac{6!}{3!} = 120$$

In generalizing this discussion, let us say that CANADA has 6 letters of several different "types": 3 A's, 1 C, 1 N, and 1 D. The number of permutations is:

$$\frac{6!}{3!\ 1!\ 1!\ 1!}$$

The general form of this result is given at the top of the next page.

Teaching Notes, p. 564C

Warm-Up Exercises

1. List all the possible arrangements of the digits in the number 1234.
1234 1243 1324 1342
1423 1432 2134 2143
2314 2341 2413 2431
3124 3142 3214 3241
3412 3421 4123 4132
4213 4231 4312 4321

2. List all the distinguishable (different) arrangements of the digits in the number 1134.
1134 1314 1341 1143
1413 1431 3114 3141
3411 4113 4131 4311

3. How many possible arrangements of two 1's are possible? of r 1's?
1; 1

4. Compare the number of arrangements in Exercise 1 with the number in Exercise 2. Then guess how to determine the number of arrangements of n objects if r of them are identical.
In Ex. 1, there are $4! = 24$ arrangements; in Ex. 2, there are $\frac{4!}{2!} = 12$. In general, there are $\frac{n!}{r!}$ arrangements of n objects if r of them are identical.

Motivating the Section

In this section, students will extend their understanding of permutations to cover more complex situations.

Mathematical Note

Point out that the formula given at the top of the page assumes that

$n_1 + n_2 + n_3 + \ldots + n_k = n.$

When discussing circular permutations (that is, arrangements of objects in a circle), stress that the important consideration is the position of the objects *relative to each other.* For example, in the circular arrangements shown at the bottom of the page, A can serve as a reference point for B, C, and D: In *all* four arrangements, B is always on A's left, C is always opposite A, and D is always on A's right; thus, the four arrangements are the same.

Example Note

In discussing Example 1, point out that the 1! factors in the denominator do not affect the calculation and thus can be omitted.

The Number of Permutations of Things Not All Different

Let S be a set of n elements of k different types. Let n_1 = the number of elements of type 1, n_2 = the number of elements of type 2, ..., n_k = the number of elements of type k. Then the number of distinguishable permutations of the n elements is:

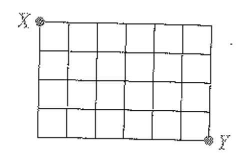
$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

Example 1 How many permutations are there of the letters of MASSACHUSETTS?

Solution Of the 13 letters of MASSACHUSETTS, there are 4 S's, 2 A's, 2 T's, 1 M, 1 C, 1 H, 1 U, and 1 E. Thus, the number of permutations is:

$$\frac{13!}{4!\,2!\,2!\,1!\,1!\,1!\,1!\,1!} = 64{,}864{,}800$$

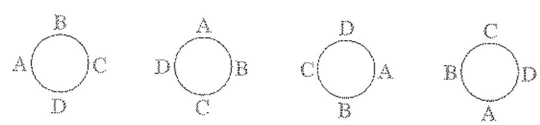
Example 2 The grid shown at the right represents the streets of a city. A person at point X is going to walk to point Y by always traveling south or east. How many routes from X to Y are possible?



Solution **Method 1** To get from X to Y, the person must travel 4 blocks south (S) and 6 blocks east (E). One possible route can be symbolized by the "word" SSESEEESEE. Other routes can be symbolized by other 10-letter words having 4 S's and 6 E's. The number of these "words" is $\frac{10!}{4!\,6!} = 210$.

Method 2 Every route from X to Y covers 10 blocks of which 4 must be south. The number of ways to choose which 4 of the 10 blocks are to be south is ${}_{10}C_4 = \frac{10!}{6!\,4!} = 210.$

Thus far we have considered only *linear* permutations, but permutations may also be *circular* (or *cyclic*). For example, the diagrams below show that seating four people around a circular table is different from seating them in a row.



These circular permutations are the *same*, because in each one, A is to the right of B, who is to the right of C, who is to the right of D.

ABCD DABC CDAB BCDA

These linear permutations are *different.*

Example 3 How many circular permutations are possible when seating four people around a table?

Solution The diagrams at the bottom of the preceding page show that 1 circular permutation corresponds to 4 linear permutations. Thus:

$$\begin{aligned}\text{number of circular permutations} &= \frac{1}{4}\,(\text{number of linear permutations})\\ &= \frac{1}{4}\cdot 4!\\ &= 3! = 6\end{aligned}$$

CLASS EXERCISES

In Exercises 1–4, find the number of permutations of the letters of each word.

1. a. MALE 24 b. MALL 12
2. a. MOUSE 120 b. MOOSE 60
3. a. VERIFY 720 b. VIVIFY 180
4. a. SHAKEUPDOWN 39,916,800 b. SHAKESPEARE 1,663,200
5. Find the number of permutations of the letters of: Answers will vary.
   a. your last name b. your full name
6. *Reading* Of the two solutions given for Example 2, which do you prefer? Why? Answers will vary.
7. Refer to the diagram for Example 2 and suppose that the street corner just north of Y is Z. How many ways are there to walk from X to Z if you always travel south or east? 84
8. All but one of the following linear permutations correspond to the same circular permutation. Which one does not? BCDAE

   CDEAB EABCD BCDAE DEABC

WRITTEN EXERCISES

In Exercises 1–4, find the number of permutations of the letters of the given word.

A

1. MISSOURI 10,080
2. PENNSYLVANIA 39,916,800
3. MISSISSIPPI 34,650
4. CONNECTICUT 1,663,200
5. Each of the 10 finalists in the state spelling bee contest receives a prize. The prizes are six \$25 bonds, three \$50 bonds, and one \$100 bond. In how many ways can these prizes be given? 840
6. The 12 workers in a cafeteria crew rotate among three kinds of jobs. In how many ways can the crew be assigned the jobs of 2 cooks, 7 servers, and 3 dishwashers? 7920

Mathematical Note

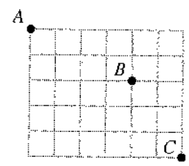
You may wish to help students generalize the formula for circular permutations of n objects. Since each circular permutation may be written as a linear permutation in n ways, there are $\frac{n!}{n} = (n-1)!$ circular permutations (see Exercise 12, page 586).

Additional Examples

1. How many different integers can be formed using all the digits of 253,225?

   There are 6 digits, of which 3 are two's and 2 are fives. The number of integers is $\frac{6!}{3!2!} = \frac{6\cdot 5\cdot 4}{2\cdot 1} = 60$.

2. A person walks from A to B, and then from B to C, always traveling south or east on the streets shown. How many paths are possible?

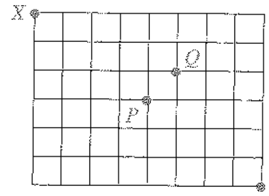


The route from A to B consists of 6 blocks, of which 4 are eastward and 2 are southward. There are $\frac{6!}{4!2!} = 15$ such routes. Similarly, there are $\frac{5!}{2!3!} = 10$ routes from B to C. Then, by the multiplication principle, there are $15 \cdot 10 = 150$ routes from A to C via B.

7. In the English language, *antidisestablishmentarianism* is one of the longest words. If a computer could print out every permutation of this word at the rate of one word per second, how long would it take? 9.71×10^{14} years

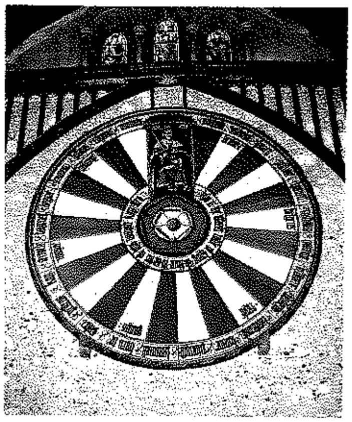
8. Which of the 50 states has the least number of permutations of the letters of its name? OHIO

In Exercises 9 and 10, a person bicycles along the city streets shown at the right by always traveling south and east.



9. Find the number of possible routes from:
   a. X to Y 3003 b. X to Q 21
   c. Q to Y 35 d. X to Y via Q 735

10. To the nearest tenth of a percent, what percent of the routes from X to Y pass through P? 40.8%



11. How many circular permutations are possible when seating 5 people around a circular table? 24

12. The photo shows the mythical Round Table of King Arthur. Note that the table indicates where King Arthur sat as well as where 24 knights sat. In how many ways can n people be seated around a similar table if:
   a. a head of the table is designated?
   b. *no* head of the table is designated?
   a. $n!$ b. $(n-1)!$

B 13. **Meteorology** A typical January in Boston has 3 days of snow, 12 days of rain, and 16 days without precipitation. In how many ways can such weather be distributed throughout the month? [*Hint*: Think of a weather distribution pattern as a 31-letter word made up of S's (snow), R's (rain), and N's (no precipitation).] 1.37×10^{11}

14. In the card game of bridge, 52 cards are dealt to 4 people who are called North, East, South, and West. A 52-letter "word" consisting of 13 N's, 13 E's, 13 S's, and 13 W's will tell you who has what cards. For example, if a word begins NEEWS . . . , then North has the ace of spades, East has the 2 and 3 of spades, West has the 4 of spades, South has the 5 of spades, and so on. How many such arrangements are there? (Compare your answer with the method shown in Exercise 29 of Section 15-3.) 5.36×10^{28}

15. A teacher's grade book alphabetically lists 25 students in a class. This teacher, who does not give pluses and minuses with letter grades, announces that on the last test there were 5 A's, 10 B's, 6 C's, 3 D's, and 1 E.
   a. Explain how a 25-letter "word" could tell who got what grade.
   b. In how many ways can these grades be distributed to the 25 students? 8.25×10^{12}

15. a. Pair the position of ... name from the grade-book list with the letter in the same position in the 25 letter "word."

Suggested Assignments

Discrete Math
Day 1: 585/3, 5, 9, 11, 13, 16, 17
Day 2: 587/1–13
Day 3: 587/14–24

Supplementary Materials

Alternative Assessment, 47–48

16. A group of 4 people orders from a restaurant menu that lists 3 main courses: fish, chicken, and steak. How many combinations of main-course orders are possible in this group? (*Hint*: You can form "words" by using 4 x's to tally the orders and 2 bars to separate the three types of orders. For example, the "word" xx |x| x represents 2 orders of fish, 1 order of chicken, and 1 order of steak. Similarly, the "word" xxx | | x represents 3 orders of fish, 0 orders of chicken, and 1 order of steak.) 15

17. A group of 12 friends goes to a cinema complex that is showing 6 different movies. If the group splits up into subgroups based on movie preferences, how many subgroup combinations are possible? (See the hint for Exercise 16.) 6188

C 18. The word ABSTEMIOUS contains the 5 vowels A, E, I, O, and U in alphabetical order. How many permutations of this word have the vowels in alphabetical order? 15,120

MIXED COMBINATORICS EXERCISES

The following exercises are counting problems like those from the last four sections. They test your ability to choose the appropriate counting technique.

A 1. Three identical door prizes are to be given to three lucky people in a crowd of 100. In how many ways can this be done? 161,700

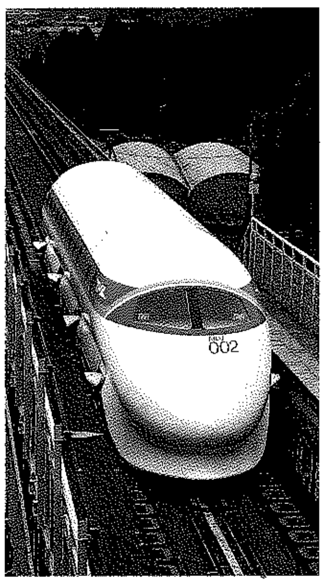
2. The license plates in a certain state consist of 3 letters followed by 3 nonzero digits. How many such license plates are possible? 12,812,904

3. How many 4-digit numbers **(a)** contain no 0's? **(b)** contain no 1's? **(c)** begin with an even digit and end with an odd digit? 6561; 5832; 2000

4. A student must take four final exams, scheduled by computer, during the morning and afternoon testing periods on Monday through Friday of exam week. If the order of the student's four exams is important, in how many ways can a computer schedule the exams? 5040

5. **a.** A railway has 30 stations. On each ticket, the departure station and the destination station are printed. How many different tickets are possible? 870
   b. If a ticket can be used in either direction between two stations, how many different tickets are needed? 435

6. In how many ways can the letters of each of the following words be arranged?
   a. RADISH 720
   b. SQUASH 360
   c. TOMATO 180



Cooperative Learning

The Mixed Combinatorics Exercises are good ones to use in small-group work. Consider devoting a class period to "jigsawing" the B- and C-level exercises. A "jigsaw" activity works as follows: Divide the class into groups of five students, and have the students in each group number themselves from 1 to 5 (perhaps by order of birthdays). All number 1's are given the same problem to solve, number 2's a different problem, and so on. After some time spent individually on their assigned problems (perhaps overnight), students gather into groups of 1's, 2's, and so on, and discuss the solution to their common problem. Students then return to their original groups, where they teach the group's members how to solve the problem.

7. There are 3 roads from town A to town B, 5 roads from town B to town C, and 4 roads from town C to town D. How many ways are there to go from A to D via B and C? How many different round trips are possible? 60; 3600

8. A teacher must pick 3 high school students from a class of 30 to prepare and serve food at the junior high school picnic. How many choices are possible? 4060

9. All students at John Jay High School must take at least one of the school's three science courses: biology, chemistry, and physics. In order to project the future enrollment in these courses, the principal sent questionnaires to 297 junior high school students and got these results:

132 intend to take biology and chemistry.
107 intend to take chemistry and physics.
88 intend to take biology and physics.
43 intend to take only biology.
55 intend to take only chemistry.
38 intend to take only physics.

How many students intend to take all three science courses? 83

10. A radio show plays the top 10 musical hits of the previous week. During the first full week of last January, these 10 hits were chosen from 70 possibilities, and they were played in order of increasing popularity. How many possible orders were there? *Note*: Since calculators cannot evaluate large factorials, use the approximation given by the formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. 1.44×10^{18}

B 11. The locks on the gymnasium lockers have dials with numbers from 0 to 39. Each locker combination consists of 3 numbers.
   a. Pythagoras remembers that his numbers are 8, 15, and 17, but he can't remember in which order they appear. If he can try one possibility every 10 s, what is the maximum amount of time that it would take him to find the right combination? 1 min
   b. Hypatia cannot remember any of her combination numbers, but she does remember that exactly two of them are the same. If she can try one possibility every 10 s, what is the maximum amount of time that it would take her to find the right combination? 13 h

12. In how many ways can 8 jackets of different styles be hung:
   a. on a straight bar? 40,320
   b. on a circular rack? 5040

13. A town council consists of 8 members including the mayor.
   a. How many different committees of 4 can be chosen from this council? 70
   b. How many of these committees include the mayor? 35
   c. How many do not include the mayor? 35
   d. Verify that the answer to part (a) is the sum of the answers to part (b) and part (c). 35 + 35 = 70

14. Repeat Exercise 13 if the council has 9 members including the mayor. See below.

15. If you have a \$1 bill, a \$5 bill, a \$10 bill, and a \$20 bill, how many different sums of money can you make using one or more of these bills? 15

14. a. 126 b. 56 c. 70 d. 56 + 70 = 126

16. **Communications** In Morse code, letters, digits, and various punctuation marks are represented by a sequence of dots and dashes. Sequences can be from 1 unit to 6 units in length. How many such sequences are possible? 126

17. The Pizza Place offers pepperoni, mushrooms, sausages, onions, anchovies, and peppers as toppings for their regular plain pizza. How many different pizzas can be made? 64

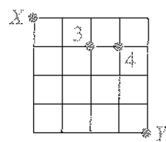
18. **a.** How many 4-letter "words" can be formed by using the 8 letters of the word TRIANGLE? 1680
    b. How many of the "words" formed in part (a) have no vowels? 120
    c. How many of the "words" formed in part (a) have at least one vowel? 1560

19. Five boys and five girls stand in a line. How many arrangements are possible **(a)** if all of the boys stand in succession? **(b)** if the boys and girls stand alternately? 86,400; 28,800

20. Answer Exercise 19 if the 5 boys and 5 girls stand in a circle instead of a line. 14,400; 2880

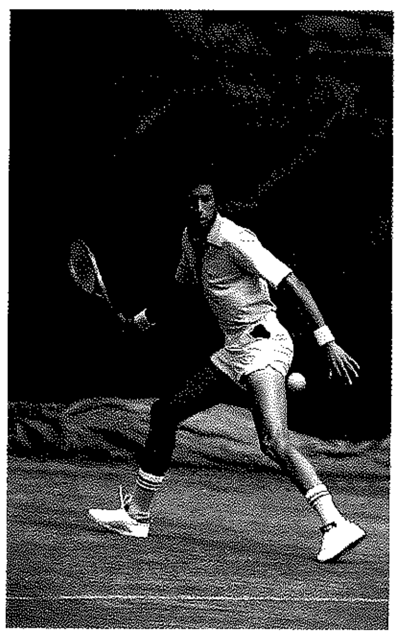
21. How many 5-digit numbers contain at least one 3? 37,512

22. In the diagram at the right, all paths go from X toward Y and consist only of south and east steps. For example, there are 3 paths that go from X to the point marked with a 3. Similarly, there are 4 paths that go from X to the point marked with a 4.



    a. Explain how the numbers 3 and 4 are obtained.
    b. Copy the grid and label each point with the number of paths that go from X to that point.

C 23. **Sports** In the World Series, two teams, A and B, play each other until one team has won 4 games. For example, the "word" ABBAAA represents a 6-game series in which team A wins games 1, 4, 5, and 6.
    a. Explain why the number of different 6-game series won by team A is $\frac{5!}{3!\,2!}$.
    b. Without actually listing the various series between team A and team B, show that there are 70 different sequences of games possible.

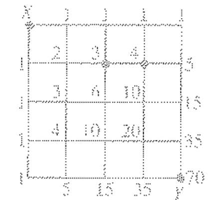


24. **Sports** In the tennis championship between players A and B at Wimbledon, the first player to win 3 sets is champion. Find the number of different ways for player A to win the championship. 10

Additional Answers Assorted Exercises

22. a. Of the 3 blocks that must be traveled, 1 must be south; ${}_3C_1 = 3$. Of the 4 blocks that must be traveled, 1 must be south; ${}_4C_1 = 4$.

b.



23. a. The team winning the series must win the last game of the series. In a six-game series, of the remaining 5 games, A wins 3. Which 3 of the 5 games is unimportant. Therefore, there are ${}_5C_3 = \frac{5!}{2!3!}$ different series.

b. A wins 4 games: ${}_3C_3 =$ 1 way;
A wins 5 games: ${}_4C_3 =$ 4 ways;
A wins 6 games: ${}_5C_3 =$ 10 ways;
A wins 7 games: ${}_6C_3 =$ 20 ways.
A parallel analysis can be made for B's winning the series. Thus, there are $2(1 + 4 + 10 + 20) =$ 70 different sequences of games possible.

Teaching Notes, p. 564D

Warm-Up Exercises

1. Expand $(a + b)^1$, $(a + b)^2$, $(a + b)^3$, and $(a + b)^4$.
$a + b$; $a^2 + 2ab + b^2$; $a^3 + 3a^2b + 3ab^2 + b^3$; $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

2. Replace each combination in the array below with its value. Compare the results with your answers to Exercise 1. What do you notice?

$${}_1C_0 \quad {}_1C_1$$
$${}_2C_0 \quad {}_2C_1 \quad {}_2C_2$$
$${}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3$$
$${}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4$$

1 1
1 2 1
1 3 3 1
1 4 6 4 1

The values are the coefficients of the terms in the expansions of Exercise 1.

3. Use the pattern in Exercise 2 to give the next row of the array and then evaluate each entry.
${}_5C_0 = 1$; ${}_5C_1 = 5$; ${}_5C_2 = 10$; ${}_5C_3 = 10$; ${}_5C_4 = 5$; ${}_5C_5 = 1$

4. Use your answers from Exercise 3 to write the expansion of $(a + b)^5$.
$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

15-5 The Binomial Theorem; Pascal's Triangle

Objective *To use the binomial theorem and Pascal's triangle.*

Our goal in this section is to derive a formula for expanding $(a + b)^n$ for positive integers n. For small values of n we get the following expansions:

$$(a + b)^1 = 1a + 1b$$
$$(a + b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$
$$(a + b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

From these examples we can see that the first term in the expansion of $(a + b)^n$ is always $1a^nb^0$. In successive terms the exponents of a decrease by 1 and the exponents of b increase by 1, so that the sum of the two exponents in a term is always n. The coefficients of the terms also have a pattern, which can be seen by studying the array of numbers below. The first five rows of this array are like the array above except that the a's, b's, and plus signs have been omitted.

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

This array is called **Pascal's triangle**, named for the French mathematician Blaise Pascal (1623–1662). Because the numbers in the array are the coefficients of the terms in the expansion of $(a + b)^n$, they are called **binomial coefficients**. Notice that (except for the 1's) each number is the sum of the two numbers just above it. Hence, from the fifth row of the triangle, we can quickly form the sixth row, as shown above. You should now be able to write the expansion of $(a + b)^6$.

Blaise Pascal

So far, it might seem that to get the numbers in the sixth row of Pascal's triangle you must first know the numbers in the fifth row, but this is not necessary. As the computations at the top of the next page show, each number in the sixth row can be calculated directly by formula.

$$\text{6th row}\begin{cases} {}_6C_0 & {}_6C_1 & {}_6C_2 & {}_6C_3 & {}_6C_4 & {}_6C_5 & {}_6C_6 \\ 1 & \frac{6}{1} & \frac{6 \cdot 5}{1 \cdot 2} & \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} & \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} & \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} & \frac{6!}{6!} \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{cases}$$

You may be wondering why combinations have anything to do with the sixth row of Pascal's triangle and the expansion of $(a + b)^6$. To see why, consider $(a + b)^6$ in factored form:

$$(a + b)^6 = (a + b)(a + b)(a + b)(a + b)(a + b)(a + b)$$

The term a^6 in the expansion of $(a + b)^6$ is obtained by multiplying the a's in the 6 factors. The term containing a^5b is obtained by multiplying an a from 5 of the factors and a b from 1 of them. Since there are ${}_6C_1 = 6$ ways to choose the factor that contributes the b, there are 6 of these a^5b terms. Likewise, the term containing a^4b^2 is obtained by multiplying an a from 4 of the factors and a b from 2 of the factors. Since there are ${}_6C_2 = 15$ ways to choose the 2 factors that contribute the b's, there are 15 of these a^4b^2 terms. A similar argument can be used to prove the following theorem.

The Binomial Theorem

If n is a positive integer, then:

$$(a + b)^n = {}_nC_0a^nb^0 + {}_nC_1a^{n-1}b^1 + {}_nC_2a^{n-2}b^2 + {}_nC_3a^{n-3}b^3 + \cdots + {}_nC_na^0b^n$$

Equivalently:

$$(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots + 1a^0b^n$$

Example Give the first four terms in the expansion of $(x - 2y)^{10}$ in simplified form.

Solution First, we use the binomial theorem to find the first four terms in the expansion of $(a + b)^{10}$:

$$a^{10} + 10a^9b + \frac{10 \cdot 9}{1 \cdot 2}a^8b^2 + \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3}a^7b^3, \text{ or}$$
$$a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3$$

Then, substituting x for a and $-2y$ for b in the expression above, we obtain the first four terms in the expansion of $(x - 2y)^{10}$:

$$x^{10} + 10x^9(-2y) + 45x^8(-2y)^2 + 120x^7(-2y)^3, \text{ or}$$
$$x^{10} - 20x^9y + 180x^8y^2 - 960x^7y^3$$

Motivating the Section

Ask students to multiply to find the coefficient of the x^2y^3-term in the expansion of $(x + y)^5$. (10) After students have completed their work, explain that the skills they learn in this section will provide them with a shortcut that makes use of what they have learned about combinations.

Additional Examples

1. Find the value of $(0.98)^7$ to the nearest hundredth by considering the expansion of $(1 - 0.02)^7$.
   $(a + b)^7 = a^7 + 7a^6b + \frac{7 \cdot 6}{1 \cdot 2}a^5b^2 + \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}a^4b^3 + \cdots = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + \cdots$
   Let $a = 1$ and $b = -0.02$.
   Then: $(1 - 0.02)^7 = 1^7 + 7(1)^6(-0.02) + 21(1)^5(-0.02)^2 + 35(1)^4(-0.02)^3 + \cdots = 1 - 0.14 + 0.0084 - 0.00028 + \cdots \approx 0.87$
2. In the expansion of $(x^2 + 3)^9$, find the term containing x^{10}.
   Since $(x^2)^5 = x^{10}$ and the first term is $(x^2)^9$, the required term is the fifth.
   ${}_9C_4(x^2)^{9-4}(3)^4 = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}(x^2)^5(81) = 10{,}206x^{10}$.

Error Analysis

Students are frequently confused by references to the "fourth" term of $(a + b)^n$ as ${}_nC_3a^{n-3}b^3$, since there are no 4's in the expression. Help them by pointing out that there are $n + 1$ terms in all, that the exponents of b will run from 0 to n (which accounts for the $n + 1$ terms), and that the exponent of b in the kth term is $k - 1$. Some students may want to have an expression for the kth term; namely:

$${}_nC_{k-1}a^{n-(k-1)}b^{k-1}$$

Additional Answers Class Exercises

1. The numbers in Pascal's triangle have symmetry in a vertical line drawn through the center of the triangle.

Using Technology

Students may wish to write a program to print the first n rows of Pascal's triangle. One possible program in BASIC would be:

```
10 INPUT "NUMBER OF
   ROWS: "; N
20 FOR R =1 TO N
30 LET C = 1
40 FOR T = 1 TO R + 1
50 PRINT C; " ";
60 LET C = C *
   (R - T + 1)/T
70 NEXT T
80 PRINT
90 NEXT R
100 END
```

A program that uses arrays can also be written. Such programs will help students complete Exercises 27 and 28.

CLASS EXERCISES

1. *Visual Thinking* Study Pascal's triangle (page 590). Describe the symmetry that the numbers in the triangle have.
2. Give the seventh row of Pascal's triangle. 1 7 21 35 35 21 7 1
3. **a.** Find the first 4 numbers in the eighth row of Pascal's triangle. 1 8 28 56
   b. State the first 4 terms of the expansion of $(x + y)^8$. $x^8, 8x^7y, 28x^6y^2, 56x^5y^3$
   c. State the first 4 terms of the expansion of $(x - y)^8$. $x^8, -8x^7y, 28x^6y^2, -56x^5y^3$

In Exercises 4–6, use Pascal's triangle to give the expansion of each binomial.

4. $(a - b)^3$ $a^3 - 3a^2b + 3ab^2 - b^3$ **5.** $(a + b)^4$ See below. **6.** $(a - b)^4$ See below.
7. Find **(a)** the third term in the expansion of $(x + y)^6$, and **(b)** the fourth term in the expansion of $(x + y)^9$. $15x^4y^2$; $84x^6y^3$

5. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ 6. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

WRITTEN EXERCISES

In Exercises 1–12, give the expansion of each binomial. Simplify your answers.

A

1. a. $(a + b)^3$	**b.** $(20 + 1)^3$	**c.** $(20 - 1)^3$
2. a. $(x + y)^4$	**b.** $(10 + 1)^4$	**c.** $(10 - 1)^4$
3. a. $(a + b)^5$	**b.** $(a - b)^5$	**c.** $(2a + 1)^5$
4. a. $(p + q)^6$	**b.** $(p - q)^6$	**c.** $(3p - 2)^6$
5. a. $(x + y)^7$	**b.** $(x - y)^7$	**c.** $(x^2 - 2y)^7$
6. a. $(a + b)^8$	**b.** $(a - b)^8$	**c.** $(2a - b^2)^8$
7. $(x^2 - y^2)^3$	**8.** $(2x^2 - 1)^4$	**9.** $(x^2 + 1)^5$
10. $\left(1 + \frac{x}{2}\right)^3$	**11.** $\left(x + \frac{1}{x}\right)^6$	**12.** $\left(2x - \frac{1}{x}\right)^8$

In Exercises 13–16, find the first four terms of the expansion of the given expression. Do *not* simplify your answers.

13. $(a^2 - b)^{100}$ **14.** $(3p + 2q)^{20}$
15. $(\sin x + \sin y)^{10}$ **16.** $(\sin x - \cos y)^{30}$
17. Find the value of $(1.01)^5$ to the nearest hundredth by considering the expansion of $(1 + 0.01)^5$. 1.05
18. Find the value of $(0.99)^5$ to the nearest hundredth by considering the expansion of $(1 - 0.01)^5$. 0.95

B

19. In the expansion of $(a + b)^{12}$, what is the coefficient **(a)** of a^8b^4? **(b)** of a^4b^8? 495; 495
20. In the expansion of $(a + b)^{20}$, what is the coefficient **(a)** of $a^{17}b^3$? **(b)** of a^3b^{17}? 1140; 1140

21. In the expansion of $\left(x^2 + \frac{2}{x}\right)^{12}$, find and simplify the constant term. 126,720

22. In the expansion of $(a^3 - 2)^{10}$, find the term containing a^{18}. $3360a^{18}$

23. A municipal council consists of a mayor and n councilors. The council needs to choose a committee of k members.
   a. How many committees are possible if any council member can be chosen? ${}_{n+1}C_k$
   b. How many committees are possible if the mayor must be included? ${}_nC_{k-1}$
   c. How many committees are possible if the mayor must not be included? ${}_nC_k$
   d. Use parts (a), (b), and (c) to complete the following equation:

$${}_{n+1}C_k = {}_?C_{k-1} + {}_?C_k \quad {}_{n+1}C_k = {}_nC_{k-1} + {}_nC_k$$

   e. Use algebra to prove that the equation in part (d) is correct.

24. Rewrite the binomial theorem using sigma (Σ) notation. $\sum_{k=0}^{n} {}_nC_k a^{n-k} b^k$

For Exercises 25 and 26, use the alternate form of Pascal's triangle shown at the right. Note the single 1 at the top of the triangle. This is sometimes called the "zeroth row" because it represents the expansion of $(a + b)^0$.

1				
1	**1**			
1	**2**	**1**		
1	**3**	**3**	**1**	
1	**4**	**6**	**4**	**1**

25. In what column of Pascal's triangle do the triangular numbers (see page 491) occur? Explain why they occur. 3

26. a. What special sequence of numbers results from summing the numbers in each row of Pascal's triangle? The nonnegative integral powers of 2
   b. Consider the diagonals that originate at a 1 on the left side of Pascal's triangle and have a slope of 1. What special sequence of numbers results from summing the numbers along each diagonal? Fibonacci sequence

27. ***Investigation*** Copy Pascal's triangle as given on page 590. Use a highlighting marker to color all the even numbers in the triangle. What patterns do you notice? (*Note*: You may need to add quite a few rows to the triangle before any patterns emerge.)

28. ***Investigation*** Repeat Exercise 27, but this time color all the numbers that are multiples of 3. What patterns do you notice?

29. a. Show that the expansion of $(\cos\theta + i\sin\theta)^3$ simplifies to:
$$(\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta)$$
   b. Use De Moivre's theorem (see page 408) to find $(\cos\theta + i\sin\theta)^3$.
   c. Use parts (a) and (b) to show that:
$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\text{and } \sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$$

30. Use De Moivre's theorem and the expansion of $(\cos\theta + i\sin\theta)^4$ to show that:
$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$\text{and } \sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

Suggested Assignments

Discrete Math
Day 1: 592/1–13 odd
Day 2: 592/17–23

Supplementary Materials

Alternative Assessment, 48
Student Resource Guide, 143–145

Additional Answers Written Exercises

27. Pascal's triangle for which even numbers are highlighted shows a pattern of inverted equilateral triangles of various sizes. Specifically, in each row n, where $n = 2^k$ and $k = 1, 2, 3, \ldots$, an inverted triangle having $n - 1$ numbers on a side begins. Moreover, to the left and to the right of each inverted triangle is the same pattern of triangles that appears immediately above the inverted triangle.

28. Pascal's triangle for which numbers divisible by three are highlighted shows a pattern of inverted equilateral triangles of various sizes. Specifically, at each row $n = 3^k$ for $k = 1, 2, 3, \ldots$, there begins a trio of equilateral triangles having $n - 1$ numbers on a side. (The triangles in each trio are themselves arranged in a triangular pattern.) Moreover, in the regions between the triangles of each trio is the same pattern of triangles that appears immediately above the uppermost triangle in the trio.

Tests, 48–49
Alternative Assessment, 70
Student Resource Guide, 146–147

In the binomial theorem, if we substitute 1 for a and x for b, we get:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$

If n is not a positive integer, the right side is an infinite series with sum $(1 + x)^n$ when $|x| < 1$. (A proof requires calculus.) Use this fact to do Exercises 31–34.

C **31.** Show that $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$. (As a check on your work, notice that the right side of the equation is an infinite geometric series with first term 1 and ratio $-x$. Thus, the series has sum $\frac{1}{1-(-x)} = (1 + x)^{-1}$ when $|x| < 1$.)

32. Show that $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$.

33. Show that $(1 + x)^{1/2} \approx 1 + \frac{1}{2}x$ when $|x|$ is small. Use the result to approximate $\sqrt{1.04}$ and $\sqrt{0.98}$. 1.02; 0.99

34. Show that $(1 + x)^{1/3} \approx 1 + \frac{1}{3}x$ when $|x|$ is small. Use the result to approximate $\sqrt[3]{1.12}$ and $\sqrt[3]{67}$. $\left(\textit{Hint: } \sqrt[3]{67} = \sqrt[3]{64\left(1 + \frac{3}{64}\right)}\right)$ 1.04; 4.06

Chapter Summary

1. If sets A and B are subsets of a universal set U, then:
   (1) the *intersection* of A and B, denoted $A \cap B$, consists of those elements in *both* A and B,
   (2) the *union* of A and B, denoted $A \cup B$, consists of those elements in *either* A or B,
   (3) the *complement* of A, denoted $\overline{A}$, consists of those elements of U that are *not* in A.

   Venn diagrams are used to illustrate a universal set and its subsets, along with their intersections, unions, and complements.
2. Four basic counting principles are presented.
   a. *The inclusion-exclusion principle*: For any sets A and B,
   $$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$
   b. *The multiplication principle*: If an action can be performed in n_1 ways, and for each of these ways another action can be performed in n_2 ways, then the two actions can be performed together in n_1n_2 ways.
   c. *The addition principle*: If two actions are mutually exclusive, and the first can be done in n_1 ways and the second in n_2 ways, then one action *or* the other can be done in $n_1 + n_2$ ways.
   d. *The complement principle*: If A is a subset of a universal set U, then
   $$n(A) = n(U) - n(\overline{A}).$$

3. The choice of r objects from a set of n objects is called a *permutation* if the order of choosing is important and a *combination* if the order is unimportant. The number of possible choices in each case is given by:

$$_nP_r = \frac{n!}{(n-r)!} \quad \text{and} \quad _nC_r = \frac{n!}{(n-r)!r!}$$

4. Let S be a set of n elements of k different types. Let n_1 = the number of elements of type 1, n_2 = the number of elements of type 2, . . . , n_k = the number of elements of type k. Then the number of distinguishable permutations of the n elements is $\dfrac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$.

5. For a positive integer n, the *binomial theorem* gives the expansion of $(a + b)^n$:

$$(a+b)^n = {}_nC_0a^nb^0 + {}_nC_1a^{n-1}b^1 + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_na^0b^n$$

Pascal's triangle can be used to find the *binomial coefficients*, $_nC_k$.

Key vocabulary and ideas

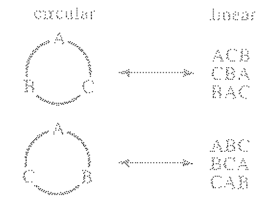
Venn diagram (p. 565)
intersection (p. 565)
union (p. 565)
complement (p. 566)
inclusion-exclusion principle (p. 566)
multiplication principle (p. 571)
addition principle (p. 573)
complement principle (p. 574)
permutation, combination (p. 578)
permutations with repetition (p. 584)
linear and circular permutations (p. 584)
Pascal's triangle (p. 590)
binomial coefficient (p. 590)
binomial theorem (p. 591)

Chapter Test

1. Of the 320 children at a movie, 190 buy popcorn and 245 buy something to drink. If 151 children buy both, how many buy neither? 36 15-1
2. A basketball team has 4 guards, 5 forwards, and 3 centers on its roster. In how many ways can a team consisting of 1 left guard, 1 right guard, 1 left forward, 1 right forward, and 1 center be formed? 720 15-2
3. A bank customer selects a password consisting of 4 different letters or 4 different digits. How many different passwords are possible? 363,840
4. A bookshelf has space for 5 books. If 7 different books are available, how many different arrangements can be made on the shelf? 2520 15-3
5. Six people meet at a party, and each pair of people shakes hands. How many handshakes are there? 15
6. A boat has 3 red, 3 blue, and 2 yellow flags with which to signal other boats. All 8 flags are flown in various sequences to denote different messages. How many such sequences are possible? 560 15-4
7. ***Writing*** Write a paragraph contrasting linear and circular permutations. Illustrate the difference between them with a specific example.
8. In the expansion of $(x - 2)^{15}$, find the coefficients of x^{13} and x^{12}. 420; −3640 15-5

Additional Answers Chapter Test

7. For each circular arrangement of n objects, there corresponds n linear permutations. For example, if 3 football players are in a huddle, there are only $2! = 2$ ways in which they can be arranged. If the same players are seated on a bench, however, there are 3 linear arrangements for each circular arrangement (so that there are $3 \cdot 2! = 3! = 6$ linear permutations in all).



In general, there are $n!$ linear arrangements of n objects and $(n - 1)!$ circular arrangements.