Chapter Thirteen

SEQUENCES AND SERIES

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13.1 SEQUENCES

Elections for president of the United States have been held every four years since 1792, that is, in 1792, 1796, 1800, 1804, and so on up to the year 2004. Any ordered list of numbers, such as

$$1792, 1796, 1800, 1804, \ldots, 2004$$

is called a *sequence*, and the individual numbers are the *terms* of the sequence. A sequence can be a finite list, such as the sequence of past presidential election years, or it can be an infinite list, such as the sequence of positive integers

$$1, 2, 3, 4, \ldots.$$

Example 1

(a) $0, 1, 4, 9, 16, 25, \ldots$ is the sequence of squares of integers.
(b) $2, 4, 8, 16, 32, \ldots$ is the sequence of positive integer powers of 2.
(c) $3, 1, 4, 1, 5, 9, \ldots$ is the sequence of digits in the decimal expansion of π.
(d) $3.9, 5.3, 7.2, 9.6, 12.9, 17.1, 23.1, 38.6, 50.2$ is the sequence of U.S. population figures, in millions, for the first 10 census reports (1790 to 1880).
(e) $3.5, 4.2, 5.1, 5.9, 6.7, 8.1, 9.4, 10.6, 10.1, 7.1, 3.8, 2.1, 1.4, 1.1$ is the sequence of pager subscribers in Japan, in millions, from the years 1989 to 2002.

Notation for Sequences

We denote the terms of a sequence by

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

so that a_1 is the first term, a_2 is the second term, and so on. We use a_n to denote the n^{th} or *general term* of the sequence. If there is a pattern in the sequence, we may be able to find a formula for a_n.

Example 2 Find the first three terms and the 98^{th} term of the sequence.

(a) $a_n = 1 + \sqrt{n}$ (b) $b_n = (-1)^n \dfrac{n}{n+1}$

Solution

(a) $a_1 = 1 + \sqrt{1}$, $a_2 = 1 + \sqrt{2} \approx 2.414$, $a_3 = 1 + \sqrt{3} \approx 2.732$, and $a_{98} = 1 + \sqrt{98} \approx 10.899$.
(b) $b_1 = (-1)^1 \dfrac{1}{1+1} = -\dfrac{1}{2}$, $b_2 = (-1)^2 \dfrac{2}{2+1} = \dfrac{2}{3}$, $b_3 = (-1)^3 \dfrac{3}{3+1} = -\dfrac{3}{4}$, and $b_{98} = (-1)^{98} \dfrac{98}{98+1} = \dfrac{98}{99}$. This sequence is called *alternating* because the terms alternate in sign.

A sequence can be thought of as a function whose domain is a set of integers. Each term of the sequence is an output value for the function, so $a_n = f(n)$.

Example 3 List the first 5 terms of the sequence $a_n = f(n)$, where $f(x) = 500 - 10x$.

Solution Evaluate $f(x)$ for $x = 1, 2, 3, 4, 5$:

$$a_1 = f(1) = 500 - 10 \cdot 1 = 490 \quad \text{and} \quad a_2 = f(2) = 500 - 10 \cdot 2 = 480.$$

Similarly, $a_3 = 470$, $a_4 = 460$, and $a_5 = 450$.

Arithmetic Sequences

You buy a used car that has already been driven 15,000 miles and drive it 8000 miles per year. The odometer registers 23,000 miles 1 year after your purchase, 31,000 miles after 2 years, and so on. The yearly odometer readings form a sequence a_n whose terms are

$$15{,}000, \ \ 23{,}000, \ \ 31{,}000, \ \ 39{,}000, \ \ \ldots.$$

Each term of the sequence is obtained from the previous term by adding 8000; that is, the difference between successive terms is 8000. A sequence in which the difference between pairs of successive terms is a fixed quantity is called an *arithmetic sequence*.

Example 4 Which of the following sequences are arithmetic?

(a) $9, 5, 1, -3, -7$ (b) $3, 6, 12, 24, 48$
(c) $2, 2+p, 2+2p, 2+3p$ (d) $10, 5, 0, 5, 10$

Solution

(a) Each term is obtained from the previous term by subtracting 4. This sequence is arithmetic.
(b) This sequence is not arithmetic: each terms is twice the previous term. The differences are 3, 6, 12, 24.
(c) This sequence is arithmetic: p is added to each term to obtain the next term.
(d) This is not arithmetic. The difference between the second and first terms is -5, but the difference between the fifth and fourth terms is 5.

We can write a formula for the general term of an arithmetic sequence. Look at the sequence $2, 6, 10, 14, 18, \ldots$ in which the terms increase by 4, and observe that

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 6 = 2 + 1 \cdot 4 \\ a_3 &= 10 = 2 + 2 \cdot 4 \\ a_4 &= 14 = 2 + 3 \cdot 4. \end{aligned}$$

When we get to the n^{th} term, we have added $(n-1)$ copies of 4, so that $a_n = 2 + (n-1)4$. In general:

> For $n \geq 1$, the n^{th} term of an arithmetic sequence is
>
> $$a_n = a_1 + (n-1)d,$$
>
> where a_1 is the first term, and d is the difference between consecutive terms.

Example 5 (a) Write a formula for the general term of the odometer sequence of the car that is driven 8000 miles per year and had gone 15,000 miles when it was bought.
(b) What is the car's mileage seven years after its purchase?

Solution

(a) The odometer reads 15,000 miles initially, so $a_1 = 15{,}000$. Each year the odometer reading increases by 8000, so $d = 8000$. Thus, $a_n = 15{,}000 + (n-1)8000$.
(b) Seven years from the date of purchase is the start of the 8^{th} year, so $n = 8$. The mileage is

$$a_8 = 15{,}000 + (8-1) \cdot 8000 = 71{,}000.$$

Arithmetic Sequences and Linear Functions

You may have noticed that the arithmetic sequence for the car's odometer reading looks like a linear function. The formula for the n^{th} term, $a_n = 15{,}000 + (n-1)8000$, can be simplified to $a_n = 7000 + 8000n$, a linear function with slope $m = 8000$ and initial value $b = 7000$. However, for a sequence we consider only positive integer inputs, whereas a linear function is defined for all values of n. We can think of an arithmetic sequence as a linear function whose domain has been restricted to the positive integers.

Geometric Sequences

You are offered a job at a salary of \$40,000 for the first year with a 5% pay raise every year. Under this plan, your annual salaries form a sequence with terms

$$
\begin{aligned}
a_1 &= 40{,}000 \\
a_2 &= 40{,}000(1.05) = 42{,}000 \\
a_3 &= 42{,}000(1.05) = 40{,}000(1.05)^2 = 44{,}100 \\
a_4 &= 44{,}100(1.05) = 40{,}000(1.05)^3 = 46{,}305,
\end{aligned}
$$

and so on, where each term is obtained from the previous one by multiplying by 1.05. A sequence in which each term is a constant multiple of the preceding term is called a *geometric sequence*. In a geometric sequence, the ratio of successive terms is constant.

Example 6 Which of the following sequences are geometric?

(a) $5, 25, 125, 625, \ldots$ (b) $-8, 4, -2, 1, -\frac{1}{2}, \ldots$ (c) $12, 6, 4, 3, \ldots$

Solution

(a) This sequence is geometric. Each term is 5 times the previous term. Note that the ratio of any term to its predecessor is 5.
(b) This sequence is geometric. The ratio of any term to the previous term is $-\frac{1}{2}$.
(c) This sequence is not geometric. The ratios of successive terms are not constant: $\dfrac{a_2}{a_1} = \dfrac{6}{12} = \dfrac{1}{2}$, but $\dfrac{a_3}{a_2} = \dfrac{4}{6} = \dfrac{2}{3}$.

As for arithmetic sequences, there is a formula for the general term of a geometric sequence. Consider the sequence $8, 2, \frac{1}{2}, \frac{1}{8}, \ldots$ in which each term is $\frac{1}{4}$ times the previous term. We have

$$
\begin{aligned}
a_1 &= 8 \\
a_2 &= 2 = 8\left(\frac{1}{4}\right) \\
a_3 &= \frac{1}{2} = 8\left(\frac{1}{4}\right)^2 \\
a_4 &= \frac{1}{8} = 8\left(\frac{1}{4}\right)^3.
\end{aligned}
$$

When we get to the n^{th} term, we have multiplied 8 by $(n-1)$ factors of $\frac{1}{4}$, so that $a_n = 8\left(\frac{1}{4}\right)^{n-1}$. In general, For $n \geq 1$, the n^{th} term of a geometric sequence is

$$a_n = a_1 r^{n-1},$$

where a_1 is the first term, and r is the ratio of consecutive terms.

Example 7

(a) Write a formula for the general term of the salary sequence that starts at \$40,000 and increases by 5% each year.
(b) What is your salary after 10 years on the job?

Solution

(a) Your starting salary is \$40,000, so $a_1 = 40{,}000$. Each year your salary increases by 5%, so $r = 1.05$. Thus, $a_n = 40{,}000(1.05)^{n-1}$.
(b) After 10 years on the job, you are at the start of your 11^{th} year, so $n = 11$. Your salary is

$$a_{11} = 40{,}000(1.05)^{11-1} \approx 65{,}156 \text{ dollars.}$$

Geometric Sequences and Exponential Functions

The formula for the salary sequence, $a_n = 40{,}000(1.05)^{n-1}$, looks like a formula for an exponential function, $f(t) = a \cdot b^t$. A geometric sequence is an exponential function whose domain is restricted to the positive integers. For many applications, this restricted domain is more realistic than an interval of real numbers. For example, salaries are usually increased once a year, rather than continuously.

Exercises and Problems for Section 13.1

Exercises

Are the sequences in Exercises 1–4 arithmetic?

1. $2, 7, 11, 14, \ldots$

2. $2, -5, -12, -19, \ldots$

3. $2, 7, 12, 17, \ldots$

4. $2, -5, -11, -16, \ldots$

Are the sequences in Exercises 5–7 arithmetic? For those that are, give a formula for the n^{th} term.

5. $6, 9, 12, 15, \ldots$

6. $1, -1, 2, -2, \ldots$

7. $-1, -1.1, -1.2, -1.3, \ldots$

Are the sequences in Exercises 8–11 geometric?

8. $2, 6, 18, 54, \ldots$

9. $2, 4, 8, 12, \ldots$

10. $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$

11. $2, 0.2, 0.02, 0.002, \ldots$

Are the sequences in Exercises 12–17 geometric? For those that are, give a formula for the n^{th} term.

12. $4, 12, 36, 108, \ldots$

13. $4, 1, \frac{1}{4}, \frac{1}{8}, \ldots$

14. $2, -4, 8, -16, \ldots$

15. $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$

16. $4, 0.4, 0.04, 0.004, \ldots$

17. $1, \dfrac{1}{1.2}, \dfrac{1}{(1.2)^2}, \dfrac{1}{(1.2)^3}, \ldots$

Problems

In Problems 18–23, write out the first four terms of each sequence and state if it is geometric.

18. $a_n = 2^n$

19. $a_n = \frac{2n+1}{n+2}$

20. $a_n = \left(-\frac{1}{3}\right)^n$

21. $a_n = \cos(n\pi)$

22. $a_n = n^2 - n$

23. $a_n = \frac{1}{\sqrt{n}}$

24. An arithmetic sequence has first term of 10 and difference of 5; what is the tenth term?

25. An arithmetic sequence has first term of 5 and difference of 10. After how many terms will the sequence exceed 1000?

In Problems 26–29, find the 5^{th}, 50^{th}, n^{th} term of the arithmetic sequences.

26. $3, 5, 7, \ldots$

27. $6, 7.2, \ldots$

28. $a_1 = 2.1$, $a_3 = 4.7$

29. $a_3 = 5.7$, $a_6 = 9$

In Problems 30–33, find the 6^{th} and n^{th} of the geometric sequences.

30. $1, 2, 4, \ldots$

31. $7, 5.25, \ldots$

32. $a_1 = 3$, $a_3 = 48$

33. $a_2 = 6$, $a_4 = 54$

34. During 2004, about 81 million barrels of oil a day were consumed worldwide.[1] Over the previous decade, consumption had been rising at 1.2% a year; assume that it continues to increase at this rate.

(a) Write the first four terms of the sequence a_n giving daily oil consumption n years after 2003; give a formula for the general term a_n.
(b) In what year is consumption expected to exceed 100 million barrels a day?

35. In 2004, US natural gas consumption was 646.7 billion cubic meters. Asian consumption was 367.7 billion cubic meters.[2] During the previous decade, US consumption increased by 0.6% a year, while Asian consumption grew by 7.9% a year. Assume these rates continue into the future.

(a) Give the first four terms of the sequence, a_n, giving US consumption of natural gas n years after 2003.
(b) Give the first four terms of a similar sequence b_n showing Asian gas consumption.
(c) According to this model, when will Asian yearly gas consumption exceed US consumption?

36. The population[3] of Nevada grew from 2.2 million in 2002 to 2.4 million in 2005. Assuming a constant percent growth rate:

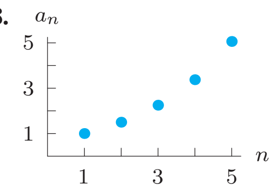
(a) Find a formula for a_n, the population in millions n years after 2005.
(b) When is the population predicted to reach 10 million?

37. Florida's population[4] was 17.960 million in 2005 and 17.613 million in 2004. Assuming the population continues to increase at the same percentage rate:

(a) Find the first three terms of the sequence a_n giving the population, in millions, n years after 2005.
(b) Write a formula for a_n.
(c) What is the doubling time of the population?

The graphs in Problems 38–41 represent either an arithmetic or a geometric sequence; decide which. For an arithmetic sequence, say if the common difference, d, is positive or negative. For a geometric sequence, say if the common ratio, r, is greater or smaller than 1.

38.



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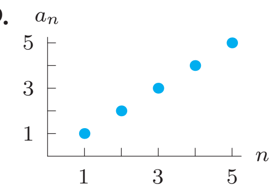

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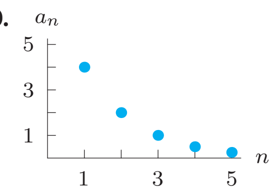
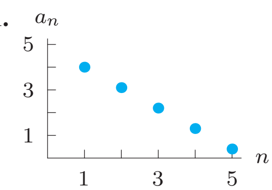


41.



A sequence a_n can be defined by a *recurrence relation*, which gives a_n in terms of the previous term, a_{n-1}, and the first term a_1. In Problems 42–45, find the first four terms of the sequence and a formula for the general term.

42. $a_n = 2a_{n-1}$; $a_1 = 3$

43. $a_n = a_{n-1} + 5$; $a_1 = 2$

44. $a_n = -a_{n-1}$; $a_1 = 1$

45. $a_n = 2a_{n-1} + 1$; $a_1 = 3$

46. A geometric sequence has first term of 10 and ratio of -0.2. After how many terms will the sequence have absolute value less than 10^{-7}?

47. The sequence defined by

$$p_{n+1} = 2p_n - \frac{p_n^2}{200}, \quad p_0 = 150,$$

is called a discrete logistic equation. Such a sequence is often used to model a population p_n, where the initial population is p_0.

(a) Compute the first 6 terms of the sequence. What do you conclude?
(b) Show that if $0 < p_0 < 200$, then p_n increases but never exceeds 200.

[1] www.bp.com/downloads, Statistical Review of World Energy 2005, accessed January 15, 2006.
[2] www.bp.com/downloads, Statistical Review of World Energy 2005, accessed January 15, 2006.
[3] health2k.state.nv.us and www.city-data.com, accessed December 26, 2005.
[4] www.floridacharts.com, accessed December 26, 2005.

48. The Fibonacci sequence starts with $1, 1, 2, 3, 5, \ldots$, and each term is the sum of the previous two terms.

(a) Write the next three terms in the sequence.
(b) Write an expression for a_n in terms of a_{n-1} and a_{n-2}.
(c) Suppose r_n is the ratio a_n/a_{n-1} and r_{n-1} is the ratio a_{n-1}/a_{n-2}. Using your answer to part (b), find a formula for r_n in terms of r_{n-1} and without any of the as.
(d) The terms r_n form another sequence. Suppose r_n tends to a fixed value, r as n increases without bound; that is, $r_n \to r$ as $n \to \infty$. Use your answer to part (c) to find an equation for r. Solve this equation. The number that you find is called the *golden ratio*.

49. Some people believe they can make money from a chain letter (they are usually disappointed). A chain letter works roughly like this: A letter arrives with a list of four names attached and instructions to mail a copy to four more friends and to send \$1 to the top name on the list. When you mail the four letters, you remove the top name (to whom the money was sent) and add your own name to the bottom of the list.

(a) If no one breaks the chain, how much money do you receive?
(b) Let d_n be the number of dollars you receive if there are n names on the list instead of 4, but you still mailed to four friends. Find a formula for d_n.

50. For a positive integer n, let a_n be the fraction of the US population with income less than or equal to \$$n$ thousand dollars.

(a) Which is larger, a_{40} or a_{50}? Why?
(b) What does the quantity $a_{50} - a_{40}$ represent in terms of US population?
(c) Is there any value of n with $a_n = 0$? Explain.
(d) What happens to the value of a_n as n increases?

13.2 DEFINING FUNCTIONS USING SUMS: ARITHMETIC SERIES

Domestic Deaths from AIDS n years after 1980

Table 13.1 shows the number of US AIDS deaths[5] that occurred each year since 1980, where a_n is the number of deaths in year n, and $n = 1$ corresponds to 1981. For each year, we can calculate the total number of death from AIDS since 1980. For example, in 1985

$$\text{Total number of AIDS deaths (1981-1985)} = 159 + 463 + 1508 + 3505 + 6972 = 12{,}607.$$

We write S_n to denote the sum of the first n terms of a sequence. In this example, 1985 corresponds to $n = 5$, so we have $S_5 = 12{,}607$.

Table 13.1 *US deaths from AIDS each year from 1981 to 2003*

n	a_n	n	a_n	n	a_n	n	a_n	n	a_n
1	159	6	12,110	11	36,616	16	38,025	21	18,524
2	463	7	16,412	12	41,094	17	21,999	22	17,557
3	1508	8	21,119	13	45,598	18	18,397	23	18,017
4	3505	9	27,791	14	50,418	19	17,172		
5	6972	10	31,538	15	51,117	20	15,245		

Example 1 Find and interpret S_8 for the AIDS sequence.

[5] www.cdc.gov/hiv/stats/hasr1301/table28.htm, from the HIV/AIDS Surveillance Report, 2001, Vol 13, No. 1, p. 34, US Department of Health and Human Services, Centers for Disease Control and Prevention, Atlanta, and www.cdc.gove/hiv/PUBS/Facts/At-A-Glance.htm, accessed January 14, 2006.

Solution Since S_8 is the sum of the first 8 terms of the sequence, we have.

$$\begin{aligned} S_8 &= \underbrace{a_1 + a_2 + a_3 + a_4 + a_5}_{S_5=12{,}607} + a_6 + a_7 + a_8 \\ &= 12{,}607 + 12{,}110 + 16{,}412 + 21{,}119 \\ &= 62{,}248. \end{aligned}$$

Here, S_8 is the number of deaths from AIDS from 1980 to 1988.

The sum of the terms of a sequence is called a *series*. We write S_n for the sum of the first n terms of the sequence, called the n^{th} *partial sum*. We see that S_n is a function of n, the number of terms in the partial sum. In this section we see how to evaluate functions defined by sums.

Arithmetic Series

Landscape timbers are large beams of wood used to landscape gardens. To make the terrace in Figure 13.1, one timber is set into the slope, followed by a stack of two, then a stack of three, then a stack of four. The stacks are separated by earth.

The total number of timbers in 4 stacks is S_4, the sum of the number of timbers in each stack:

$$\text{Total number of timbers} = S_4 = 1 + 2 + 3 + 4 = 10.$$

For a larger terrace using 5 stacks of timbers, the total number is given by

$$S_5 = 1 + 2 + 3 + 4 + 5 = 15.$$

For an even larger terrace using 6 stacks, the total number is given by

$$S_6 = 1 + 2 + 3 + 4 + 5 + 6 = 21.$$

For a terrace made from n stacks of landscape timbers, S_n, the total number of timbers needed is a function of n, so

$$S_n = f(n) = 1 + 2 + \cdots + n.$$

The symbol $\cdots$ means that all the integers from 1 to n are included in the sum.

Notice that each stack of landscape timbers contains one more timber than the previous one. Thus, the number of timbers in each stack,

$$1, 2, 3, 4, 5, \ldots$$

is an arithmetic sequence, so S_n is the sum of the terms of an arithmetic sequence. Such a sum is called an *arithmetic series*.

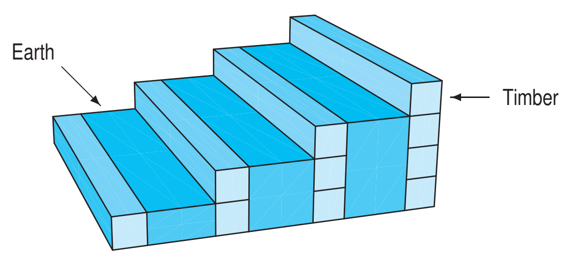


Figure 13.1: A slope terraced for planting using landscape timbers

The Sum of an Arithmetic Series

We now find a formula for the sum of an arithmetic series. A famous story concerning this series[6] is told about the great mathematician Carl Friedrich Gauss (1777–1855), who as a young boy was asked by his teacher to add the numbers from 1 to 100. He did so almost immediately:

$$S_{100} = 1 + 2 + 3 + \cdots + 100 = 5050.$$

Of course, no one really knows how Gauss accomplished this, but he probably did not perform the calculation directly, by adding 100 terms. He might have noticed that the terms in the sum can be regrouped into pairs, as follows:

$$\begin{aligned} S_{100} = 1 + 2 + \cdots + 99 + 100 &= \underbrace{(1 + 100) + (2 + 99) + \cdots + (50 + 51)}_{50 \text{ pairs}} \\ &= \underbrace{101 + 101 + \cdots + 101}_{50 \text{ terms}} \qquad \text{Each pair adds to 101} \\ &= 50 \cdot 101 = 5050. \end{aligned}$$

The approach of pairing numbers works for the sum from 1 to n, no matter how large n is. Provided n is an even number, we can write

$$\begin{aligned} S_n = 1 + 2 + \cdots + (n-1) + n &= \underbrace{(1 + n) + (2 + (n-1)) + (3 + (n-2)) + \cdots}_{\frac{1}{2}n \text{ pairs}} \\ &= \underbrace{(1 + n) + (1 + n) + \cdots}_{\frac{1}{2}n \text{ pairs}} \qquad \text{Each pair adds to } 1 + n \end{aligned}$$

so we have the formula:

$$\boxed{S_n = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1).}$$

Example 2 Check this formula for S_n with $n = 100$.

Solution Using the formula, we get the same answer, 5050, as before:

$$S_{100} = 1 + 2 + \cdots + 100 = \frac{1}{2} \cdot 100(100 + 1) = 50 \cdot 101 = 5050.$$

A similar derivation shows that this formula for S_n also holds for odd values of n.

To find a formula for the sum of a general arithmetic series, we first assume that n is even and write

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_n \\ &= \underbrace{(a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \cdots}_{\frac{1}{2}n \text{ pairs}}. \end{aligned}$$

[6]As told by E.T. Bell, *The Men of Mathematics*, p. 221 (New York: Simon and Schuster, 1937), the series involved was arithmetic, but more complicated than this one.

We pair the first term with the last term, the second term with the next to last term, and so on, just as Gauss may have done. Each pair of terms adds up to the same value, just as each of Gauss's pairs added to 101. Using the formula for the terms of an arithmetic sequence, $a_n = a_1 + (n-1)d$, the first pair, $a_1 + a_n$, can be written as

$$a_1 + a_n = a_1 + \underbrace{a_1 + (n-1)d}_{a_n} = 2a_1 + (n-1)d,$$

and the second pair, $a_2 + a_{n-1}$, can be written as

$$a_2 + a_{n-1} = \underbrace{a_1 + d}_{a_2} + \underbrace{a_1 + (n-2)d}_{a_{n-1}} = 2a_1 + (n-1)d.$$

Both the first two pairs have the same sum: $2a_1 + (n-1)d$. The remaining pairs also all have the same sum, so

$$S_n = \underbrace{(a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \cdots}_{\frac{1}{2}n \text{ pairs}} = \frac{1}{2}n\left(2a_1 + (n-1)d\right).$$

The same formula gives the sum of the series when n is odd. See Problem 45 on page 548.

The **sum**, S_n, of the first n terms of the **arithmetic series** with $a_n = a_1 + (n-1)d$ is

$$S_n = \frac{1}{2}n(a_1 + a_n) = \frac{1}{2}n(2a_1 + (n-1)d).$$

Example 3 Calculate the sum $1 + 2 + \cdots + 100$ using the formula for S_n.

Solution Here, $a_1 = 1$, $n = 100$, and $d = 1$. We get the same answer as before:

$$S_{100} = \frac{1}{2}n\left(2a_1 + (n-1)d\right) = \frac{1}{2} \cdot 100\left(2 \cdot 1 + (100-1) \cdot 1\right) = 50 \cdot 101 = 5050.$$

Summation Notation

The symbol Σ is used to indicate addition. This symbol, pronounced *sigma*, is the Greek capital letter for *S*, which stands for sum. Using this notation, we write

$$\sum_{i=1}^{n} a_i \qquad \text{to stand for the sum } a_1 + a_2 + \cdots + a_n.$$

The Σ tells us we are adding some numbers. The a_i tells us that the numbers we are adding are called a_1, a_2, and so on. The sum begins with a_1 and ends with a_n because the subscript i starts at

$i = 1$ (at the bottom of the Σ sign) and ends at $i = n$ (at the top of the Σ sign):

This tells us that the sum ends at a_n

$$\sum_{i=1}^{n} a_i \longleftarrow$$

This tells us that the numbers we are adding are called $a_1, a_2, a_3, \ldots$

This tells us that the sum starts at a_1

Example 4 Write $S_n = f(n)$, the total number of landscape timbers in n stacks, using sigma notation, and give a formula for S_n.

Solution Since $S_n = f(n) = 1 + 2 + \cdots + n$, we have $a_i = i$. We start at $i = 1$ and end at $i = n$. Thus, $S_n = \sum_{i=1}^{n} i$. The formula for the sum of an arithmetic series with $a_1 = d = 1$ gives

$$S_n = f(n) = \sum_{i=1}^{n} i = \frac{1}{2}n(n+1).$$

Example 5 Use sigma notation to write the sum of the first 20 positive odd numbers. Evaluate this sum.

Solution The odd numbers form an arithmetic sequence: 1, 3, 5, 7, ... with $a_1 = 1$ and $d = 2$. The i^{th} odd number is

$$a_i = 1 + (i-1)2 = 2i - 1.$$

(As a check: $a_1 = 1 + (1-1)2 = 1$ and $a_2 = 1 + (2-1)2 = 3$.) Thus, we have

$$\text{Sum of the first 20 odd numbers} = \sum_{i=1}^{20} a_i = \sum_{i=1}^{20}(2i-1).$$

We evaluate the sum using $n = 20$, $a_1 = 1$, and $d = 2$:

$$\text{Sum} = \frac{1}{2}n\left(2a_1 + (n-1)d\right) = \frac{1}{2}\cdot 20\left(2\cdot 1 + (20-1)\cdot 2\right) = 400.$$

Example 6 If air resistance is neglected, a falling object travels 16 ft during the first second, 48 ft during the next, 80 ft during the next, and so on. These distances form the arithmetic sequence 16, 48, 80, ... In this sequence, $a_1 = 16$ and $d = 32$.

(a) Find a formula for the n^{th} term in the sequence of distances. Calculate the fifth and tenth terms.
(b) Calculate S_1, S_2, and S_3, the total distance an object falls in 1, 2, and 3 seconds, respectively.
(c) Give a formula for S_n, the distance fallen in n seconds.

Solution

(a) The n^{th} term is $a_n = a_1 + (n-1)d = 16 + 32(n-1) = 32n - 16$. Thus, the fifth term is $a_5 = 32(5) - 16 = 144$. The value of $a_{10} = 32(10) - 16 = 304$.

(b) Since $a_1 = 16$ and $d = 32$,

$$\begin{aligned} S_1 &= a_1 = 16 \text{ feet} \\ S_2 &= a_1 + a_2 = a_1 + (a_1 + d) = 16 + 48 = 64 \text{ feet} \\ S_3 &= a_1 + a_2 + a_3 = 16 + 48 + 80 = 144 \text{ feet.} \end{aligned}$$

(c) The formula for S_n is $S_n = \frac{1}{2}n\,(2a_1 + (n-1)d)$. We can check our answer to part (b) using this formula:

$$S_1 = \frac{1}{2}\cdot 1(2\cdot 16 + 0\cdot 32) = 16, \quad S_2 = \frac{1}{2}\cdot 2(2\cdot 16 + 1\cdot 32) = 64, \quad S_3 = \frac{1}{2}\cdot 3(2\cdot 16 + 2\cdot 32) = 144.$$

Example 7 An object falls from 1000 feet starting at time $t = 0$ seconds. What is its height, h, in feet above the ground at $t = 1, 2, 3$ seconds? Show these values on a graph of height against time.

Solution At time $t = 0$, the height $h = 1000$ feet. At time $t = 1$, the object has fallen $S_1 = 16$ feet, so

$$h = 1000 - 16 = 984 \text{ feet.}$$

At time $t = 2$, the object has fallen a total distance of $S_2 = 64$ feet, so

$$h = 1000 - 64 = 936 \text{ feet.}$$

At time $t = 3$, the object has fallen a total distance of $S_3 = 144$ feet, so

$$h = 1000 - 144 = 856 \text{ feet.}$$

These heights are marked on the graph in Figure 13.2.

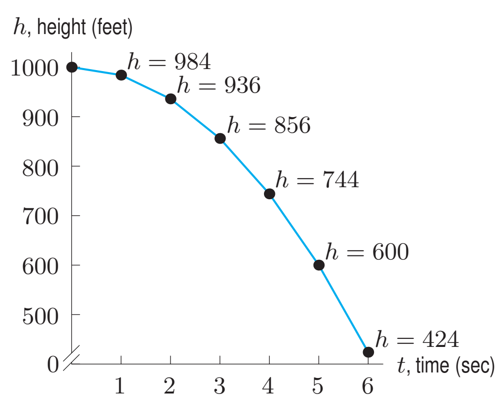


Figure 13.2: Height of a falling object

Exercises and Problems for Section 13.2

Exercises

Are the series in Exercises 1–4 arithmetic?

1. $2 + 4 + 8 + 16 + \cdots$

2. $10 + 8 + 6 + 4 + 2 + \cdots$

3. $1 + 2 + 4 + 5 + 7 + 8 + \cdots$

4. $-\frac{1}{3} + \frac{2}{3} + \frac{5}{3} + \frac{8}{3} + \cdots$

Expand the sums in Exercises 5–10. (Do not evaluate.)

5. $\sum_{i=-1}^{5} i^2$

6. $\sum_{i=10}^{20} (i+1)^2$

7. $\sum_{k=0}^{5} 2k + 1$

8. $\sum_{j=1}^{6} 3(j-3)$

9. $\sum_{j=2}^{10} (-1)^j$

10. $\sum_{n=1}^{7} (-1)^{n-1} 2^n$

In Exercises 11–14, write the sum using sigma notation.

11. $3 + 6 + 9 + 12 + 15 + 18 + 21$

12. $10 + 13 + 16 + 19 + 22$

13. $1/2 + 1 + 3/2 + 2 + 5/2 + 3 + 7/2 + 4$

14. $30 + 25 + 20 + 15 + 10 + 5$

15. **(a)** Use sigma notation to write the sum $2 + 4 + 6 + \cdots + 20$ of the first 10 even numbers.
(b) Without a calculator, find the sum in part (a).

In Exercises 16–20, complete the tables with the terms of the arithmetic series $a_1, a_2, \ldots, a_n$, and the sequence of partial sums, $S_1, S_2, \ldots, S_n$. State the values of a_1 and d where $a_n = a_1 + (n-1)d$.

16.

n	1	2	3	4	5	6	7	8
a_n	2	7	12	17				
S_n	2	9	21	38				

17.

n	1	2	3	4	5	6	7	8
a_n	3	7	11					
S_n	3	10						

18.

n	1	2	3	4	5	6	7	8
a_n	7			16				
S_n								

19.

n	1	2	3	4	5	6	7	8
a_n	2							
S_n	2	13	33	62				

20.

n	1	2	3	4	5	6	7	8
a_n								
S_n						201	273	356

21. Find the sum of the first 1000 integers: $1 + 2 + 3 + \cdots + 1000$.

Without using a calculator, find the sum of the series in Problems 22–29.

22. $\sum_{i=1}^{50} 3i$

23. $\sum_{i=1}^{30} (5i + 10)$

24. $\sum_{n=0}^{15} \left(2 + \frac{1}{2}n\right)$

25. $\sum_{n=0}^{10} (8 - 4n)$

26. $\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + \cdots + 25\sqrt{2}$

27. $-101 - 91 - 81 - 71 - 61 - \cdots - 11 - 1$

28. $26.5 + 24.5 + 22.5 + \cdots + 2.5 + 0.5$

29. $-3.01 - 3.02 - 3.03 - \cdots - 3.35$

Problems

30. Jenny decides to raise money for her local charity by encouraging people to lay quarters on the floor of a classroom in the shape of an equilateral triangle. She puts one quarter in the first row, two in the second, three in the third, and so on. Her target is to raise \$400. How many rows does she need?

31. Find the thirtieth positive multiple of 5 and the sum of the first thirty positive multiples of 5.

32. For the AIDS data in Table 13.1 on page 541,

(a) Find and interpret in terms of AIDS
  (i) The partial sums S_5, S_6, S_7, S_8.
  (ii) $S_6 - S_5, S_7 - S_6, S_8 - S_7$.

(b) Use your answer to part (a) (ii) to explain the value of $S_{n+1} - S_n$ for any positive integer n.

33. Table 13.2 shows US Census figures, in millions. Interpret these figures as partial sums, S_n of a sequence, a_n, where n is the number of decades since 1940, so $S_1 = 150.7$, $S_2 = 179.3$, $S_3 = 203.3$, and so on.

(a) Find and interpret in terms of population
  (i) S_4, S_5, S_6
  (ii) a_2, a_5, a_6
  (iii) $a_6/10$

(b) For any positive integer n, what are the meanings of S_n, a_n, and $a_n/10$ in terms of population?

Table 13.2

Year	1950	1960	1970	1980	1990	2000
Population	150.7	179.3	203.3	226.6	248.7	281.4

Simplify the expressions in Problems 34–37.

34. $\displaystyle\sum_{i=1}^{5} i^2 - \sum_{j=0}^{4} (j+1)^2$

35. $\displaystyle\sum_{i=4}^{20} i - \sum_{j=4}^{20} (-2j)$

36. $\displaystyle\sum_{i=1}^{20} 1$

37. $\displaystyle\sum_{i=1}^{15} i^3 - \sum_{j=3}^{15} j^3$

Problems 38–41 refer to the falling object of Example 6 on page 545.

38. Calculate the distance, S_7, the object falls in 7 seconds.

39. **(a)** Find the total distance that the object falls in 4, 5, 6 seconds.

(b) The object falls from 1000 feet at time $t = 0$. Calculate its height at $t = 4$, $t = 5$, $t = 6$ seconds. Explain how you can check your answer using Figure 13.2.

40. Find a formula for $f(n)$, the distance fallen by the object in n seconds.

41. If the object falls from 1000 feet, how long does it take to hit the ground?

42. A boy is dividing M&Ms between himself and his sister. He gives one to his sister and takes one for himself. He gives another to his sister and takes two for himself. He gives a third one to his sister and takes three for himself, and so on.

(a) On the n^{th} round, how many M&Ms does the boy give his sister? How many does he take himself?

(b) After n rounds, how many M&Ms does his sister have? How many does the boy have?

43. An auditorium has 30 seats in the first row, 34 seats in the second row, 38 seats in the third row, and so on. If there are twenty rows in the auditorium, how many seats are there in the last row? How many seats are there in the auditorium?

44. **(a)** Show that $n^3 - (n-1)^3 = 3n^2 - 3n + 1$.

(b) Write $n^3 = n^3 - (n-1)^3 + (n-1)^3 - (n-2)^3 + (n-2)^3 - \ldots - 2^3 + 2^3 - 1^3 + 1^3 - 0^3$ and then show

$$n^3 = \sum_{j=1}^{j=n} (3j^2 - 3j + 1).$$

(c) Use the result $\sum_{j=1}^{j=n} j = \frac{1}{2}n(n+1)$ to show that

$$\sum_{j=1}^{j=n} j^2 = \frac{n}{6}(n+1)(2n+1).$$

45. In the text we showed how to calculate the sum of an arithmetic series with an even number of terms. Consider the arithmetic series

$$5 + 12 + 19 + 26 + 33 + 40 + 47 + 54 + 61.$$

Here, there are $n = 9$ terms, and the difference between each term is $d = 7$. Adding these terms directly, we find that their sum is 297. In this problem we find the sum of this arithmetic sequence in two different ways. We then use our results to obtain a general formula for the sum of an arithmetic series with an odd number of terms.

(a) The sum of the first and last terms is $5 + 61 = 66$, the sum of the second and next-to-last terms is $12 + 54 = 66$, and so on. Find the sum of this arithmetic series by pairing off terms in this way. Notice that since the number of terms is odd, one of them will be unpaired.

(b) This arithmetic series can be thought of as a series of eight terms $(5 + \cdots + 54)$ plus an additional term (61). Use the formula we found for the sum of an arithmetic series containing an even number of terms to find the sum of the given arithmetic series.

(c) Find a formula for the sum of an arithmetic series with n terms where n is odd. Let a_1 be the first term in the series, and let d be the difference between consecutive terms. Show that the two approaches used in parts (a) and (b) give the same result, and show that your formula is the same as the formula given for even values of n.

13.3 FINITE GEOMETRIC SERIES

In the previous section, we studied *arithmetic series*. An arithmetic series is the sum of terms in a sequence in which each term is obtained by adding a constant to the preceding term. In this section, we study another type of sum, a *geometric series*. In a geometric series, each term is a constant multiple of the preceding term.

Bank Balance

A person deposits \$2000 every year in an IRA that pays 6% interest per year, compounded annually. After the first deposit (but before any interest has been earned), the balance in the account in dollars is

$$B_1 = 2000.$$

After 1 year has passed, the first deposit has earned interest, so the balance becomes $2000(1.06)$ dollars. Then the second deposit is made and the balance becomes

$$\begin{aligned} B_2 &= \underbrace{2^{\text{nd}}\text{ deposit}}_{2000} + \underbrace{1^{\text{st}}\text{ deposit with interest}}_{2000(1.06)} \\ &= 2000 + 2000(1.06) \text{ dollars.} \end{aligned}$$

After 2 years have passed, the third deposit is made, and the balance is

$$\begin{aligned} B_3 &= \underbrace{3^{\text{rd}}\text{ deposit}}_{2000} + \underbrace{2^{\text{nd}}\text{ dep. with 1 year interest}}_{2000(1.06)} + \underbrace{1^{\text{st}}\text{ deposit with 2 years interest}}_{2000(1.06)^2} \\ &= 2000 + 2000(1.06) + 2000(1.06)^2. \end{aligned}$$

Let B_n be the balance in dollars after n deposits. Then we see that

$$\begin{aligned} &\text{After 4 deposits} \quad & B_4 &= 2000 + 2000(1.06) + 2000(1.06)^2 + 2000(1.06)^3 \\ &\text{After 5 deposits} \quad & B_5 &= 2000 + 2000(1.06) + 2000(1.06)^2 + 2000(1.06)^3 + 2000(1.06)^4 \\ & & &\vdots \\ &\text{After } n \text{ deposits} \quad & B_n &= 2000 + 2000(1.06) + 2000(1.06)^2 + \cdots + 2000(1.06)^{n-1}. \end{aligned}$$

Example 1 How much money is in this IRA in 5 years, right after a deposit is made? In 25 years, right after a deposit is made?

Solution After 5 years, we have made 6 deposits. A calculator gives

$$\begin{aligned} B_6 &= 2000 + 2000(1.06) + 2000(1.06)^2 + \cdots + 2000(1.06)^5 \\ &= \$13{,}950.64. \end{aligned}$$

In 25 years, we have made 26 deposits. Even using a calculator, it would be tedious to evaluate B_{26} by adding 26 terms. Fortunately, there is a shortcut. Start with the formula for B_{26}:

$$B_{26} = 2000 + 2000(1.06) + 2000(1.06)^2 + \cdots + 2000(1.06)^{25}.$$

Multiply both sides of this equation by 1.06 and add 2000, giving

$$1.06B_{26} + 2000 = 1.06\left(2000 + 2000(1.06) + 2000(1.06)^2 + \cdots + 2000(1.06)^{25}\right) + 2000.$$

We simplify the right-hand side to get

$$1.06B_{26} + 2000 = 2000(1.06) + 2000(1.06)^2 + \cdots + 2000(1.06)^{25} + 2000(1.06)^{26} + 2000.$$

Notice that the right-hand side of this equation and the formula for B_{26} have almost every term in common. We can rewrite this equation as

$$\begin{aligned}1.06B_{26} + 2000 &= \underbrace{2000 + 2000(1.06) + 2000(1.06)^2 + \cdots + 2000(1.06)^{25}}_{B_{26}} + 2000(1.06)^{26}\\ &= B_{26} + 2000(1.06)^{26}.\end{aligned}$$

Solving for B_{26} gives

$$\begin{aligned}1.06B_{26} - B_{26} &= 2000(1.06)^{26} - 2000\\ 0.06B_{26} &= 2000(1.06)^{26} - 2000\\ B_{26} &= \frac{2000(1.06)^{26} - 2000}{0.06}.\end{aligned}$$

Using a calculator to evaluate this expression for B_{26}, we find that $B_{26} = 118{,}312.77$ dollars.

Geometric Series

The formula for the IRA balance,

$$B_{26} = 2000 + 2000(1.06) + 2000(1.06)^2 + \cdots + 2000(1.06)^{25},$$

is an example of a *geometric series*. This is a finite geometric series, because there are a finite number of terms (in this case, 26). In general, a geometric series is the sum of the terms of a geometric sequence—that is, in which each term is a constant multiple of the preceding term.

A **finite geometric series** is a sum of the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{i=0}^{n-1} ar^i.$$

Notice that S_n is defined to contain exactly n terms. Since the first term is $a = ar^0$, we stop at ar^{n-1}. For instance, the series

$$50 + 50(1.06) + 50(1.06)^2 + \cdots + 50(1.06)^{25}$$

contains 26 terms, so $n = 26$. For this series, $r = 1.06$ and $a = 50$.

The Sum of a Geometric Series

The shortcut from Example 1 can be used to find the sum of a general geometric series. Let S_n be the sum of a geometric series of n terms, so that

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiply both sides of this equation by r and add a, giving

$$\begin{aligned} rS_n + a &= r\left(a + ar + ar^2 + \cdots + ar^{n-1}\right) + a \\ &= \left(ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n\right) + a. \end{aligned}$$

The right-hand side can be rewritten as

$$\begin{aligned} rS_n + a &= \underbrace{a + ar + ar^2 + \cdots + ar^{n-1}}_{S_n} + ar^n \\ &= S_n + ar^n. \end{aligned}$$

Solving the equation $rS_n + a = S_n + ar^n$ for S_n gives

$$\begin{aligned} rS_n - S_n &= ar^n - a \\ S_n(r-1) &= ar^n - a \qquad \text{factoring out } S_n \\ S_n &= \frac{ar^n - a}{r-1} \\ &= \frac{a(r^n - 1)}{r-1}. \end{aligned}$$

By multiplying the numerator and denominator by -1, this formula can be rewritten as follows:

> The sum of a finite **geometric series of n terms** is given by
>
> $$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \qquad \text{for } r \neq 1.$$

This formula is called a *closed form* of the sum.

Example 2 Use the formula for the sum of a geometric series to solve Example 1.

Solution We need to find B_6 and B_{26} where

$$B_n = 2000 + 2000(1.06) + 2000(1.06)^2 + \cdots + 2000(1.06)^{n-1}.$$

Using the formula for S_n with $a = 2000$ and $r = 1.06$, we get the same answers as before:

$$\begin{aligned} B_6 &= \frac{2000(1-(1.06)^6)}{1-1.06} = 13{,}950.64, \\ B_{26} &= \frac{2000(1-(1.06)^{26})}{1-1.06} = 118{,}312.77. \end{aligned}$$

Drug Levels in The Body

Geometric series arise naturally in many different contexts. The following example illustrates a geometric series with decreasing terms.

Example 3 A patient is given a 20-mg injection of a therapeutic drug. Each day, the patient's body metabolizes 50% of the drug present, so that after 1 day only one-half of the original amount remains, after 2 days only one-fourth remains, and so on. The patient is given a 20-mg injection of the drug every day at the same time. Write a geometric series that gives the drug level in this patient's body right after the n^{th} injection.

Solution Immediately after the 1st injection, the drug level in the body is given by

$$Q_1 = 20.$$

One day later, the original 20 mg has fallen to $20 \cdot \frac{1}{2} = 10$ mg and the second 20-mg injection is given. Right after the second injection, the drug level is given by

$$\begin{aligned} Q_2 &= \underbrace{2^{\text{nd}} \text{ injection}}_{20} + \underbrace{\text{Residue of } 1^{\text{st}} \text{ injection}}_{\frac{1}{2}\cdot 20} \\ &= 20 + 20\left(\frac{1}{2}\right). \end{aligned}$$

Two days later, the original 20 mg has fallen to $(20 \cdot \frac{1}{2}) \cdot \frac{1}{2} = 20(\frac{1}{2})^2 = 5$ mg, the second 20 mg injection has fallen to $20 \cdot \frac{1}{2} = 10$ mg, and the third 20 mg injection is given. Right after the third injection, the drug level is given by

$$\begin{aligned} Q_3 &= \underbrace{3^{\text{rd}} \text{ injection}}_{20} + \underbrace{\text{Residue of } 2^{\text{nd}} \text{ injection}}_{20\cdot\frac{1}{2}} + \underbrace{\text{Residue of } 1^{\text{st}} \text{ injection}}_{20\cdot\frac{1}{2}\cdot\frac{1}{2}} \\ &= 20 + 20\left(\frac{1}{2}\right) + 20\left(\frac{1}{2}\right)^2. \end{aligned}$$

Continuing, we see that

$$\begin{aligned} &\text{After } 4^{\text{th}} \text{ injection} \quad Q_4 = 20 + 20\left(\frac{1}{2}\right) + 20\left(\frac{1}{2}\right)^2 + 20\left(\frac{1}{2}\right)^3 \\ &\text{After } 5^{\text{th}} \text{ injection} \quad Q_5 = 20 + 20\left(\frac{1}{2}\right) + 20\left(\frac{1}{2}\right)^2 + \cdots + 20\left(\frac{1}{2}\right)^4 \\ &\qquad\qquad\vdots \\ &\text{After } n^{\text{th}} \text{ injection} \quad Q_n = 20 + 20\left(\frac{1}{2}\right) + 20\left(\frac{1}{2}\right)^2 + \cdots + 20\left(\frac{1}{2}\right)^{n-1}. \end{aligned}$$

This is another example of a geometric series. Here, $a = 20$ and $r = 1/2$ in the geometric series formula

$$Q_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

To calculate the drug level for a specific value of n, we use the formula for the sum of a geometric series.

Example 4 What quantity of the drug remains in the patient's body after the 10^{th} injection?

Solution After the 10^{th} injection, the drug level in the patient's body is given by

$$Q_{10} = 20 + 20\left(\frac{1}{2}\right) + 20\left(\frac{1}{2}\right)^2 + \cdots + 20\left(\frac{1}{2}\right)^9.$$

Using the formula for S_n with $n = 10$, $a = 20$, and $r = 1/2$, we get

$$Q_{10} = \frac{20(1-(\frac{1}{2})^{10})}{1-\frac{1}{2}} = 39.961 \text{ mg}.$$

Exercises and Problems for Section 13.3

Exercises

How many terms are there in the series in Exercises 1–2? Find the sum.

1. $\displaystyle\sum_{j=5}^{18} 3 \cdot 2^j$

2. $\displaystyle\sum_{k=2}^{20} (-1)^k 5(0.9)^k$

Find the sum of the series in Exercises 3–7.

3. $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{2^{10}}$

4. $5 + 15 + 45 + 135 + \cdots + 5(3^{12})$

5. $1/125 + 1/25 + 1/5 + \cdots + 625$

6. $\displaystyle\sum_{n=1}^{10} 4(2^n)$

7. $\displaystyle\sum_{k=0}^{7} 2\left(\frac{3}{4}\right)^k$

In Exercises 8–11, decide which of the following are geometric series. For those which are, give the first term and the ratio between successive terms. For those which are not, explain why not.

8. $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{128}$

9. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{256}$

10. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{100}$

11. $5 - 10 + 20 - 40 + 80 - \cdots - 2560$

Write each of the sums in Exercises 12–15 in sigma notation.

12. $1 + 4 + 16 + 64 + 256$

13. $3 - 9 + 27 - 81 + 243 - 729$

14. $2 + 10 + 50 + 250 + 1250 + 6250 + 31250$

15. $32 - 16 + 8 - 4 + 2 - 1$

Evaluate the sums in Exercises 16–19.

16. $\displaystyle\sum_{n=0}^{n=4} (0.1)^n$

17. $\displaystyle\sum_{n=0}^{n=5} \frac{3}{2^n}$

18. $\displaystyle\sum_{j=0}^{j=n-1} e^{jx}$

19. $\displaystyle\sum_{n=0}^{n=N} (-1)^n$

Problems

20. Write $ar^3 + ar^5 + ar^7 + ar^9 + ar^{11}$ in $\sum$ notation.

21. Worldwide consumption of oil was about 81 billion barrels in 2004.[7] Assume that consumption continues to increase at 1.2% per year, the rate for the previous decade.

(a) Write a sum representing the total oil consumption for 25 years, starting with 2004.

(b) Evaluate this sum.

22. Figure 13.3 shows the quantity of the drug atenolol in the blood as a function of time, with the first dose at time $t = 0$. Atenolol is taken in 50-mg doses once a day to lower blood pressure.

(a) If the half-life of atenolol in the blood is 6.3 hours, what percentage of the atenolol present at the start of a 24-hour period is still there at the end?

[7] www.bp.com/downloads, Statistical Review of World Energy 2005, accessed January 15, 2006.

(b) Find expressions for the quantities $Q_0, Q_1, Q_2, Q_3, \ldots$, and Q_n shown in Figure 13.3. Write the expression for Q_n in closed-form.

(c) Find expressions for the quantities $P_1, P_2, P_3, \ldots$, and P_n shown in Figure 13.3. Write the expression for P_n in closed-form.

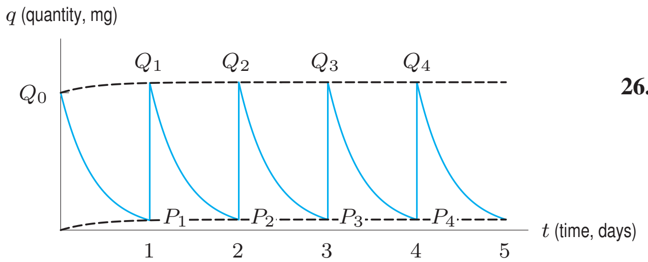


Figure 13.3

23. Annual deposits of \$3000 are made into a bank account earning 5% interest per year. What is the balance in the account right after the 15^{th} deposit if interest is calculated

(a) Annually **(b)** Continuously

24. A deposit of \$1000 is made once a year, starting today, into a bank account earning 3% interest per year, compounded annually. If 20 deposits are made, what is the balance in the account on the day of the last deposit?

25. What effect does doubling each of the following quantities (leaving other quantities the same) have on the answer to Problem 24? Is the answer doubled, more than doubled, or less than doubled?

(a) The deposit
(b) The interest rate
(c) The number of deposits made

26. To save for a new car, you put \$500 a month into an account earning interest at 3% per year, compounded continuously.

(a) How much money do you have 2 years after the first deposit, right before you make a deposit?
(b) When does the balance first go over \$10,000?

27. A bank account in which interest is earned at 4% per year, compounded annually, starts with a balance of \$50,000. Payments of \$1000 are made out of the account once a year for ten years, starting today. Interest is earned right before each payment is made.

(a) What is the balance in the account right after the tenth payment is made?
(b) Assume that the tenth payment exhausts the account. What is the largest yearly payment that can be made from this account?

13.4 INFINITE GEOMETRIC SERIES

In this section, we look at geometric series with an infinite number of terms and see under what circumstances they have a finite sum.

Long-term Drug Level in the Body

Suppose the patient from Example 3 on page 552 receives injections over a long period of time. What happens to the drug level in the patient's body? To find out, we calculate the drug level after 10, 15, 20, and 25 injections:

$$Q_{10} = \frac{20(1-(\frac{1}{2})^{10})}{1-\frac{1}{2}} = 39.960938 \text{ mg},$$

$$Q_{15} = \frac{20(1-(\frac{1}{2})^{15})}{1-\frac{1}{2}} = 39.998779 \text{ mg},$$

$$Q_{20} = \frac{20(1-(\frac{1}{2})^{20})}{1-\frac{1}{2}} = 39.999962 \text{ mg},$$

$$Q_{25} = \frac{20(1-(\frac{1}{2})^{25})}{1-\frac{1}{2}} = 39.999999 \text{ mg}.$$

The drug level appears to approach 40 mg. To see why this happens, notice that if there is exactly 40 mg of drug in the body, then half of this amount is metabolized in one day, leaving 20 mg. The next 20 mg injection replaces what was lost, and the level returns to 40 mg. We say that the *equilibrium* drug level is 40 mg.

Initially the patient has less than 40 mg of the drug in the body. Then the amount metabolized in one day is less than 20 mg. Thus, after the next 20 mg injection, the drug level is higher than it was before. For instance, if there are currently 30 mg, then after one day, half of this has been metabolized, leaving 15 mg. At the next injection the level rises to 35 mg, or 5 mg higher than where it started. Eventually, the quantity levels off to 40 mg.

Another way to think about the patient's drug level over time is to consider an *infinite geometric series*. After n injections, the drug level is given by the sum of the finite geometric series

$$Q_n = 20 + 20\left(\frac{1}{2}\right) + 20\left(\frac{1}{2}\right)^2 + \cdots + 20\left(\frac{1}{2}\right)^{n-1} = \frac{20(1-(\frac{1}{2})^n)}{1-\frac{1}{2}}.$$

What happens to the value of this sum as the number of terms approaches infinity? It does not seem possible to add up an infinite number of terms. However, we can look at the *partial sums*, Q_n, to see what happens for large values of n. For large values of n, we see that $(1/2)^n$ is very small, so that

$$Q_n = \frac{20\,(1-\text{Small number})}{1-\frac{1}{2}} = \frac{20\,(1-\text{Small number})}{1/2}.$$

We write $\to$ to mean "tends toward." Thus, as $n \to \infty$, we know that $(1/2)^n \to 0$, so

$$Q_n \to \frac{20(1-0)}{1/2} = \frac{20}{1/2} = 40 \text{ mg}.$$

The Sum of an Infinite Geometric Series

Consider the geometric series $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. In general, if $|r| < 1$, then $r^n \to 0$ as $n \to \infty$, so

$$S_n = \frac{a(1-r^n)}{1-r} \to \frac{a(1-0)}{1-r} = \frac{a}{1-r} \text{ as } n \to \infty.$$

Thus, if $|r| < 1$, the partial sums S_n approach a finite value, S, as $n \to \infty$. In this case, we say that the series *converges* to S.

For $|r| < 1$, the **sum of the infinite geometric series** is given by

$$S = a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}.$$

On the other hand, if $|r| > 1$, then we say that the series does not converge. The terms in the series get larger and larger as $n \to \infty$, so adding infinitely many of them does not give a finite sum.

What happens when $r = \pm 1$? For $|r| \geq 1$, the value of S_n does not approach a fixed number as $n \to \infty$, so we say that the infinite series does not converge. (To see why, consider what happens to 1^n and $(-1)^n$ as n increases.)

Example 1 Find the sum of the geometric series $\sum_{i=0}^{17} 7(-z)^i$ and $\sum_{i=0}^{\infty} (-z)^i$ provided $|z| < 1$.

Solution We have

$$\sum_{i=0}^{17} 7(-z)^i = 7(-z)^0 + 7(-z)^1 + 7(-z)^2 + \cdots + 7(-z)^{17}.$$

Since $(-z)^0 = 1$, this is a geometric series with $a = 7$ and $r = -z$. There are 18 terms, so

$$\sum_{i=0}^{17} 7(-z)^i = \frac{7(1-(-z)^{18})}{1-(-z)} = \frac{7(1-z^{18})}{1+z}.$$

As for $\sum_{i=0}^{\infty}(-z)^i$, this is an infinite geometric series. Since $|z| < 1$, we have

$$\sum_{i=0}^{\infty}(-z)^i = \frac{1}{1-(-z)} = \frac{1}{1+z}.$$

Present Value of a Series of Payments

When baseball player Alex Rodriguez was signed by the Texas Rangers, before being traded to the New York Yankees, he was given a signing bonus for $\$10$ million: $\$2$ million a year for five years. Of course, since much of the money was to be paid in the future, the team's owners did not have to have all $\$10$ million available on the day of the signing. How much money would the owners have to deposit in a bank account on the day of the signing in order to cover all the future payments? Assuming the account was earning interest, the owners would have to deposit less than $\$10$ million. This smaller amount is called the *present value* of $\$10$ million. We calculate the present value of Rodriguez's bonus on the day he signed.

Definition of Present Value

Let's consider a simplified version of this problem, with only one future payment: How much money would we need to deposit in a bank account today in order to have $\$2$ million in one year? At an annual interest rate of 5%, compounded annually, the deposit would grow by a factor of 1.05. Thus,

$$\text{Deposit} \cdot 1.05 = \$2 \text{ million},$$
$$\text{Deposit} = \frac{\$2{,}000{,}000}{1.05} = \$1{,}904{,}761.91.$$

We need to deposit $\$1{,}904{,}761.91$. This is the *present value* of the $\$2$ million. Similarly, if we need $\$2$ million in 2 years, the amount we need to deposit is given by

$$\text{Deposit} \cdot (1.05)^2 = \$2 \text{ million},$$
$$\text{Deposit} = \frac{\$2{,}000{,}000}{(1.05)^2} = \$1{,}814{,}058.96.$$

The $\$1{,}814{,}058.96$ is the present value of $\$2$ million payable two years from today. In general,

> The **present value**, $\$P$, of a future payment, $\$B$, is the amount that would have to be deposited (at some interest rate, r) in a bank account today to have exactly $\$B$ in the account at the relevant time in the future.

If r is the yearly interest rate (compounded annually) and if n is the number of years, then

$$B = P(1+r)^n, \quad \text{or equivalently,} \quad P = \frac{B}{(1+r)^n}.$$

Calculating the Present Value of Rodriguez's Bonus

The present value of Rodriguez's bonus represents what it was worth on the day it was signed. Suppose that he receives his money in 5 payments of $2 million each, the first payment to be made on the day the contract was signed. We calculate the present values of all 5 payments assuming that interest is compounded annually at a rate of 5% per year. Since the first payment is made the day the contract is signed, we have, in millions of dollars,

$$\text{Present value of first payment} = 2.$$

Since the second payment is made a year in the future, in millions of dollars we have:

$$\text{Present value of second payment} = \frac{2}{(1.05)^1} = \frac{2}{1.05}.$$

The third payment is made two years in the future, so in millions of dollars:

$$\text{Present value of third payment} = \frac{2}{(1.05)^2},$$

and so on. Similarly, in millions of dollars:

$$\text{Present value of fifth payment} = \frac{2}{(1.05)^4}.$$

Thus, in millions of dollars,

$$\text{Total present value} = 2 + \frac{2}{1.05} + \frac{2}{(1.05)^2} + \cdots + \frac{2}{(1.05)^4}.$$

Rewriting this expression, we see that it is a finite geometric series with $a = 2$ and $r = 1/1.05$:

$$\text{Total present value} = 2 + 2\left(\frac{1}{1.05}\right) + 2\left(\frac{1}{1.05}\right)^2 + \cdots + 2\left(\frac{1}{1.05}\right)^4.$$

The formula for the sum of a finite geometric series gives

$$\text{Total present value of contract in millions of dollars} = \frac{2\left(1 - (\frac{1}{1.05})^5\right)}{1 - \frac{1}{1.05}} \approx 9.09.$$

Thus, the total present value of the bonus is about $9.09 million dollars.

Example 2 Suppose Alex Rodriguez's bonus with the Rangers guaranteed him and his heirs an annual payment of $2 million *forever*. How much would the owners need to deposit in an account today in order to provide these payments?

Solution At 5%, the total present value of an infinite series of payments is given by the infinite series

$$\begin{aligned}\text{Total present value} &= 2 + \frac{2}{1.05} + \frac{2}{(1.05)^2} + \cdots \\ &= 2 + 2\left(\frac{1}{1.05}\right) + 2\left(\frac{1}{1.05}\right)^2 + \cdots.\end{aligned}$$

The sum of this infinite geometric series can be found using the formula:

$$\text{Total present value} = \frac{2}{1-\frac{1}{1.05}} = 42 \text{ million dollars.}$$

To see that this answer is reasonable, suppose that \$42 million is deposited in an account today, and that a \$2 million payment is immediately made to Alex Rodriguez. Over the course of a year, the remaining \$40 million earns 5% interest, which works out to \$2 million, so the next year the account again has \$42 million. Thus, it would have cost the Texas Rangers only about \$33 million more (than the \$9.09 million) to pay Rodriguez and his heirs \$2 million a year forever.

Example 3 Using summation notation, write the present value of Alex Rodriguez's bonus, first for the 5-payment case and then for the infinite sequence of payments.

Solution For the 5 payments, we have

$$\text{Total present value} = 2 + \frac{2}{1.05} + \frac{2}{(1.05)^2} + \frac{2}{(1.05)^3} + \frac{2}{(1.05)^4} = \sum_{i=0}^{4} 2\left(\frac{1}{1.05}\right)^i .$$

For the infinite sequence of payments, the series goes to ∞ instead of stopping at 4, so we have

$$\text{Total present value} = 2 + \frac{2}{1.05} + \frac{2}{(1.05)^2} + \cdots = \sum_{i=0}^{\infty} 2\left(\frac{1}{1.05}\right)^i .$$

Exercises and Problems for Section 13.4

Exercises

In Exercises 1–7, decide which of the following are geometric series. For those that are, give the first term and the ratio between successive terms. For those that are not, explain why not.

1. $1 - x + x^2 - x^3 + x^4 - \cdots$
2. $y^2 + y^3 + y^4 + y^5 + \cdots$
3. $1 + x + 2x^2 + 3x^3 + 4x^4 + \cdots$
4. $3 + 3z + 6z^2 + 9z^3 + 12z^4 + \cdots$
5. $e^x + e^{2x} + e^{3x} + e^{4x} + \cdots$
6. $1 - e^{-3x} + e^{-6x} - e^{-9x} + \cdots$
7. $1 + \sqrt{2} + 2 + 2\sqrt{2} + \cdots$
8. Find the sum of the series in Problem 2.
9. Find the sum of the series in Problem 1.

Find the sum of the series in Exercises 10–17.

10. $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$
11. $11 - 11(0.1) + 11(0.1)^2 - \cdots$
12. $\sum_{i=2}^{\infty} (0.1)^i$
13. $\sum_{i=4}^{\infty} \left(\frac{1}{3}\right)^i$
14. $\sum_{i=0}^{\infty} \frac{3^i + 5}{4^i}$
15. $\sum_{i=0}^{\infty} \frac{3^i}{2^{2i}}$
16. $\sum_{j=1}^{\infty} 7((0.1)^j + (0.2)^{j+2})$
17. $\sum_{i=1}^{\infty} x^{2i}, \quad |x| < 1$

Problems

18. A repeating decimal can always be expressed as a fraction. This problem shows how writing a repeating decimal as a geometric series enables you to find the fraction. Consider the decimal $0.232323\ldots$.
    - **(a)** Use the fact that $0.232323\ldots = 0.23 + 0.0023 + 0.000023 + \cdots$ to write $0.232323\ldots$ as a geometric series.
    - **(b)** Use the formula for the sum of a geometric series to show that $0.232323\ldots = 23/99$.

In Problems 19–23, use the method of Problem 18 to write each of the decimals as fractions.

19. $0.235235235\ldots$

20. $6.19191919\ldots$

21. $0.12222222\ldots$

22. $0.4788888\ldots$

23. $0.7638383838\ldots$

24. You have an ear infection and are told to take a 250-mg tablet of ampicillin (a common antibiotic) four times a day (every six hours). It is known that at the end of six hours, about 4% of the drug is still in the body. What quantity of the drug is in the body right after the third tablet? The fortieth? Assuming you continue taking tablets, what happens to the drug level in the long run?

25. In Problem 24 we found the quantity Q_n, the amount (in mg) of ampicillin left in the body right after the n^{th} tablet is taken.

(a) Make a similar calculation for P_n, the quantity of ampicillin (in mg) in the body right *before* the n^{th} tablet is taken.
(b) Find a simplified formula for P_n.
(c) What happens to P_n in the long run? Is this the same as what happens to Q_n? Explain in practical terms why your answer makes sense.

26. Draw a graph like that in Figure 13.3 for 250 mg of ampicillin taken every 6 hours, starting at time $t = 0$. Put on the graph the values of $Q_1, Q_2, Q_3, \ldots$ calculated in Problem 24 and the values of $P_1, P_2, P_3, \ldots$ calculated in Problem 25.

27. Basketball player Patrick Ewing received a contract from the New York Knicks in which he was offered \$3 million a year for ten years. Determine the present value of the contract on the date of the first payment if the interest rate is 7% per year, compounded continuously.

Problems 28–30 are about *bonds*, which are issued by a government to raise money. An individual who buys a \$1000 bond gives the government \$1000 and in return receives a fixed sum of money, called the *coupon*, every six months or every year for the life of the bond. At the time of the last coupon, the individual also gets the \$1000, or *principal*, back.

28. What is the present value of a \$1000 bond that pays \$50 a year for 10 years, starting one year from now? Assume interest rate is 6% per year, compounded annually.

29. What is the present value of a \$1000 bond that pays \$50 a year for 10 years, starting one year from now? Assume the interest rate is 4% per year, compounded annually.

30. **(a)** What is the present value of a \$1000 bond that pays \$50 a year for 10 years, starting one year from now? Assume the interest rate is 5% per year, compounded annually.
(b) Since \$50 is 5% of \$1000, this bond is often called a 5% bond. What does your answer to part (a) tell you about the relationship between the principal and the present value of this bond when the interest rate is 5%?
(c) If the interest rate is more than 5% per year, compounded annually, which one is larger: the principal or the value of the bond? Why do you think the bond is then described as *trading at discount*?
(d) If the interest rate is less than 5% per year, compounded annually, why is the bond described as *trading at a premium*?

CHAPTER SUMMARY

- **Sequences**
  Arithmetic: $a_n = a_1 + (n-1)d$.
  Geometric: $a_n = a_1 r^{n-1}$
- **Sigma Notation**
  $\sum_{i=i}^{n} a_i$
- **Arithmetic Series**
  Partial sum:
  $$S_n = a_1 + a_2 + \cdots + a_n = \frac{1}{2}n(2a_1 + (n-1)d).$$
- **Geometric Series**
  Finite:
  $$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \quad r \neq 1.$$
  Infinite:
  $$S = a + ar + ar^2 + \cdots = \frac{a}{1-r},$$
  converges for $|r| < 1$, does not converge for $|r| > 1$.
- **Applications**
  Bank balance; drug levels.
  Present value: $P = \dfrac{B}{(1+r)^n}$ or $P = \dfrac{B}{e^{kt}}$.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER THIRTEEN

Exercises

1. Find the sum of the first eighteen terms of the series: $8 + 11 + 14 + \cdots$. What is the eighteenth term?

2. Write the following using sigma notation: $100 + 90 + 80 + 70 + \cdots + 0$.

3. **(a)** Write the sum $\displaystyle\sum_{n=1}^{5}(4n-3)$ in expanded form.

(b) Compute the sum.

In Exercises 4–6, decide which of the following are geometric series. For those that are, give the first term and the ratio between successive terms. For those that are not, explain why not.

4. $1 + 2z + (2z)^2 + (2z)^3 + \cdots$

5. $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$

6. $1 - y^2 + y^4 - y^6 + \cdots$

7. Find the sum of the series in Problem 4.

8. Find the sum of the series in Problem 6.

9. Find the sum of the first nine terms of the series: $7 + 14 + 21 + \cdots$.

Problems

10. Figure 13.4 shows the first four members in a sequence of square-shaped grids. In each successive grid, new dots are shown in black (●). For instance, the second grid has 3 more dots than the first grid, the third has 5 more dots than the second, and the fourth has 7 more dots than the third.

(a) Write down the sequence of the total number of dots in each grid (black and white). Using the fact that each grid is a square, find a formula in terms of n for the number of dots in the n^{th} grid.

(b) Write down the sequence of the number of black dots in each grid. Find a formula in terms of n for the number of black dots in the n^{th} grid.

(c) State the relationship between the two sequences in parts (a) and (b). Use the formulas you found in parts (a) and (b) to confirm this relationship.

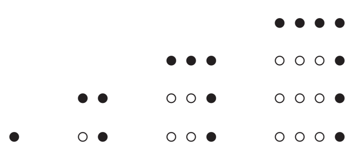


Figure 13.4

11. The numbers in the sequence 1, 4, 9, 16, ... from Problem 10 are known as *square numbers* because they describe the number of dots in successive square grids. Analogously, the numbers in the sequence 1, 3, 6, 10, ... are known as *triangular numbers* because they describe the number of dots in successive triangular patterns, as shown in Figure 13.5: Find a formula for the n^{th} triangular number.

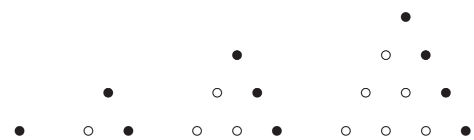


Figure 13.5

12. In a workshop, it costs \$300 to make one piece of furniture. The second piece costs a bit less, \$280. The third costs even less, \$263, and the fourth costs only \$249. The cost for each additional piece of furniture is called the *marginal cost* of production. Table 13.3 gives the marginal cost, c, and the change in marginal cost, Δc, in terms of the number of pierces of furniture, n. As the quantity produced increases, the marginal cost generally decreases and then increases again.

Table 13.3

n	1		2		3		4
c (\$)	300		280		263		249
Δc (\$)		-20		-17		-14	

(a) Assume that the arithmetic sequence -20, -17, -14, ..., continues. Complete the table for $n = 5$, 6, ..., 12.

(b) Find a formula for c_n, the marginal cost for producing the n^{th} piece of furniture. Use the fact that c_n is found by adding the terms in an arithmetic sequence. Using your formula, find the cost for producing the 12^{th} piece and the 50^{th} piece of furniture.

(c) A piece of furniture can be sold at a profit if it costs less than \$400 to make. How many pieces of furniture should the workshop make each day? Discuss.

13. A store clerk has 108 cans to stack. He can fit 24 cans on the bottom row and can stack the cans 8 rows high. Use arithmetic series to determine how he can stack the cans so that each row contains fewer cans than the row beneath it and that the number of cans in each row decreases at a constant rate.

14. A university with an enrollment of 8000 students in 2007 is projected to grow by 2% in each of the next three years and by 3.5% in each of the following seven years. Find the sequence of the university's student enrollment for the next 10 years.

15. Each person in a group of 30 shakes hands with each other person exactly once. How many total handshakes take place?

16. A bank account with a \$75,000 initial deposit is used to make annual payments of \$1000, starting one year after the initial \$75,000 deposit. Interest is earned at 2% a year, compounded annually, and paid into the account right before the payment is made.

(a) What is the balance in the account right after the 24^{th} payment?

(b) Answer the same question for yearly payments of \$3000.

17. One way of valuing a company is to calculate the present value of all its future earnings. A farm expects to sell \$1000 worth of Christmas trees once a year forever, with the first sale in the immediate future. What is the present value of this Christmas tree business? Assume that the interest rate is 4% per year, compounded continuously.

18. You inherit \$100,000 and put the money in a bank account earning 3% per year, compounded annually. You withdraw \$2000 from the account each year, right after the interest is earned. Your first \$2000 is withdrawn before any interest is earned.

(a) Compare the balance in the account right after the first withdrawal and right after the second withdrawal. Which do you expect to be higher?

(b) Calculate the balance in the account right after the 20^{th} withdrawal is made.

(c) What is the largest yearly withdrawal you can take from this account without the balance decreasing over time?

19. After breaking his leg, a patient retrains his muscles by going for walks. The first day, he manages to walk 300 yards. Each day after that he walks 50 yards farther than the day before.

(a) Write a sequence that represents the distances walked each day during the first week.

(b) How far is he walking after two weeks?

(c) How long until he is walking at least one mile?

20. Before email made it easy to contact many people quickly, groups used telephone trees to pass news to their members. In one group, each person is in charge of calling 4 people. One person starts the tree by calling 4 people. At the second stage, each of these 4 people call another 4 people. In the third stage, each of the people in stage two calls 4 people, and so on.

(a) How many people have the news by the end of the 5^{th} stage?

(b) Write a formula for the total number of members in a tree of 10 stages.

(c) How many stages are required to cover a group with 5000 members?

21. A ball is dropped from a height of 10 feet and bounces. Each bounce is $3/4$ of the height of the bounce before. Thus after the ball hits the floor for the first time, the ball rises to a height of $10(3/4) = 7.5$ feet, and after it hits the floor for the second time, it rises to a height of $7.5(3/4) = 10(3/4)^2 = 5.625$ feet.

(a) Find an expression for the height to which the ball rises after it hits the floor for the n^{th} time.

(b) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the first, second, third, and fourth times.

(c) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the n^{th} time. Express your answer in closed form.

22. You might think that the ball in Problem 21 keeps bouncing forever since it takes infinitely many bounces. This is not true! It can be shown that a ball dropped from a height of h feet reaches the ground in $\frac{1}{4}\sqrt{h}$ seconds. It is also true that it takes a bouncing ball the same amount of time to rise h feet. Use these facts to show that the ball in Problem 21 stops bouncing after

$$\frac{1}{4}\sqrt{10} + \frac{1}{2}\sqrt{10}\sqrt{\frac{3}{4}}\left(\frac{1}{1-\sqrt{3/4}}\right)$$

seconds, or approximately 11 seconds.

23. This problem illustrates how banks create credit and can thereby lend out more money than has been deposited.

Suppose that initially \$100 is deposited in a bank. Experience has shown bankers that on average only 8% of the money deposited is withdrawn by the owner at any time. Consequently, bankers feel free to lend out 92% of their deposits. Thus \$92 of the original \$100 is loaned out to other customers (to start a business, for example). This \$92 will become someone else's income and, sooner or later, will be redeposited in the bank. Then 92% of \$92, or $\$92(0.92) = \$84.64$, is loaned out again and eventually redeposited. Of the \$84.64, the bank again loans out 92%, and so on.

(a) Find the total amount of money deposited in the bank.

(b) The total amount of money deposited divided by the original deposit is called the *credit multiplier*. Calculate the credit multiplier for this example and explain what this number tells us.

24. Take a rectangle whose sides have length 1 and 2, divide it into two equal pieces, and shade one of them. Now, divide the unshaded half into two equal pieces and shade one of them. Take the unshaded area and divide it into two equal pieces and shade one of them. Continue doing this.

(a) Draw a picture that illustrates this process.

(b) Find a series that describes the shaded area after n divisions.

(c) If you continue this process indefinitely, what would the total shaded area be? Make two arguments, one using your picture, another using the series you found in part (b).

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–34 true or false? Give an explanation for your answer.

1. If $a_n = n^2 + 1$, then $a_1 = 2$.

2. The first term of a sequence must be a positive integer.

3. The sequence 1, 1, 1, 1, ... is both arithmetic and geometric.

4. The sequence $a, 2a, 3a, 4a$ is arithmetic.

5. The sequence $a, a+1, a+2, a+3$ is arithmetic.

6. If the difference between successive terms of an arithmetic sequence is d, then $a_n = a_1 + dn - d$.

7. The sequence $3x^2, 3x^4, 3x^6, \ldots$ is geometric.

8. The sequence $2, 4, 2, 4, 2, 4, \ldots$ is alternating.

9. For all series $S_1 = a_1$.

10. The accumulation of college credit is an example of a finite series.

11. The sum formula for a finite series with n terms is $S_n = \frac{1}{2}n(n+1)$.

12. There are four terms in $\displaystyle\sum_{i=0}^{4} a_i$.

13. $\displaystyle\sum_{i=1}^{n} 3 = 3n$.

14. The sum of a finite arithmetic series is the average of the first and last terms, times the number of terms.

15. If C_n denotes your college credits earned by the end of year n, then C_4 is your credits earned during your fourth year of college.

16. If C_n denotes your college credits earned by the end of year n, then $C_3 - C_2$ is your credits earned in year 2.

17. For any series S_n, we must have $S_3 \le S_4$.

18. If $a_n = (-1)^{n+1}$ then the sequence is alternating.

19. The sum formula for a finite geometric series, $S_n = \dfrac{a(1-r^n)}{1-r}$, can be simplified by dividing by $(1-r)$ to get $S_n = a(1-r^{n-1})$.

20. With the same interest rate, a bank balance after 5 years of \$2000 annual payments is the same as the balance after by 10 years of \$1000 annual payments.

21. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$ is a geometric series.

22. $2000(1.06)^3 + 2000(1.06)^4 + \cdots + 2000(1.06)^{10}$ is a geometric series.

23. $5000 + 2000(1.06) + 2000(1.06)^2 \cdots + 2000(1.06)^{10}$ is a geometric series.

24. $5000 + 2000(1.06) + 2000(1.06)^2 \cdots + 2000(1.06)^{10} = 3000 + \dfrac{2000(1-1.06^{11})}{-0.06}$.

25. Payments of \$2000 are made yearly into an account earning 5% interest compounded annually. If the interest rate is doubled, the balance at the end of 20 years is doubled.

26. The present value of a sequence of payments is the amount that you would need to invest at this moment in order to exactly make the future payments.

27. If the present value of a sequence of payments is 1 million dollars, then the present value of a sequence of payments which are doubled in size, would be 2 million.

28. The series $\displaystyle\sum_{i=0}^{\infty} \frac{1}{i^2}$ is geometric.

29. For $Q_n = \sum_{i=1}^{n}(-1)^i$, we see $Q_2 = 0$, $Q_4 = 0$, $Q_6 = 0$, etc. Thus, the series converges to a sum of $Q = \sum_{i=1}^{\infty}(-1)^i = 0$.

30. A patient takes 100 mg of a drug each day; 5% of the amount of the drug in the body decays each day. Then, the long-run level of the drug in the body is 2000 mg.

31. If $S = a + ar + ar^2 + \cdots$ and $|a| < 1$ then the infinite series converges.

32. It is possible to add an infinite number of positive terms and get a finite sum.

33. The sum of an infinite arithmetic series with difference $d = 0.01$ converges.

34. All infinite geometric series converge.