Finding Probabilities

16-1 Introduction to Probability

Objective *To find a sample space of an experiment and the probability of an event or either of two events.*

Probability theory is the branch of mathematics that deals with uncertainty. Although the foundations of probability theory were laid in the 1500s and 1600s by mathematicians who were interested in questions about gambling, the subject has since become associated with many other fields, such as meteorology and genetics.

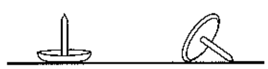
At its most basic level, probability theory assigns a number (between 0 and 1, inclusive) to an *event* as a means of indicating the *probability*, or likelihood, of the occurrence of the event. For example, if a weather report states that the probability of precipitation tomorrow is 60%, then in the past, whenever weather conditions similar to the current conditions existed, it rained 3 out of 5 times.

The probability of an event is determined either *empirically* or *theoretically*. The precipitation probability mentioned above is an example of an empirical probability because it is based on previous observations of the weather. Another example of empirical probability is found in the following activity.

Activity

For this activity, you should work with a partner. You and your partner will need a cup and a thumbtack with a flat head. (All the thumbtacks used by the class should be alike.)

a. Repeat the following at least 20 times: Place the thumbtack in the cup, shake it, and "pour" it onto your desk. Your partner should then record whether the tack lands "point up" or "point down," as shown at the right.



point up　　point down

b. Based on your record of results from part (a), what would you say is the probability that the tack lands "point up"?

c. Combine the results from part (a) for all the pairs of partners in the class. Based on the combined results, what would you say is the probability that the tack lands "point up"?

d. Of the two probabilities from parts (b) and (c), which do you think is a better predictor of the tack's "behavior"? Explain your reasoning.

e. Describe how changing the shape of the tack would affect the probability that it lands "point up."

As Mark Twain reportedly said, "Everybody talks about the weather, but nobody ever does anything about it!" *Probability theory,* however, and records of similar conditions allow us to at least make reasonable weather predictions.

Teaching Notes, p. 596B

Warm-Up Exercises

1. You toss a dime twice. Copy and complete the table below to show all possible outcomes. You can get "heads" or "tails" on each toss.

First toss	Second toss
H	H
H	T
T	H
T	T

2. Which principle explains the number of possible outcomes? The mult. principle; there are 2 outcomes (H or T) for each toss, so there are $2 \cdot 2 = 4$ outcomes in all.

3. Find the probability that you get the same outcome on both tosses? $\frac{2}{4} = \frac{1}{2}$

4. Find the probability that you get at least one "heads." $\frac{3}{4}$

5. Find the probability that you get "heads" on the first toss. $\frac{1}{2}$

Motivating the Section

Ask students to identify some situations in which probabilities are used. (Two such situations are weather forecasts (precipitation probabilities) and sports contests (odds of winning).)

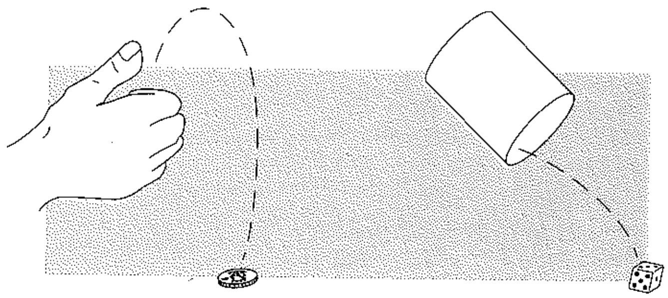
Error Analysis

Emphasize the phrase "exactly one" in the definition of a sample space and ask students to give examples of the two different errors possible if that phrase is ignored: a set that has no entry corresponding to a possible outcome and a set that has more than one entry for a possible outcome. For the experiment of selecting a random coin from the set of all United States coins, the first error is represented by {penny, nickel, dime, quarter}, which omits the half dollar and the silver dollar. The second error is represented by {copper coin, silver coin, coin from Philadelphia mint, coin from Denver mint} since both mints produce copper and silver coins.

Communication Note

You may need to point out to students that "die" is the singular of "dice."

Although empirical probabilities are unavoidable in many applied areas of probability theory, in this chapter we will usually consider events whose probabilities can be determined simply by reasoning about the events. For example, by assuming that the mass of a coin is evenly distributed so that each of the two sides (which are called "heads" and "tails") has an equal chance of turning up when the coin is flipped, we can say that the theoretical probability of getting "heads" is $\frac{1}{2}$. Likewise, the shape and composition of a die lead us to believe that each of the six faces has an equal chance of turning up, so the theoretical probability of rolling, say, a "3" is $\frac{1}{6}$.



At this point, let us generalize our discussion of theoretical probability. We refer to an action having various outcomes that occur unpredictably (such as tossing a coin or rolling a die) as an **experiment**. A **sample space** of an experiment is a set S such that each outcome of the experiment corresponds to exactly one element of S. An **event** is any subset of a sample space. If a sample space of an experiment contains n equally likely outcomes and if m of the n outcomes correspond to some event A, then the **probability** of event A, denoted $P(A)$, is:

$$P(A) = \frac{m}{n}$$

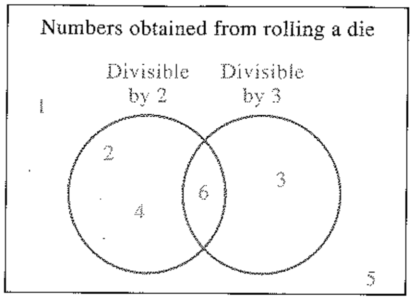
If all n outcomes correspond to event A, then the event is certain to occur and $P(A) = \frac{n}{n} = 1$. Similarly, if no outcomes correspond to event A, then the event is certain *not* to occur and $P(A) = \frac{0}{n} = 0$.

Example 1 Suppose a die is rolled. Give a sample space for this experiment. Then find the probability of rolling a prime number.

Solution A sample space for the die-rolling experiment is {1, 2, 3, 4, 5, 6}. Of the six equally likely outcomes in the sample space, three outcomes (2, 3, and 5) correspond to the event "rolling a prime number." Thus:

$$P(\text{prime number}) = \frac{3}{6} = \frac{1}{2}$$

To determine the probability of the occurrence of either one of two different events, we can use a Venn diagram (introduced in Section 15-1). For example, in the roll of a die, consider the events "rolling a number divisible by 2" and "rolling a number divisible by 3." The Venn diagram at the right shows these two events. Notice that one number, 6, is common to both events, so that:



$$P\binom{\text{divisible}}{\text{by 2 } or \text{ 3}} = P\binom{\text{divisible}}{\text{by 2}} + P\binom{\text{divisible}}{\text{by 3}} - P\binom{\text{divisible}}{\text{by 2 } and \text{ 3}}$$

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{2}{3}$$

Note that this result is based on the inclusion-exclusion principle (page 566).

Probability of Either of Two Events

For any two events A and B,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Two events A and B that cannot occur simultaneously are called **mutually exclusive** events. For example, in the roll of a die, the events "rolling a 3" and "rolling an even number" are mutually exclusive. Moreover:

$$P(3 \text{ or even number}) = P(3) + P(\text{even number})$$

$$= \frac{1}{6} + \frac{3}{6} = \frac{2}{3}$$

This result, generalized below, is a special case of the result stated above.

Probability of Either of Two Mutually Exclusive Events

If events A and B are mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B).$$

The event "not A" occurs when event A does not. Since events A and "not A" are mutually exclusive and since one or the other is certain to occur, we have:

$$P(A \text{ or not } A) = 1$$

$$P(A) + P(\text{not } A) = 1$$

$$P(\text{not } A) = 1 - P(A)$$

Review Note

Ask students how a Venn diagram can be drawn for mutually exclusive events. Be sure you get both answers: drawing two circles that do not overlap, and drawing two overlapping circles for which the intersection is empty.

Mathematical Note

Students should recognize that events A and B are mutually exclusive if and only if $P(A \text{ and } B) = 0$.

Problem Solving

The discussion at the bottom of the page offers students a good problem-solving technique: If finding the probability of an event is too difficult to do directly, try finding the probability of the event's complement and subtracting the probability from 1.

Additional Examples

1. Suppose you toss a coin four times.
   a. How many different equally likely outcomes are possible?
   b. Find the probability of obtaining no heads.
   c. Find the probability of obtaining at least one head.
   d. Find the probability of obtaining exactly one head.

   a. On each toss, there are two outcomes, heads and tails. By the multiplication principle, there are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ possible outcomes, each of which is equally likely to occur.

   b. $P(\text{TTTT}) = \frac{1}{16}$

   c. The event of obtaining at least one head is the complement of the event in part (b), so $P(\text{at least one head}) = 1 - \frac{1}{16} = \frac{15}{16}$

   d. $P(\text{exactly one head}) = P(\text{HTTT, THTT, TTHT, TTTH}) = \frac{4}{16} = \frac{1}{4}$

2. Each of five cards is labeled with a letter A, B, C, D, or E. Two cards are chosen without the first card being replaced.
   a. List all the possible outcomes.
   b. Find the probability that both letters chosen are consonants.

   a. $S = \{AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED\}$

Example 2 As shown below, a standard deck of playing cards consists of 52 cards, with 13 cards in each of four *suits* (clubs, spades, diamonds, and hearts). Clubs and spades are black cards, and diamonds and hearts are red cards. Also, jacks, queens, and kings are called *face cards*.

Clubs (♣): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king
Spades (♠): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king
Diamonds (♦): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king
Hearts (♥): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

If the deck is well-shuffled, what is the probability that the top card is:
a. a black ace? **b.** not a black ace? **c.** a diamond face card?

Solution **a.** Since the cards are well-shuffled, any one of them has the same chance of being the top card. With 2 black aces in the deck,

$$P(\text{black ace}) = \frac{2}{52} = \frac{1}{26}.$$

b. $P(\text{not a black ace}) = 1 - P(\text{black ace})$

$$= 1 - \frac{1}{26} = \frac{25}{26}$$

c. With 3 diamond face cards in the deck,

$$P(\text{diamond face card}) = \frac{3}{52}.$$

In Example 2, notice that the card-choosing experiment has such sample spaces as:

S_1 = the set of 52 cards listed in Example 2
S_2 = {club, spade, diamond, heart}
S_3 = {black card, red card}
S_4 = {spade, non-spade}

We often refer to the elements of a sample space as *sample points*. For example, the outcome of choosing, say, the 7 of clubs corresponds to the sample point "club" in the sample space S_2, to the sample point "black card" in the sample space S_3, and to the sample point "non-spade" in the sample space S_4.

If you were asked for the probability of drawing a spade from a standard deck, sample space S_4 would not be helpful, because its two sample points "spade" and "non-spade" are not equally likely. (Thus, it would be incorrect to say that the probability of a spade is $\frac{1}{2}$.) On the other hand, the four sample points of sample space S_2 *are* equally likely, so we can say $P(\text{spade}) = \frac{1}{4}$. We can also use the 52 equally likely sample points of S_1 and say $P(\text{spade}) = \frac{13}{52} = \frac{1}{4}$.

Example 3 Suppose a red die and a white die are rolled. What is the probability that the sum of the numbers showing on the dice is 9 or 10?

Solution The table below gives a sample space consisting of 36 ordered pairs (r, w) where r and w are the numbers showing on the red die and the white die.

		White die					
		1	2	3	4	5	6
	1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
Red	3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
die	4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

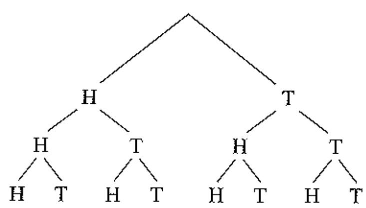
Sum = 9 Sum = 10

Since four of the equally likely sample points give a sum of 9 and three give a sum of 10, $P(\text{sum is 9 or 10}) = \frac{4}{36} + \frac{3}{36} = \frac{7}{36}$.

Sometimes it is convenient to use a tree diagram to obtain a sample space, as shown in the next example.

Example 4 Suppose you toss a coin three times. What is the probability that exactly two of the tosses result in "heads"?

Solution The tree diagram at the right illustrates the experiment of tossing a coin three times and obtaining a "head" (H) or a "tail" (T) on each toss. By following each path of the tree, you get:



Sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Thus, $P(2 \text{ "heads"}) = P(\text{HHT or HTH or THH}) = \frac{3}{8}$.

The chances of winning a sporting event are often given as *odds* instead of as a probability. If the odds in favor of a team winning are 6 to 5, this means that the team's winning chances to its losing chances are in a 6 to 5 ratio. In other words, the probability of winning is $\frac{6}{6+5} = \frac{6}{11}$, and the probability of losing is $\frac{5}{11}$.

Additional Examples cont.

b. $P(BC, BD, CB, CD, DB, DC) = \frac{6}{20} = \frac{3}{10}$. (Note that there are three consonants, so there are $3 \cdot 2 = 6$ ways of choosing two without replacement.)

Using Technology

A computer or programmable calculator can be used to simulate any of the situations described in the examples of this section. For instance, the following program simulates Example 4:

```
10  PRINT "HOW MANY
    TRIALS";
20  INPUT N
30  LET S = 0
40  FOR I = 1 TO N
50  LET H = 0
60  FOR T = 1 TO 3
70  IF RND(1) < 0.5 THEN
    LET H = H + 1
80  NEXT T
90  IF H = 2 THEN LET S =
    S + 1
100 NEXT I
110 PRINT S;" OUT OF ";N;
    "TRIALS RESULTED IN
    EXACTLY TWO HEADS."
120 PRINT "PROBABILITY OF
    GETTING TWO HEADS IN
    THREE TOSSES OF A
    COIN IS ABOUT "; S/N
130 END
```

Of course, this program gives an *empirical* probability. Students who use this program should see that the empirical probability approximates the theoretical probability better as the number of trials increases.

Students may wish to write simulation programs for the other examples in this section.

Assessment Note

Have students translate into odds the probabilities of getting 0, 1, 2, or 3 heads when three coins are tossed. (The odds in favor are 1:7 for 0 heads and 3 heads, and 3:5 for 1 head and 2 heads.)

Additional Answers Class Exercises

9. S is not a sample space because some outcomes, such as queen of hearts, correspond to more than one element in S.

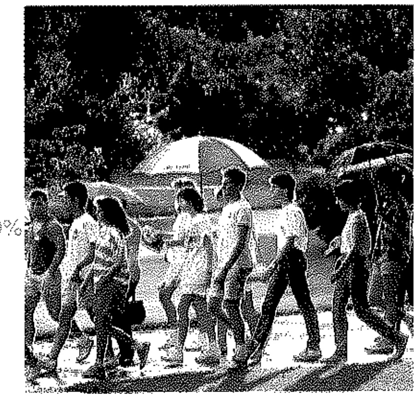
CLASS EXERCISES

1. As in Example 1, suppose a die is rolled. Find each probability.
a. P(perfect square) $\frac{1}{3}$ **b.** P(factor of 60) 1 **c.** P(negative number) 0

2. If a card is drawn at random from a standard deck of 52 cards, what is the probability of getting:
a. the queen of hearts? $\frac{1}{52}$ **b.** a heart? $\frac{1}{4}$ **c.** a queen? $\frac{1}{13}$
d. a red card? $\frac{1}{2}$ **e.** a face card? $\frac{3}{13}$ **f.** a red face card? $\frac{3}{26}$

3. If two dice are rolled, what is the probability that both show the same number? $\frac{1}{6}$

4. If two dice are rolled, find the probability of getting:
a. a sum of 3 $\frac{1}{18}$
b. a sum of 4 $\frac{1}{12}$
c. a sum of 3 or 4 $\frac{5}{36}$



5. **Meteorology** If the probability of rain tomorrow is 40%, what is the probability of no rain tomorrow? 60%

6. **Occupational Safety** If the probability of no accidents in a manufacturing plant during one month is 0.82, what is the probability of at least one accident? 0.18

7. ***Discussion*** In the solution of Example 3, another possible sample space is the set of the 11 possible sums for the two dice:

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Since two of these 11 possibilities correspond to a sum of 9 or 10, some people might be tempted to say that the probability of a sum of 9 or 10 is $\frac{2}{11}$. Discuss why this reasoning is incorrect. The possible sums are not equally likely.

8. A penny, a nickel, and a dime are each tossed.
a. Does the set $S = \{0, 1, 2, 3\}$, which gives the four different numbers of "heads" that could come up, satisfy the definition of a sample space? Yes
b. What is wrong with reasoning that since one of the four sample points in part (a) corresponds to 2 "heads," then $P(2 \text{ "heads"}) = \frac{1}{4}$? The sample points are not equally likely.

9. A card is picked at random from a standard deck. Explain why the set S = {club, spade, red card, face card} is *not* a sample space.

10. Comment on the following reasoning:

There are 3 states whose names begin with the letter C (California, Colorado, and Connecticut); call them "C-states." Likewise, there are 3 "O-states" (Ohio, Oklahoma, and Oregon). Thus, if a person is chosen at random from the U.S. population, that person has the same probability of being from a "C-state" as from an "O-state." This is incorrect because population is not evenly distributed among the states.

WRITTEN EXERCISES

For Exercises 1–4, suppose a card is drawn from a well-shuffled standard deck of 52 cards. Find the probability of drawing each of the following.

A

1. **a.** a black card $\frac{1}{2}$ **b.** a spade $\frac{1}{4}$ **c.** not a spade $\frac{3}{4}$
2. **a.** a black face card $\frac{3}{26}$ **b.** a black jack $\frac{1}{26}$ **c.** not a black jack $\frac{25}{26}$
3. **a.** a red diamond $\frac{1}{4}$ **b.** a black diamond 0 **c.** not a black diamond 1
4. **a.** a jack $\frac{1}{13}$ **b.** a jack or king $\frac{2}{13}$ **c.** neither jack nor king $\frac{11}{13}$
5. Mr. and Mrs. Smith each bought 10 raffle tickets. Each of their three children bought 4 tickets. If 4280 tickets were sold in all, what is the probability that the grand prize winner is:
   a. Mr. or Mrs. Smith? $\frac{1}{214}$ **b.** one of the 5 Smiths? $\frac{4}{535}$ **c.** none of the Smiths? $\frac{531}{535}$
6. One of the integers between 11 and 20, inclusive, is picked at random. What is the probability that the integer is:
   a. even? $\frac{1}{2}$ **b.** divisible by 3? $\frac{3}{10}$ **c.** a prime? $\frac{2}{5}$
7. New York City is divided into five boroughs: Manhattan, Queens, the Bronx, Brooklyn, and Staten Island. Suppose that a New York City telephone number is randomly chosen.
   a. Explain why the probability that it is a Manhattan telephone number is *not* $\frac{1}{5}$.
   b. What do you need to know in order to find the correct probability in part (a)?

8. Suppose that a member of the U.S. Senate and a member of the U.S. House of Representatives are randomly chosen to be photographed with the President. Explain why $\frac{1}{50}$ is the probability that the senator is from Iowa, and why $\frac{1}{50}$ is *not* the probability that the representative is from Iowa.

In Exercises 9–12, use the table on page 601, which gives the 36 equally likely outcomes when two dice are rolled. Find the probability of each event.

9. **a.** Sum is 6. $\frac{5}{36}$ **b.** Sum is 7. $\frac{1}{6}$ **c.** Sum is 8. $\frac{5}{36}$
10. **a.** Sum is even. $\frac{1}{2}$ **b.** Sum is 12. $\frac{1}{36}$ **c.** Sum is less than 12. $\frac{35}{36}$
11. The two dice show different numbers. $\frac{5}{6}$
12. The red die shows a greater number than the white die. $\frac{5}{12}$

Additional Answers Written Exercises

7. a. There are not an equal number of phone numbers in each borough.
   b. The number of phone numbers in Manhattan and the number of phone numbers in New York City
8. Each state has an equal number of senators, but the number of representatives varies according to the population of the state.

Suggested Assignments

Discrete Math
603/1, 5, 7, 9, 15, 17, 25, 27, 29

Supplementary Materials

Alternative Assessment, 49
Activities Book, 44–45

Additional Answers Written Exercises

13. b. $\frac{1}{4}$

14. b. $\frac{1}{3}$

19. a. {{Alvin, Bob}, {Alvin, Carol}, {Alvin, Donna}, {Bob, Carol}, {Bob, Donna}, {Carol, Donna}}
b. $\frac{1}{6}$
c. $\frac{1}{2}$

20. a. {(black, black), (black, red), (red, black), (red, red)}
b. After a red card is drawn, a black card is more likely to be drawn because there are 26 black cards left and only 25 red cards.

21. The inclusion-exclusion principle states that "For any sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$." If A and B are events in a sample space of s equally likely outcomes, then dividing both sides of the above equation by s will turn each term into a probability: $\frac{n(A \cup B)}{s} = \frac{n(A)}{s} + \frac{n(B)}{s} - \frac{n(A \cap B)}{s}$ becomes $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, which is the equation for the probability of either of two events.

13. A die is rolled and a coin is tossed.
a. Make a tree diagram showing the 12 possible outcomes of this experiment.
b. Find the probability that the die's number is even and the coin is "heads."

14. The numbers 1, 2, 3, and 4 are written on separate slips of paper and placed in a hat. Two slips of paper are then randomly drawn, one after the other and without replacement.
a. Make a tree diagram showing the 12 possible outcomes of this experiment.
b. Find the probability that the sum of the numbers picked is 6 or more.

15. Suppose that the odds in favor of the National League's winning the All-Star Game are 4 to 3.
a. What is the probability that the National League wins? $\frac{4}{7}$
b. What is the probability that the American League wins? $\frac{3}{7}$

16. Suppose that the odds in favor of the incumbent's winning an election are 2 to 3.
a. What is the probability that the incumbent wins? $\frac{2}{5}$
b. What is the probability that the incumbent's challenger wins? $\frac{3}{5}$

B **17.** Suppose you roll two dice, each of which is a regular octahedron with faces numbered 1 to 8.
a. What is the probability that the sum of the numbers showing is 2? $\frac{1}{64}$
b. What is the probability that the sum is 3? $\frac{1}{32}$
c. What sum is most likely to appear? 9

18. Suppose you roll two dice, each of which is a regular dodecahedron with faces numbered 1 to 12.
a. What is the probability that the sum of the numbers showing is 24? $\frac{1}{144}$
b. What is the probability that the sum is 23? $\frac{1}{72}$
c. What sum is most likely to appear? 13

19. From a group consisting of Alvin, Bob, Carol, and Donna, two people are to be randomly selected to serve on a committee.
a. Give a sample space for this experiment.
b. Find the probability that Bob and Carol are selected.
c. Find the probability that Carol is not selected.

20. From a standard deck of cards, two cards are randomly drawn, one after the other and without replacement. The color of each card is noted.
a. Give a sample space for this experiment.
b. Explain why two red cards are less likely to be drawn than a red card and then a black card.

21. ***Reading*** Explain how the inclusion-exclusion principle (see page 566) can be used to find the probability of either of two events (see page 599).

22. Of the 1260 households in a small town, 632 have dogs, 568 have cats, and 114 have both types of pet. If a household is chosen at random, what is the probability that the household has either type of pet? $\frac{181}{210}$
23. The letters of the word TEXAS are arranged in a random order. What is the probability that the letters spell TAXES? $\frac{1}{120}$
24. The letters of the word COSINE are arranged in a random order. What is the probability that the letters spell SONICE? $\frac{1}{720}$
25. A number k is randomly chosen from $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. What is the probability that the expression $x^2 - 2x + k$ can be written as a product of two linear factors, each with integral coefficients? $\frac{3}{8}$
26. A number c is randomly chosen from $\{1, 2, 3, 4, 5, 6\}$. What is the probability that the graph of $y = x^2 - 4x + c$ intersects the x-axis? $\frac{2}{3}$
27. If a 3-letter "word" is formed by randomly choosing 3 letters from the word OCEAN, what is the probability that it is composed only of vowels? $\frac{1}{10}$
28. If a 3-letter "word" is formed by randomly choosing 3 letters from the word PAINTED, what is the probability that it is composed only of vowels? $\frac{1}{35}$
29. A number between 100 and 999, inclusive, is chosen at random. What is the probability that it contains **(a)** no 0's? **(b)** at least one 0? a. $\frac{81}{100}$ b. $\frac{19}{100}$
30. A number between 1000 and 9999, inclusive, is chosen at random. What is the probability that it contains **(a)** no 9's? **(b)** at least one 9? a. $\frac{81}{125}$ b. $\frac{44}{125}$

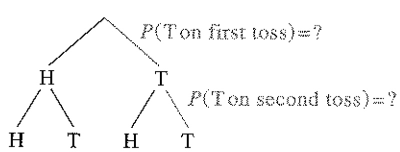
16-2 Probability of Events Occurring Together

Objective *To find the probability of events occurring together and to determine whether two events are independent.*

If an experiment involves two or more events occurring together (whether successively or simultaneously), a tree diagram is a useful way to calculate the probabilities of the possible outcomes.

Activity

The tree diagram at the right illustrates an experiment in which a coin is tossed twice. (In the diagram, H and T represent "heads" and "tails," respectively.)



a. Use the tree diagram to give a sample space for the experiment. {HH, HT, TH, TT}
b. Use your answer to part (a) to find $P(\text{TT})$. $\frac{1}{4}$
c. What are the probabilities indicated in red in the tree diagram? $\frac{1}{2}$; $\frac{1}{2}$
d. How could you obtain your answer to part (b) using your answers to part (c)? Multiply.
e. If the coin is tossed a third time, use the method from part (d) to find $P(\text{TTT})$. $\frac{1}{8}$

Teaching Notes, p. 596B

Warm-Up Exercises

A pile of cards consists of three red cards, two blue cards, and one white card.

1. You draw one card from the pile without looking. What is the probability that the card is:
   a. red? $\frac{1}{2}$ **b.** blue? $\frac{1}{3}$
   c. white? $\frac{1}{6}$
2. Assume that on your first drawing the card you selected was white and that it *was not* placed back in the pile. You then draw another card (without looking) from the remaining cards in the pile. What is the probability that this card is:
   a. red? $\frac{3}{5}$ **b.** blue? $\frac{2}{5}$
   c. white? 0
3. Assume again that the card you selected on the first drawing was white, but that it *was* placed back in the pile before you draw a second time. What is the probability that this second card is:
   a. red? $\frac{1}{2}$ **b.** blue? $\frac{1}{3}$
   c. white? $\frac{1}{6}$
4. Compare your answers for Exercises 2 and 3 to those for Exercise 1. Does replacement of the card selected in the first drawing affect the probabilities for the color of the card selected in the second drawing? yes

Motivating the Section

Pose the following question: If a person in this school building is chosen at random, which of the following events has the *least* probability: (a) the person is a teacher, (b) the person wears glasses, or (c) the person is a teacher who wears glasses? Students should choose event (c), because both events (a) and (b) must occur for event (c) to occur. The probability of events occurring together is the topic of this section.

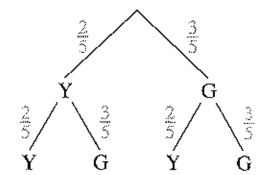
Example Note

Before discussing Example 2, ask students if they think that not replacing the balls will increase the probability, decrease it, or leave it unchanged.

As the activity on the preceding page shows, if you know the probabilities that correspond to the branches of a tree diagram for some experiment, then you can find the probability of a particular outcome by multiplying the probabilities along the path leading to that outcome. We use this approach in the following examples.

Example 1 Two yellow balls and three green balls are placed in a jar. One ball is randomly chosen, its color is noted, and the ball is put back in the jar. This procedure is repeated for a second ball. Find the probability that both balls are the same color.

Solution The probability of a yellow (Y) first ball is $\frac{2}{5}$, and the probability of a green (G) first ball is $\frac{3}{5}$. Because the first ball is put back in the jar before the second ball is chosen, the second ball has the same probabilities of being yellow and of being green as the first ball. These probabilities are shown next to the branches of the tree diagram at the right. Thus, the probability of getting two yellow balls is:



$$P(\text{YY}) = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$$

Similarly:

$$P(\text{YG}) = \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}$$

$$P(\text{GY}) = \frac{3}{5} \cdot \frac{2}{5} = \frac{6}{25}$$

$$P(\text{GG}) = \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}$$

Note that the sum of these probabilities is 1.

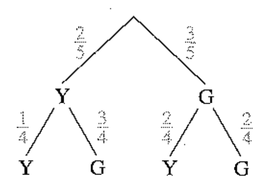
From the above calculations, we find that

$$\begin{aligned} P(\text{both same color}) &= P(\text{YY}) + P(\text{GG}) \\ &= \frac{4}{25} + \frac{9}{25} = \frac{13}{25}. \end{aligned}$$

Example 2 Rework Example 1 if the first ball is *not* put back in the jar before the second ball is chosen.

Solution In this situation, the probability of a yellow (Y) first ball is still $\frac{2}{5}$, and the probability of a green (G) first ball is still $\frac{3}{5}$. The probabilities for the second ball, however, are different this time.

If the first ball chosen is yellow, then there are 1 yellow and 3 green balls left, so that the probability of a yellow second ball is $\frac{1}{4}$ and the probability of a green second ball is $\frac{3}{4}$. If the first ball is green, however, then there are 2 yellow and 2 green balls left, so that the probability of a yellow second ball is $\frac{2}{4}$ and the probability of a green second ball is also $\frac{2}{4}$. Thus:



$$P(\text{both same color}) = P(\text{YY}) + P(\text{GG})$$
$$= \frac{2}{5} \cdot \frac{1}{4} + \frac{3}{5} \cdot \frac{2}{4} = \frac{8}{20} = \frac{2}{5}$$

In Examples 1 and 2, we denote the probability that the second ball is yellow given that the first ball is yellow by:

$$P(\text{second is Y} \mid \text{first is Y})$$

This probability is called a **conditional probability**, because it gives the probability that the second ball is yellow on the condition that the first ball is yellow. Other conditional probabilities are considered below.

In Example 1: $P(\text{second is G} \mid \text{first is G}) = \frac{3}{5}$

$P(\text{second is G} \mid \text{first is Y}) = \frac{3}{5}$

Note that these are equal.

In Example 2: $P(\text{second is G} \mid \text{first is G}) = \frac{2}{4}$

$P(\text{second is G} \mid \text{first is Y}) = \frac{3}{4}$

Note that these are different.

In Example 1, as the calculations on the preceding page show, the probability that the second ball is green is $\frac{3}{5}$ whether or not the first ball is green. Since the probability that the second ball is green does *not* depend on the color of the first ball, the two events "second ball is green" and "first ball is green" are called *independent events*. This is not the case, however, in Example 2 where the probability that the second ball is green *does* depend on whether or not the first ball is green.

In general, two events A and B are **independent** if and only if the occurrence of A does not affect the probability of the occurrence of B. In other words, events A and B are independent if and only if

$$P(B \mid A) = P(B).$$

Assessment Note

After discussing Examples 1 and 2, you may wish to ask students to give two different methods for calculating the probability of getting balls of different colors. (Method 1: $P(\text{different color}) = P(\text{YG}) + P(\text{GY})$; method 2: $1 - P(\text{same color})$)

Application

An important application of the probability of two events occurring together is the design of backup safety systems for space travel, nuclear power plants, and so on. For example, two computer systems are often used in controlling space vehicles. The second computer shadows the first and is ready to take control in the event that the first computer malfunctions. Suppose the probability of a computer failing is 0.01%. What is the probability that the mission will have correct computer support? $(1 - (0.0001)^2$. This answer assumes computer failure on the second machine to be independent of failure on the first.)

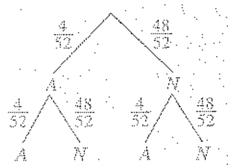
Example Note

In discussing Example 3, you should point out that the independence of the two events has nothing to do with how many actions are performed. In this case, a single action—drawing a card—is performed and two events—"jack" and "spade"—are defined for the action.

Additional Examples

1. Two cards are drawn from the top of a well-shuffled deck of cards. If A represents an ace and N represents a "non-ace," find $P(AA)$, $P(AN)$, $P(NA)$, and $P(NN)$ in the following cases:
   - **a.** the first card is replaced and the cards are shuffled before the second card is drawn.
   - **b.** the cards are drawn consecutively without replacement.

a.



$P(AA) = \frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$

$P(AN) = \frac{4}{52} \cdot \frac{48}{52} = \frac{12}{169}$

$P(NA) = \frac{48}{52} \cdot \frac{4}{52} = \frac{12}{169}$

$P(NN) = \frac{48}{52} \cdot \frac{48}{52} = \frac{144}{169}$

Examples 1 and 2 suggest the following rules for events occurring together.

Probability of Events Occurring Together

Rule 1. For any two events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A).$$

Rule 2. If events A and B are independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

Example 3 A card is randomly drawn from a standard deck.

a. Show that the events "jack" and "spade" are independent.

b. Show how Rule 2 given above can be used to find the probability of drawing the jack of spades.

Solution **a.** Since 13 of the 52 cards are spades, $P(\text{spade}) = \frac{13}{52}$.

Since 1 of the 4 jacks is a spade, $P(\text{spade} \mid \text{jack}) = \frac{1}{4}$.

Thus, since $P(\text{spade} \mid \text{jack}) = P(\text{spade})$, the events "jack" and "spade" satisfy the condition for independence stated at the bottom of the preceding page.

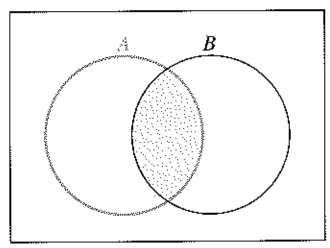
b. If we think of drawing the jack of spades as the simultaneous occurrence of the events "jack" and "spade," Rule 2 gives:

$$P(\text{jack and spade}) = P(\text{jack}) \cdot P(\text{spade}) = \frac{1}{13} \cdot \frac{1}{4} = \frac{1}{52}$$

We can write Rule 1 given above as:

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

In this form, the rule gives us a means of calculating conditional probability. Another way to calculate $P(B \mid A)$ can be seen by picturing events A and B as sets in a Venn diagram, as shown at the right. Since event A is known to have happened, we can ignore everything outside circle A in the diagram. The probability of B given A is just the fraction of A's sample points that are in B. We therefore have the rule



$$P(B \mid A) = \frac{n(A \cap B)}{n(A)}$$

which is equivalent to the one above.

Example 4 Each student in a class of 30 studies one foreign language and one science. The students' choices are shown in the table below.

	Chemistry (C)	Physics (P)	Biology (B)	Totals
French (F)	7	4	3	14
Spanish (S)	1	6	9	16
Totals	8	10	12	30

a. Find the probability that a randomly chosen student studies chemistry.
b. Find the probability that a randomly chosen student studies chemistry given that the student studies French.
c. Are the events "student studies chemistry" and "student studies French" independent?

Solution

a. Since 8 of the 30 students study chemistry, $P(\text{C}) = \frac{8}{30} = \frac{4}{15}$.

b. Of the 14 students who study French, 7 study chemistry. Thus:

$$P(\text{C} \mid \text{F}) = \frac{n(\text{C} \cap \text{F})}{n(\text{F})} = \frac{7}{14} = \frac{1}{2}$$

c. Parts (a) and (b) show that $P(\text{C} \mid \text{F}) \neq P(\text{C})$, so the events "student studies chemistry" and "student studies French" are *not* independent.

CLASS EXERCISES

1. ***Reading*** Find an instance where each of the two rules given at the top of the preceding page is used in Examples 1 or 2.
2. In Example 3, are the events "jack" and "face card" independent? No
3. Consider the class of 30 students described in Example 4.
   a. Find the probability that a randomly chosen student studies Spanish. $\frac{8}{15}$
   b. Find the probability that a randomly chosen student studies biology given that the student studies Spanish. $\frac{9}{16}$
   c. Are the events "student studies Spanish" and "student studies biology" independent? No

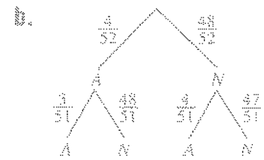
WRITTEN EXERCISES

A

1. From a box containing 3 red balls and 5 green balls, 2 balls are randomly picked, one after the other and without replacement.
   a. Draw a tree diagram showing the probabilities of each branch of the tree.
   b. Are the events "second ball is red" and "first ball is red" independent? No
   c. Find the probability that both balls are the same color. c. $\frac{13}{28}$ d. $\frac{15}{28}$
   d. Find the probability that one ball is red and one is green.

Additional Examples cont.

b.



$P(AA) = P(A \mid A) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$

$P(AN) = P(N \mid A) = \frac{4}{52} \cdot \frac{48}{51} = \frac{16}{221}$

$P(NA) = P(A \mid N) = \frac{48}{52} \cdot \frac{4}{51} = \frac{16}{221}$

$P(NN) = P(N \mid N) = \frac{48}{52} \cdot \frac{47}{51} = \frac{188}{221}$

2. Two dice are rolled. Show that the events "both numbers are the same" and "sum is 6" are *not* independent events. Refer to Example 3 in Section 16-1.

$P(\text{numbers are the same}) = \frac{6}{36} = \frac{1}{6}$.
$\{\text{sum is } 6\} = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$, so $P(\text{numbers are the same} \mid \text{sum is } 6) = \frac{1}{5}$.
Since $P(\text{numbers are the same}) \neq P(\text{numbers are the same} \mid \text{sum is } 6)$, the events are *not* independent.

Additional Answers Class Exercises

1. Rule 1 is used in Example 2 when finding $P(YY)$ and $P(GG)$. Rule 2 is used in Example 1 when finding $P(YY)$ and $P(GG)$.

Exercise Note

Students must use the fact that events are independent if and only if $P(B|A) = P(B)$ to answer parts (a) and (b) of Exercise 8. The fact that the answers are different might surprise some students.

Additional Answers Written Exercises

2. b. No
c. $\frac{1}{2}$
d. $\frac{1}{2}$
3. b. Yes
c. $\frac{17}{32}$
d. $\frac{15}{32}$
4. b. Yes
c. $\frac{5}{8}$
d. $\frac{3}{8}$
7. a. $P(F) = P(F|D) = \frac{3}{13}$
b. $P(R) = \frac{1}{2}$, $P(R|D) = 1$; $P(R) \neq P(R|D)$
10. b. $\frac{1}{216}$, $\frac{5}{72}$, $\frac{25}{72}$, $\frac{125}{216}$

Suggested Assignments

Discrete Math
Day 1: 609/1, 5, 7, 11, 13–18
Day 2: 612/21–25, 27, 29

Supplementary Materials

Alternative Assessment, 50

2. Repeat Exercise 1 if the box contains 1 red ball and 3 green balls.
3. Repeat Exercise 1 if the first ball chosen is put back in the box and mixed with the other balls before the second ball is picked.
4. Repeat Exercise 2 if the box contains 1 red ball and 3 green balls and if the first ball chosen is put back in the box and mixed with the other balls before the second ball is picked.
5. Two cards are dealt from the top of a well-shuffled standard deck of cards.
   a. Draw a tree diagram showing the probabilities of a heart (H) and non-heart (N) for each of the two cards.
   b. Find $P(\text{HH})$, $P(\text{HN})$, $P(\text{NH})$, and $P(\text{NN})$. $\frac{1}{17}$, $\frac{13}{68}$, $\frac{13}{68}$, $\frac{19}{34}$
6. A coin is slightly bent so that the probability of getting "heads" on a toss is 0.55 instead of 0.50.
   a. Draw a tree diagram showing the probabilities for two tosses of the coin. Use H for "heads" and T for "tails."
   b. Find $P(\text{HH})$, $P(\text{HT})$, $P(\text{TH})$, and $P(\text{TT})$. 0.3025, 0.2475, 0.2475, 0.2025
7. A card is randomly chosen from a standard deck.
   a. Show that the events "face card" and "diamond" are independent.
   b. Show that the events "red card" and "diamond" are *not* independent.
8. The following 6 cards are placed face down on a table: ace, 2, and 3 of hearts; ace and 2 of clubs; and ace of spades. One card is randomly chosen.
   a. Are the events "ace" and "heart" independent? No
   b. Are the events "ace" and "club" independent? Yes
9. A penny, a nickel, and a dime are tossed one after the other.
   a. Draw a tree diagram showing the probabilities of a "head" (H) and "tail" (T) for each toss.
   b. Find $P(3 \text{ "heads"})$, $P(\text{exactly 2 "heads"})$, $P(\text{exactly 1 "head"})$, and $P(\text{no "heads"})$. $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$
10. A die is rolled 3 times.
   a. Draw a tree diagram showing the probabilities of a six (S) and non-six (N) for each roll.
   b. Find $P(3 \text{ sixes})$, $P(\text{exactly 2 sixes})$, $P(\text{exactly 1 six})$, and $P(\text{no sixes})$.
11. The letters of the word COMPUTER are written on separate slips of paper and placed in a hat. Three letters are then randomly drawn from the hat, one after the other and without replacement. To find the probability that all 3 letters are vowels, two methods can be used.

Method 1: $P(3 \text{ vowels}) = \dfrac{\text{number of 3-letter words having 3 vowels}}{\text{number of 3-letter words}}$

Method 2: $P(3 \text{ vowels}) = P(\text{first is vowel}) \times P(\text{second is vowel} \mid \text{first is vowel}) \times P(\text{third is vowel} \mid \text{first and second are vowels})$

   a. Use both methods to find $P(3 \text{ vowels})$. $\frac{1}{56}$
   b. Using whichever method you prefer, find the probability that all 3 letters are consonants. $\frac{5}{28}$

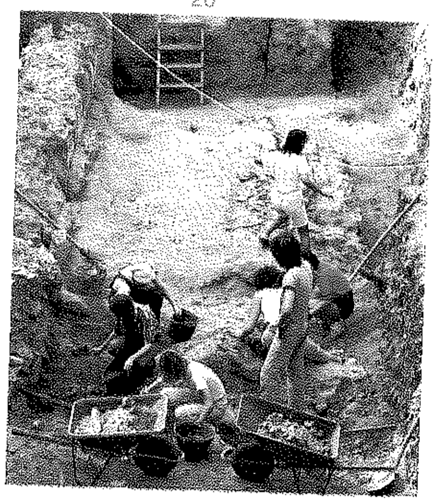
12. In a math class with 13 girls and 11 boys, the teacher randomly selects four students to put homework problems on the chalkboard. What is the probability that all are girls? (*Hint*: Use one of the methods described in Exercise 11.) $\frac{65}{966}$

In Exercises 13–18, use the table below, which gives the areas of concentration of the 400 students at a small college.

Area of concentration	Class				
	Freshman	Sophomore	Junior	Senior	Totals
Natural sciences	50	35	33	29	147
Social sciences	20	25	28	24	97
Humanities	40	40	39	37	156
Totals	110	100	100	90	400

13. If a student is selected at random, what is the probability that:
   a. the student's area of concentration is the natural sciences? $\frac{147}{400}$
   b. the student is a freshman in the social sciences? $\frac{1}{20}$

14. If a sophomore is selected at random, what is the probability that his or her area of concentration is the humanities? $\frac{2}{5}$



15. a. What is the probability that a senior's area of concentration is the natural sciences? $\frac{29}{90}$
   b. What is the probability that a student in the natural sciences is a senior? $\frac{29}{147}$

16. a. What is the probability that a junior's area of concentration is the humanities? $\frac{39}{100}$
   b. What is the probability that a student in the humanities is a junior? $\frac{39}{156}$

17. A student is selected at random. Are the events "student is a junior" and "student is in the humanities" independent? Yes

18. A student is selected at random. Are the events "student is a junior" and "student is in the natural sciences" independent? No

B 19. ***Reading*** Given a sample space S with subsets A and B, use $n(S)$ to show that the rule $P(B \mid A) = \frac{n(A \cap B)}{n(A)}$ is equivalent to the rule $P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$.

20. ***Reading*** Read again the two rules given at the top of page 608. Explain why Rule 2 is a special case of Rule 1.
If A and B are independent, $P(B \mid A) = P(B)$.

Additional Answers Written Exercises

19. $P(B \mid A) = \frac{n(A \cap B)}{n(A)} = \frac{n(A \cap B)}{n(A)} \cdot \frac{\frac{1}{n(S)}}{\frac{1}{n(S)}} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(A)}{n(S)}} = \frac{P(A \text{ and } B)}{P(A)}$

Additional Answers Written Exercises

21. a. $\frac{11}{50}$; $\frac{11}{850}$
b. $\frac{2}{17}$
c. $\frac{1}{5525}$
d. $\frac{4324}{5525}$
e. $\frac{1201}{5525}$
f. $\frac{48}{221}$

22. a. $\frac{33}{66,640}$
b. $\frac{33}{16,660}$
c. $\frac{35,673}{54,145}$
d. $\frac{18,472}{54,145}$

23. a. $\frac{1}{64}$
b. $\frac{1}{8}$
c. $\frac{1}{2197}$
d. $\frac{1728}{2197}$
e. $\frac{469}{2197}$
f. $\frac{36}{169}$

24. a. $\frac{1}{1024}$
b. $\frac{1}{256}$
c. $\frac{248,832}{371,293}$
d. $\frac{122,461}{371,293}$

30. a. Probability that team X wins the championship $= f(p) = 20p^7 - 70p^6 + 108p^5 - 95p^4 + 52p^3 - 18p^2 + 4p$
b. When $p = 0.5$, $f(p) = 0.5$.

21. Three cards are dealt from a well-shuffled standard deck.

a. Complete the calculation: $P(3 \text{ clubs}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{?}{?} = ?$

b. Find the probability that all 3 cards are red.

c. Find the probability that all 3 cards are aces.

d. Find the probability that none of the cards is an ace.

e. Find the probability that at least one of the cards is an ace.

f. Find the probability that either one or two of the cards is an ace.

22. Five cards are dealt from a well-shuffled standard deck.

a. Find the probability that all 5 cards are spades. (*Hint*: See Exercise 21(a).)

b. Find the probability that all 5 cards are of the same suit.

c. Find the probability that none of the cards is an ace.

d. Find the probability that at least one card is an ace.

23. Repeat Exercise 21 if each card is replaced before the next is drawn.

24. Repeat Exercise 22 if each card is replaced before the next is drawn.

25. Three people are randomly chosen. Find the probability of each event.

a. All were born on different days of the week. a. $\frac{30}{49}$ b. $\frac{19}{49}$

b. At least two people were born on the same day of the week.

26. A die is rolled 3 times. Find the probability of each event.

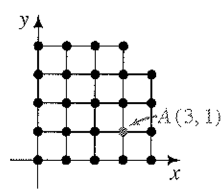
a. All 3 numbers are different. a. $\frac{5}{9}$ b. $\frac{4}{9}$

b. At least 2 of the numbers are the same.

27. A bag contains 3 red marbles and 2 black marbles. Two persons, A and B, take turns drawing a marble from the bag without replacing any of the marbles drawn. The first person to draw a black marble wins. What is the probability that person A, who draws first, wins? $\frac{3}{5}$

28. Repeat Exercise 27 if the bag contains 4 red marbles and 1 black marble. $\frac{3}{5}$

29. **Sports** The championship series of the National Basketball Association consists of a series of at most 7 games between two teams X and Y. The first team to win 4 games is the champion and the series is over. At any time before or after a game, the status of the series can be recorded as a point (x, y) on the grid shown at the right. The point $A(3, 1)$, for example, means that team X has won 3 games and team Y has won 1 game. From point A, the series can end in a championship for team X in 3 ways (X, YX, YYX). If you assume that team X has a probability of 0.6 of winning each and every remaining game, then the probability that team X becomes champion from point A is $P(\text{X}) + P(\text{YX}) + P(\text{YYX}) = 0.6 + (0.4)(0.6) + (0.4)(0.4)(0.6) = 0.936$.



a. Find the probability that team Y becomes champion from point A. 0.064

b. If team X has won 1 game and team Y has won 3 games, find the probability that team Y becomes champion. 0.784

c. If team X has won 2 games and team Y has won 1 game, find the probability that team X becomes champion. 0.8208

C **30. Sports** Refer to Exercise 29 and suppose that the *home* team has a probability p of winning each game and a probability $q = 1 - p$ of losing.

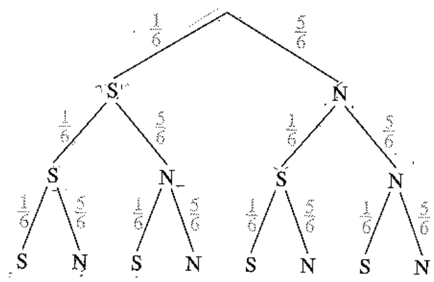
a. If team X is the home team for games 1, 2, 6, and 7, what is the probability that team X wins the championship? Give the answer in terms of p only.

b. Use a computer or graphing calculator to find the approximate value of p that makes team X's probability of winning the championship 0.5.

16-3 The Binomial Probability Theorem

Objective *To use the binomial probability theorem to find the probability of a given outcome on repeated independent trials of a binomial experiment and to approximate the probability when the trials are not independent.*

Consider the experiment in which a die is rolled three times. On each roll, the probability of getting a six (S) is $p = \frac{1}{6}$ and the probability of getting a non-six (N) is $1 - p = \frac{5}{6}$. These probabilities are shown in the tree diagram at the right. We can use the tree diagram to create the table below, which summarizes the results of the experiment.



	3 sixes	2 sixes			1 six			0 sixes
Outcome	SSS	SSN	SNS	NSS	SNN	NSN	NNS	NNN
Probability	$\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}$	$\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}$	$\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}$	$\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}$	$\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}$	$\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}$	$\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}$	$\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}$
	$\left(\frac{1}{6}\right)^3$	$3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$			$3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$			$\left(\frac{5}{6}\right)^3$

Now compare: ↓ ↓ ↓ ↓

$$[p + (1 - p)]^3 = p^3 \quad + \quad 3p^2(1 - p) \quad + \quad 3p(1 - p)^2 \quad + (1 - p)^3$$

If the die were rolled 4 times instead of 3, we would have the following distribution of probabilities:

Outcome	4 sixes	3 sixes	2 sixes	1 six	0 sixes
Probability	$\left(\frac{1}{6}\right)^4$	$4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)$	$6\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$	$4\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3$	$\left(\frac{5}{6}\right)^4$

↓ ↓ ↓ ↓ ↓

$$[p + (1 - p)]^4 = p^4 \ + 4p^3(1 - p) + 6p^2(1 - p)^2 + 4p(1 - p)^3 + (1 - p)^4$$

Teaching Notes, p. 596C

Warm-Up Exercises

Write the expansion of each binomial, simplifying your answer. Then substitute $\frac{2}{3}$ for x and $\frac{1}{3}$ for y in the expansion and evaluate the sum.

1. $(x + y)^2$ $x^2 + 2xy + y^2$; $\frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$
2. $(x + y)^3$ $x^3 + 3x^2y + 3xy^2 + y^3$; $\frac{8}{27} + \frac{12}{27} + \frac{6}{27} + \frac{1}{27} = 1$
3. $(x + y)^4$ $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$; $\frac{16}{81} + \frac{32}{81} + \frac{24}{81} + \frac{8}{81} + \frac{1}{81} = 1$

Evaluate each expression for $p = \frac{1}{4}$ and $q = \frac{3}{4}$.

4. ${}_7C_5 \cdot p^5 \cdot q^2$ $\frac{189}{16,384}$
5. ${}_6C_3 \cdot p^3 \cdot q^3$ $\frac{135}{1024}$
6. ${}_5C_3 \cdot p^3 \cdot q^2$ $\frac{45}{512}$

Motivating the Section

A good introduction to this section might be to consider the problem of jury selection. Suppose that the general population eligible for jury duty is 52% women. In an 8-person jury we would expect four women; three women would be less likely, and one woman even more unusual. But how unusual? What is the probability of that happening?

Assessment Note

Ask students to decide whether the binomial probability theorem can be applied to each of the following experiments with five repeated trials.

1. Flipping a coin five times, counting heads. (Yes)
2. Drawing five cards without replacement from a deck, counting aces. (No; not independent)
3. Rolling a die five times, counting face values. (No; more than two outcomes per trial)
4. Shooting five foul shots in basketball, counting baskets made. (Debatable; is each trial independent?)

Additional Examples

1. A quiz consists of 6 true-false questions. If you guess the answer to each question, what is the probability that you get at least 4 correct answers?
   $P(\text{at least 4 correct}) = P(4 \text{ correct}) + P(5 \text{ correct}) + P(6 \text{ correct}) = {}_6C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2 + {}_6C_5 \cdot \left(\frac{1}{2}\right)^5 \cdot \frac{1}{2} + {}_6C_6 \cdot \left(\frac{1}{2}\right)^6 = \frac{15}{64} + \frac{6}{64} + \frac{1}{64} = \frac{22}{64} \approx 0.344$
2. A jar contains 2 red marbles and 3 blue marbles. Three marbles are randomly drawn. Find the probability of 1, 2, or 3 blue marbles if the drawing is done **(a)** with replacement and **(b)** without replacement.

The two illustrations on the preceding page can be generalized to give the following theorem, which is based on the binomial theorem (page 591).

The Binomial Probability Theorem

Suppose an experiment consists of a sequence of n repeated independent trials, each trial having two possible outcomes, A or not A. If on each trial, $P(A) = p$ and $P(\text{not } A) = 1 - p$, then the binomial expansion of $[p + (1 - p)]^n$,

$$ {}_nC_np^n + \cdots + {}_nC_kp^k(1-p)^{n-k} + \cdots + {}_nC_0(1-p)^n, $$

gives the following probabilities for the occurrences of A:

Outcome	n A's	$\cdots$	k A's	$\cdots$	0 A's
Probability	${}_nC_np^n$	$\cdots$	${}_nC_kp^k(1-p)^{n-k}$	$\cdots$	${}_nC_0(1-p)^n$

Example 1 A coin is tossed 10 times. What is the probability of exactly 4 "heads"?

Solution Substitute $n = 10$, $k = 4$, and $p = 1 - p = \frac{1}{2}$ in the theorem above.

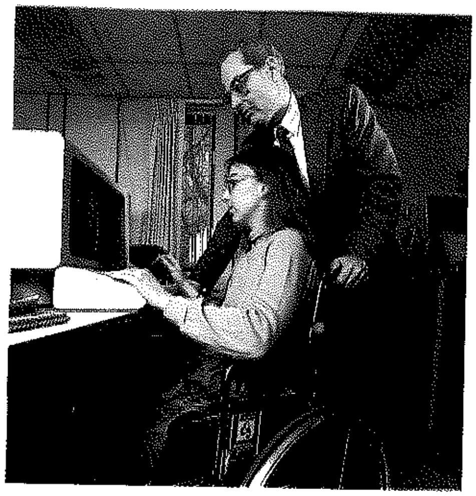
$$ P(4 \text{ out of 10 tosses result in "heads"}) = {}_{10}C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6 = 210\left(\frac{1}{2}\right)^{10} = \frac{210}{1024} \approx 0.205 $$

Example 2 In a survey of the 1000 students in a school, 950 indicated that they were right-handed. Find the probability that at least one of four randomly chosen students from the school is left-handed.

Solution Technically speaking, this is not a binomial experiment because the sampling of the four students is done without replacement. In this experiment the probability of the first student's being right-handed is $\frac{950}{1000} = 0.95$. The probability of the second student's being right-handed is either $\frac{949}{999}$ or $\frac{950}{999}$ depending on whether the first student is right-handed. These two probabilities (as well as the probabilities of the third and fourth student's being right-handed) are so close to 0.95 that we will use the binomial probability theorem even though the theorem assumes a constant 0.95 probability. Thus:

$$ \begin{aligned} P(\text{at least one left-handed}) &= 1 - P(\text{all right-handed}) \\ &\approx 1 - {}_4C_4(0.95)^4 \\ &\approx 1 - 0.815 = 0.185 \end{aligned} $$

Example 3 Of the computer operators who work for a large temporary employment agency, 60% have a car. If 5 of the computer operators are randomly chosen to work on a job that requires car transportation, what is the probability that at least 4 have cars?



Solution As was the case in Example 2, this is not a binomial experiment because the sampling of the 5 computer operators is done without replacement. Nevertheless, if there is a large number of computer operators, the probability of choosing each of the 5 is close enough to 0.6 that we can use the binomial probability theorem to obtain a good approximation of the probability that at least 4 have cars. Thus:

$$\begin{aligned} P(\text{at least 4 have cars}) &= P(\text{exactly 4 have cars}) + P(\text{5 have cars}) \\ &\approx {}_5C_4(0.6)^4(0.4)^1 + {}_5C_5(0.6)^5 \\ &\approx 0.259 + 0.078 = 0.337 \end{aligned}$$

CLASS EXERCISES

1. In the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^4$, shown below, the first term represents the probability of getting 4 "heads" in 4 tosses of a coin. What probabilities do the other terms represent?

$$\left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$$

2. ***Discussion*** In your own words, tell how to find the probability of getting exactly 3 fives when 4 dice are rolled.
3. A coin is tossed 9 times and comes up "heads" each time. What is the probability that it will come up "heads" on the next toss? $\frac{1}{2}$
4. **a.** A jar contains 50 red marbles and 50 white marbles. Two marbles are randomly drawn, one after the other and without replacement. What is the probability of drawing 2 red marbles?
   b. Explain why the binomial probability theorem does not apply in part (a).
   c. What does the binomial probability theorem give as an *approximation* of the probability in part (a)?

Additional Examples cont.

a. $P(1 \text{ blue}) = {}_3C_1\cdot\frac{3}{5}\cdot\left(\frac{2}{5}\right)^2 = \frac{36}{125} = 0.288$
$P(2 \text{ blue}) = {}_3C_2\cdot\left(\frac{3}{5}\right)^2\cdot\frac{2}{5} = \frac{54}{125} = 0.432$
$P(3 \text{ blue}) = {}_3C_3\cdot\left(\frac{3}{5}\right)^3 = \frac{27}{125} = 0.216$

b. $P(1 \text{ blue}) = P(BRR) + P(RBR) + P(RRB) = \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3} + \frac{2}{5}\cdot\frac{3}{4}\cdot\frac{1}{3} + \frac{2}{5}\cdot\frac{1}{4}\cdot\frac{3}{3} = \frac{3}{10} = 0.3$
$P(2 \text{ blue}) = P(BBR) + P(BRB) + P(RBB) = \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3} + \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3} + \frac{2}{5}\cdot\frac{3}{4}\cdot\frac{2}{3} = \frac{3}{5} = 0.6$
$P(3 \text{ blue}) = \frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3} = \frac{1}{10} = 0.1$

Additional Answers Class Exercises

1. 3 heads and 1 tail, 2 heads and 2 tails, 1 head and 3 tails, 4 tails
2. Find the second term in $\left(\frac{1}{6}+\frac{5}{6}\right)^4$.
4. a. $\frac{49}{198}$
   b. The first and second drawings are not independent trials.
   c. $\frac{1}{4}$

Error Analysis

Students often believe that the probability of 2 successes in 4 trials is the same as that of 3 successes in 6 trials since each represents half the trials being successful. You can help students see the error in this reasoning by first calculating the probability of getting exactly one head in two tosses of a coin (0.5) and then asking if they feel the probability is the same for getting *exactly* 10 heads in 20 tosses.

Additional Answers Written Exercises

7. a. $\frac{1}{4}$

b. $\frac{9}{16}$, $\frac{3}{8}$, $\frac{1}{16}$

Suggested Assignments

Discrete Math
616/7–19 odd, 20

Supplementary Materials

Alternative Assessment, 50–51

WRITTEN EXERCISES

A

1. Make a table, like the ones on page 613, showing the 8 different ways in which "heads" (H) and "tails" (T) can occur if a coin is tossed 3 times. Then find the probability of getting:
a. 3 "heads" $\frac{1}{8}$ **b.** 2 "heads" $\frac{3}{8}$ **c.** 1 "head" $\frac{3}{8}$ **d.** 0 "heads" $\frac{1}{8}$

2. Make a table, like the ones on page 613, showing the 16 different ways in which sixes (S) and non-sixes (N) can occur when a die is rolled 4 times. Then find the probability of getting:
a. 4 sixes $\frac{1}{1296}$ **b.** 3 sixes $\frac{5}{324}$ **c.** 2 sixes $\frac{25}{216}$ **d.** 1 six $\frac{125}{324}$ **e.** 0 sixes $\frac{625}{1296}$

3. Consider the set of families with exactly 4 children. If $P(\text{child is a boy}) = \frac{1}{2}$, find the probability that one of these families, picked at random, has:
a. 4 boys $\frac{1}{16}$ **b.** 3 boys $\frac{1}{4}$ **c.** 2 boys $\frac{3}{8}$ **d.** 1 boy $\frac{1}{4}$ **e.** 0 boys $\frac{1}{16}$

4. Suppose a coin is bent so that the probability of its coming up "heads" on any toss is $\frac{2}{5}$. If the coin is tossed 3 times, find the probability of getting:
a. 3 "heads" $\frac{8}{125}$ **b.** 2 "heads" $\frac{36}{125}$ **c.** 1 "head" $\frac{54}{125}$ **d.** 0 "heads" $\frac{27}{125}$

5. What is the probability of getting exactly 2 fives in 4 rolls of a die? ≈ 0.116

6. What is the probability of getting exactly 1 three in 7 rolls of a die? ≈ 0.391

7. a. If one card is drawn from a well-shuffled standard deck, what is the probability of drawing a spade?
b. If one card is drawn from each of two well-shuffled standard decks, what is the probability of drawing 0 spades? 1 spade? 2 spades?

8. If one card is drawn from each of three well-shuffled standard decks, find the probability of drawing:
a. 3 spades $\frac{1}{64}$ **b.** 2 spades $\frac{9}{64}$ **c.** 1 spade $\frac{27}{64}$ **d.** 0 spades $\frac{27}{64}$

9. A quiz has 6 multiple-choice questions, each with 4 choices. If you guess at every question, what is the probability of getting:
a. all 6 questions right? $\frac{1}{4096}$ **b.** 5 out of 6 questions right? $\frac{9}{2048}$

10. A jar contains 4 red balls and 3 white balls, all the same size. Suppose you pull out a ball and note its color, put it back, and mix up the contents of the jar. If you do this twice more, find the probability of getting:
a. 0 red balls $\frac{27}{343}$ **b.** 1 red ball $\frac{108}{343}$ **c.** 2 red balls $\frac{144}{343}$ **d.** 3 red balls $\frac{64}{343}$

B

11. Eight out of every ten nutritionists recommend Brand X. If nutritionists A, B, and C are asked their opinions on Brand X, what is the probability that:
a. all three recommend Brand X? See below. **b.** none recommends Brand X? $\frac{1}{125}$
c. at least one recommends Brand X? $\frac{124}{125}$

12. In a certain high school, one third of the senior boys are at least 6 ft tall. In a randomly selected group of 7 senior boys, what is the probability that:
a. all are less than 6 ft tall? See below. **b.** none are less than 6 ft tall? $\frac{1}{2187}$
c. all but one are less than 6 ft tall? $\frac{448}{2187}$

11. a. $\frac{64}{125}$ 12. a. $\frac{128}{2187}$

13. In the *random-number table* shown below, the probability that each digit, 0–9, occurs in any given position is 0.1. We can use the table to perform *Monte Carlo simulations* in order to find certain probabilities empirically.

49487	52802	28667	62058	87822	14704	18519	17889	45869	14454
29480	91539	46317	84803	86056	62812	33584	70391	77749	64906
25252	97738	23901	11106	86864	55808	22557	23214	15021	54268
02431	42193	96960	19620	29188	05863	92900	06836	13433	21709
69414	89353	70724	67893	23218	72452	03095	68333	13751	37260

For example, to use Monte Carlo simulations to find the probability of getting 0 "heads" in 3 tosses of a coin, first associate the digits 0–4 with the outcome "heads" and the digits 5–9 with the outcome "tails." Then the first row of the random-number table represents 16 simulations of 3 tosses of a coin.

494|87 5|280|2 28|667| 620|58 8|782|2 14|704| 185|19 1|788|9 45|869| 144|54
HTH|TT T|HTH|H HT|TTT| THH|TT T|TTH|H HH|THH| HTT|HT H|TTT|T HT|TTT| HHH|

a. The empirical probability (0.3125) is greater than the theoretical probability (0.125).

a. Of the 16 simulations, 5 produce 0 "heads." Thus, based on 16 simulations, $P(0 \text{ "heads"}) = 0.3125$. Compare this probability to the theoretical probability of getting 0 "heads" in 3 tosses of a coin (see Exercise 1(d)).

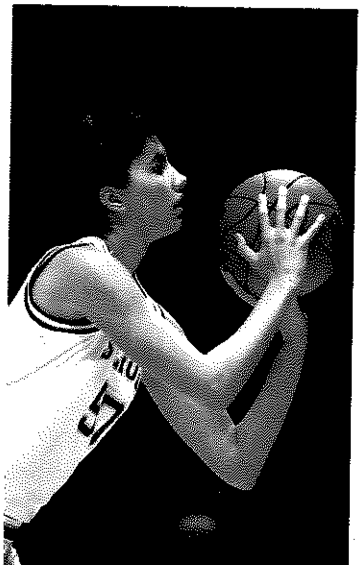
b. Begin with the last 2 digits of the first row and perform 14 more simulations. Using all 30 simulations, find the probability of getting 0 "heads." $0.1\overline{6}$

c. How many simulations can be done using the whole table? Based on these simulations, what is the probability of getting 0 "heads"? 83; ≈0.13253

d. ***Discussion*** When you use Monte Carlo simulations to approximate the theoretical probability of an event, will increasing the number of simulations always improve the accuracy of the approximation? Explain.

14. **a.** ***Discussion*** In order to use Monte Carlo simulations (see Exercise 13) to find the probabilities asked for in Exercises 2–12, how should the digits 0–9 be assigned to the possible outcomes for each exercise? (*Hint:* It may be helpful to eliminate some digits. Cross off all occurrences of those digits in the table before actually doing a simulation.)

b. Perform Monte Carlo simulations for Exercise 2 and Exercise 7. Compare the probabilities you find to the theoretical probabilities found using the binomial probability theorem.



15. **a.** **Sports** A basketball player's free-throw percent is .750. What is the probability that she scores on exactly 4 of her next 5 free throws?

b. ***Discussion*** Using the binomial probability theorem to complete part (a) assumes that the probability of a successful free throw is always .750. Is this assumption valid? Explain.

a. 0.396 b. No; the player's free-throw percent changes each time she takes a free throw.

Cooperative Learning

Exercises 13 and 14 are perhaps best done in small groups.

Exercise Note

Exercises 13 and 14 can lead to a discussion of the difference between theoretical and empirical probability. To increase the variety of empirical probabilities calculated, students can use different strings of random numbers for the Monte Carlo simulations in Exercise 14. To create a new string of random numbers, begin *anywhere* in the table and move in *any* direction from that number. Alternatively, students can use a calculator or computer with a random-number generator to create a random-number table.

Using Technology

Students also can write a computer program to conduct Monte Carlo simulations (see the Computer Exercise on the next page).

Additional Answers
Computer Exercise

```
10  PRINT "WHAT IS THE
    NUMBER OF TRIALS"
20  PRINT "IN A SIMULA-
    TION";
30  INPUT N
40  PRINT "WHAT IS THE
    PROBABILITY THAT
    THE"
50  PRINT "DESIRED OUT-
    COME OCCURS ON
    EACH TRIAL";
60  INPUT P
70  PRINT "HOW MANY
    SIMULATIONS DO YOU"
80  PRINT "WANT PER-
    FORMED";
90  INPUT S
100 DIM A(N)
110 FOR I = 0 TO N
120 LET A(I) = 0
130 NEXT I
140 FOR I = 1 TO S
150 LET T = 0
160 FOR J = 1 TO N
170 IF RND (1) <= P
    THEN LET T = T + 1
180 NEXT J
190 LET A(T) = A(T) + 1
200 NEXT I
210 PRINT "T","T/S"
220 FOR I = 0 TO N
230 PRINT I, A(I)/S
240 NEXT I
250 END
```


16. ***Investigation*** Let $n = 6$ and $P(A) = 0.5$ in the binomial probability theorem. Plot the points $(k, P(k \text{ occurrences of } A))$ for $k = 0, 1, \ldots, 6$, and describe the shape of a smooth curve that can be drawn through the points. What happens to the shape of this curve for values of $P(A)$ other than 0.5?

17. In a package of tomato seeds, 9 seeds out of 10 sprout on the average. What is the probability that of the first 10 seeds planted, 1 does not sprout? 0.387

18. One out of every 5 boxes of Rice Toasties has a secret message decoder ring. You buy 5 boxes hoping to get at least 2 rings. What are your chances? ≈0.263

Parts of Exercises 19 and 20 require the use of a calculator or computer. If you use a computer, you may need to write your own short programs.

19. a. If n people are randomly chosen, for what value of n is the probability of at least two having the same birthday equal to 1? 366 (ignoring leap year)

b. Suppose you guess that the probability in part (a) is 0.5 if $n = 10$. To test the guess, use the probability that no two of 10 people share a birthday:

$$P(\text{no match}) = P\begin{pmatrix}\text{2nd person}\\ \text{does not}\\ \text{match first}\end{pmatrix} \times P\begin{pmatrix}\text{3rd person does}\\ \text{not match either}\\ \text{of first two}\end{pmatrix} \times \cdots \times P\begin{pmatrix}\text{10th person does}\\ \text{not match any}\\ \text{of first nine}\end{pmatrix}$$

$$= \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{356}{365} = \frac{{}_{364}P_9}{365^9}$$

Thus, $P(\text{at least one match}) = 1 - P(\text{no match}) = 1 - \dfrac{{}_{364}P_9}{365^9}$.

Use a calculator to evaluate the above expression for $P(\text{at least one match})$. 0.117

c. The probability that of 10 randomly chosen people at least two have the same birthday is less than 0.5. Use the method of part (b) to determine how many people are needed for the probability to be 0.5. 23

20. a. $1 - \left(\frac{364}{365}\right)^n$

20. a. What is the probability that in a group of n randomly chosen people, at least one has the same birthday as you?

b. Find the value of n that makes the probability in part (a) approximately 0.5. 253

COMPUTER EXERCISE

Write a program that does Monte Carlo simulations (see Exercise 13) of the binomial probability theorem. Use the computer's random-number generator to simulate N independent trials with probability P that a given outcome occurs on each trial. Count the number of times T that the outcome occurs. Repeat the simulation of N trials many times and count how often $T = 0$, $T = 1, \ldots, T = N$. Convert these tallies to approximate probabilities by dividing by the number of simulations.

16-4 Probability Problems Solved with Combinations

Objective *To use combinations to solve probability problems.*

In Example 1 we show two methods of finding the probability that five cards randomly drawn from a standard deck are all hearts. Method 1 uses conditional probability. Method 2, which uses combinations, is presented because it can be used to solve many problems that are not readily solved using conditional probability.

Example 1 Five cards are drawn at random from a standard deck. Find the probability that all 5 cards are hearts.

Solution **Method 1** Since there are 13 hearts in the 52-card deck, the probability that the first card drawn is a heart is $\frac{13}{52}$. With 12 hearts among the 51 cards remaining, the probability that the second card drawn is a heart is $\frac{12}{51}$. Continuing in this way, we have:

$$P(\text{all hearts}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48}$$
$$= \frac{33}{66{,}640} \approx 0.000495$$

Method 2 The experiment of choosing 5 cards from 52 has ${}_{52}C_5$ equally likely outcomes. Of these, there are ${}_{13}C_5$ combinations that contain 5 (of the 13) hearts. Thus:

$$P(\text{all hearts}) = \frac{{}_{13}C_5}{{}_{52}C_5} = \frac{1287}{2{,}598{,}960} \approx 0.000495$$

Example 2 Five cards are drawn at random from a standard deck. Find the probability that exactly 2 are hearts.

Solution In this example it is impractical to use a card-by-card approach based on conditional probability, as in Method 1 of Example 1. The difficulty is that we do not know whether the hearts will appear as the first and second cards, as the first and third cards, or as some other two cards. Instead, we determine the number of 5-card combinations that contain exactly 2 hearts. There are ${}_{13}C_2$ choices for the 2 hearts, and there are ${}_{39}C_3$ choices for the other 3 cards. By the multiplication principle (page 571), the number of 5-card combinations containing exactly 2 hearts is ${}_{13}C_2 \cdot {}_{39}C_3$. Since there are ${}_{52}C_5$ possible 5-card combinations, we have:

$$P\begin{pmatrix}\text{2 hearts and}\\ \text{3 non-hearts}\end{pmatrix} = \frac{{}_{13}C_2 \cdot {}_{39}C_3}{{}_{52}C_5} = \frac{78 \cdot 9139}{2{,}598{,}960} \approx 0.274$$

Teaching Notes, p. 596C

Warm-Up Exercises

Three cards are drawn from a well-shuffled standard deck of 52 cards, one after the other and without replacement.

1. Find the probability of drawing:
   a. all clubs. $\frac{11}{850}$
   b. no clubs. $\frac{703}{1700}$
   c. exactly one club. (Hint: The club can occur on the first, second, or third drawing.) $\frac{741}{1700}$
2. Evaluate:
   a. $\frac{{}_{13}C_3}{{}_{52}C_3}$ $\frac{11}{850}$
   b. $\frac{{}_{39}C_3}{{}_{52}C_3}$ $\frac{703}{1700}$
   c. $\frac{{}_{13}C_1 \cdot {}_{39}C_2}{{}_{52}C_3}$ $\frac{741}{1700}$
3. Compare your answers to Exercises 1 and 2. What do you notice? 1(a) and 2(a) are the same, as are 1(b) and 2(b), and 1(c) and 2(c).
4. Use the results of Exercises 1-3 to write and evaluate an expression using combinations to find the probability of getting exactly two clubs.
   $\frac{{}_{13}C_2 \cdot {}_{39}C_1}{{}_{52}C_3} = \frac{117}{850}$

Motivating the Section

You may wish to introduce this section by doing an example whose probability can be determined by listing all possibilities, such as selecting a committee of 3 from 5 students.

Using Technology

In discussing Example 3, you may wish to point out that most calculators can give only an approximation to ${}_{52}C_{13}$. To obtain the exact value, students may need to write out the factors and do as much canceling as possible before multiplying.

Additional Examples

1. Four balls are chosen at random from a bag containing 4 white balls, 3 green balls, and 5 black balls.
   a. Find the probability of choosing 0, 1, 2, 3, or 4 black balls and show that the sum of the probabilities is 1.
   b. Find the probability of choosing 2 white balls and 2 black balls.

a. $P(\text{0 black}) = \frac{{}_7C_4}{{}_{12}C_4} = \frac{35}{495} \approx 0.071$

$P(\text{1 black}) = \frac{{}_7C_3 \cdot {}_5C_1}{{}_{12}C_4} = \frac{35 \cdot 5}{495} \approx 0.354$

$P(\text{2 black}) = \frac{{}_7C_2 \cdot {}_5C_2}{{}_{12}C_4} = \frac{21 \cdot 10}{495} \approx 0.424$

$P(\text{3 black}) = \frac{{}_7C_1 \cdot {}_5C_3}{{}_{12}C_4} = \frac{7 \cdot 10}{495} \approx 0.141$

$P(\text{4 black}) = \frac{{}_5C_4}{{}_{12}C_4} = \frac{5}{495} \approx 0.010$

$\frac{35 + 175 + 210 + 70 + 5}{495} = 1$

b. $P(\text{2 white, 2 black}) = \frac{{}_4C_2 \cdot {}_5C_2}{{}_{12}C_4} = \frac{6 \cdot 10}{495} \approx 0.121$

Example 3 Thirteen cards are dealt from a well-shuffled standard deck. What is the probability that the 13 cards contain exactly 4 aces and exactly 3 kings?

Solution

1. There are ${}_4C_4$ choices for the aces.
2. There are ${}_4C_3$ choices for the kings.
3. Since the remaining 6 cards must be chosen from the 44 cards that are neither aces nor kings, there are ${}_{44}C_6$ choices for the remaining cards.
4. There are ${}_{52}C_{13}$ choices for the 13 cards.
5. Using these results and the multiplication principle, we have:

$$P\left(\begin{array}{c}\text{13 cards with exactly}\\ \text{4 aces and exactly 3 kings}\end{array}\right) = \frac{{}_4C_4 \cdot {}_4C_3 \cdot {}_{44}C_6}{{}_{52}C_{13}} = \frac{1 \cdot 4 \cdot 7{,}059{,}052}{635{,}013{,}559{,}600} \approx 0.0000445$$

CLASS EXERCISES

1. Three marbles are picked at random from a bag containing 4 red marbles and 5 white marbles. Match each event with its probability.

Events:	*Probabilities:*
a. All 3 marbles are red.	$\frac{{}_4C_1 \cdot {}_5C_2}{{}_9C_3}$ c
b. Exactly 2 marbles are red.	$\frac{{}_4C_2 \cdot {}_5C_1}{{}_9C_3}$ b
c. Exactly 1 marble is red.	$\frac{{}_4C_3 \cdot {}_5C_0}{{}_9C_3}$ a
d. No marble is red.	$\frac{{}_4C_0 \cdot {}_5C_3}{{}_9C_3}$ d

2. Five cards are drawn at random from a standard deck. Match each event with its probability.

Events:	*Probabilities:*
a. All 4 aces are chosen.	$\frac{{}_4C_0 \cdot {}_{48}C_5}{{}_{52}C_5}$ b
b. No aces are chosen.	$\frac{{}_4C_4 \cdot {}_{48}C_1}{{}_{52}C_5}$ a
c. Exactly 4 diamonds are chosen.	$\frac{{}_4C_4 \cdot {}_4C_1}{{}_{52}C_5}$ d
d. Four aces and one jack are chosen.	$\frac{{}_{13}C_4 \cdot {}_{39}C_1}{{}_{52}C_5}$ c

3. Five cards are drawn at random from a standard deck.
   a. You can find the probability of getting at least 1 ace by calculating the sum:
   $$P\binom{\text{exactly}}{\text{1 ace}} + P\binom{\text{exactly}}{\text{2 aces}} + P\binom{\text{exactly}}{\text{3 aces}} + P(\text{4 aces})$$
   It is far easier, however, to find $1 - P(\underline{\ ?\ })$. No aces
   b. Find the probability of getting at least 1 ace using the method suggested in part (a). 0.3412

WRITTEN EXERCISES

In Exercises 1 and 2, leave your answers in terms of factorials unless directed otherwise by your teacher.

A

1. Five cards are dealt from a well-shuffled standard deck. What is the probability of getting:
   a. all hearts? $\frac{13!\,47!}{8!\,52!}$ b. no hearts? $\frac{39!\,47!}{34!\,52!}$ c. at least one heart? See below.
2. Thirteen cards are dealt from a well-shuffled standard deck. What is the probability of getting:
   a. all clubs? $\frac{13!\,39!}{52!}$ b. no clubs? $\frac{39!\,39!}{26!\,52!}$ c. at least one club? See below.
3. A bag contains 5 red marbles and 3 white marbles. If 2 marbles are randomly drawn, one after the other and without replacement, what is the probability that the number of red marbles is 0? 1? 2? (Check to see that the sum of the probabilities is 1.) $\frac{3}{28}$, $\frac{15}{28}$, $\frac{5}{14}$
4. Repeat Exercise 3 if the bag contains 6 red marbles and 2 white marbles; that is, find the probabilities of drawing 0, 1, and 2 red marbles.

5. Free concert tickets are distributed to 4 students chosen at random from 8 juniors and 12 seniors in the school orchestra. What is the probability that free tickets are received by:
   a. 4 seniors?
   b. exactly 3 seniors?
   c. exactly 2 seniors?
   d. exactly 1 senior?
   e. no seniors?
6. A town council consists of 8 Democrats, 7 Republicans, and 5 Independents. A committee of 3 is chosen by randomly pulling names from a hat. What is the probability that the committee has:
   a. 2 Democrats and 1 Republican?
   b. 3 Independents?
   c. no Independents?
   d. 1 Democrat, 1 Republican, and 1 Independent?

1. c. $1 - \frac{39!\,47!}{34!\,52!}$ 2. c. $1 - \frac{39!\,39!}{26!\,52!}$

Additional Examples cont.

2. A carton contains 200 batteries, of which 5 are defective. If a random sample of 5 batteries is chosen, what is the probability that at least one of them is defective?

$P(\text{at least one defective}) = 1 - P(\text{none defective}) = 1 - \frac{{}_{195}C_5}{{}_{200}C_5} = 1 - \frac{(195)(194)(193)(192)(191)}{(200)(199)(198)(197)(196)} \approx 0.12$

Exercise Note

The technique suggested in Class Exercise 3 is one that students should keep in mind and apply whenever they are asked to find probabilities in "at least one" situations.

Additional Answers Written Exercises

4. $\frac{1}{28}$, $\frac{3}{7}$, $\frac{15}{28}$

5. a. $\frac{33}{323} \approx 0.102$
   b. $\frac{352}{969} \approx 0.363$
   c. $\frac{616}{1615} \approx 0.381$
   d. $\frac{224}{1615} \approx 0.139$
   e. $\frac{14}{969} \approx 0.014$

6. a. $\frac{49}{285} \approx 0.172$
   b. $\frac{1}{114} \approx 0.009$
   c. $\frac{91}{228} \approx 0.399$
   d. $\frac{14}{57} \approx 0.246$

Suggested Assignments

Discrete Math
Day 1: 621/1–12
Day 2: 622/13–20

Supplementary Materials

Alternative Assessment, 51
Student Resource Guide, 148–151

Cooperative Learning

Consider using the B-level exercises for small-group work.

Additional Answers Written Exercises

16. a. 6.30×10^{-12}
b. 2.21×10^{-4}
c. 6.30×10^{-11}
d. 0.987

A committee of 3 people is to be randomly selected from the 6 people Archibald (A), Beatrix (B), Charlene (C), Denise (D), Eloise (E), and Fernando (F). The sample space below lists the 20 possible committees.

{A, B, C}	{A, C, E}	{B, C, D}	{B, E, F}
{A, B, D}	{A, C, F}	{B, C, E}	{C, D, E}
{A, B, E}	{A, D, E}	{B, C, F}	{C, D, F}
{A, B, F}	{A, D, F}	{B, D, E}	{C, E, F}
{A, C, D}	{A, E, F}	{B, D, F}	{D, E, F}

In Exercises 7–14, find the probability in each of two ways:

Method 1: Use the sample space given above.
Method 2: Use combinations.

Sample Find the probability that Archibald and Beatrix are on the committee.

Solution **Method 1** Since 4 of the 20 sample points include A and B,

$$P(\text{A and B}) = \frac{4}{20} = \frac{1}{5}.$$

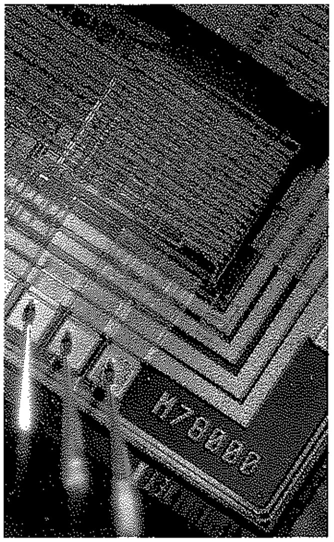
Method 2 $P(\text{A and B}) = \dfrac{{}_2C_2 \cdot {}_4C_1}{{}_6C_3} = \dfrac{1 \cdot 4}{20} = \dfrac{1}{5}$

7. Find the probability that Eloise is on the committee. $\frac{1}{2}$
8. Find the probability that Eloise and Fernando are on the committee. $\frac{1}{5}$
9. Find the probability that either Eloise or Fernando is on the committee. $\frac{4}{5}$
10. Find the probability that neither Eloise nor Fernando is on the committee. $\frac{1}{5}$
11. Find the probability that Archibald is on the committee and Beatrix is not. $\frac{3}{10}$
12. Find the probability that Archibald and Beatrix are on the committee but Charlene is not. $\frac{3}{20}$
13. Find the probability that Denise is on the committee given that Archibald is. $\frac{2}{5}$
14. Find the probability that Denise is on the committee given that neither Archibald nor Beatrix is. $\frac{3}{4}$

B

15. Thirteen cards are dealt from a well-shuffled standard deck. What is the probability of getting:
    a. all red cards? 1.64×10^{-5}
    b. 7 diamonds and 6 hearts? 4.64×10^{-6}
    c. at least 1 face card? 0.981
    d. all face cards? 0
16. Thirteen cards are dealt from a well-shuffled standard deck. What is the probability of getting:
    a. all cards from the same suit?
    b. 7 spades, 3 hearts, and 3 clubs?
    c. all of the 12 face cards?
    d. at least one diamond?
17. A committee of 4 is chosen at random from a group of 5 married couples. What is the probability that the committee includes no two people who are married to each other? $\frac{8}{21}$

18. The letters of the word ABRACADABRA are written on separate slips of paper and placed in a hat. Five slips of paper are then randomly drawn, one after the other and without replacement. What is the probability of getting:
a. all A's? a. $\frac{1}{462}$ b. $\frac{2}{11}$
b. both R's?
c. at least one B? c. $\frac{8}{11}$ d. $\frac{10}{231}$
d. one of each letter?



19. **Manufacturing** A quality control inspector randomly inspects 4 microchips in every lot of 100. If one or more microchips are defective, the entire lot is rejected for shipment. Suppose a lot contains 10 defective microchips and 90 acceptable ones. What is the probability that the lot is rejected?

20. **Manufacturing** A lot of 20 television sets consists of 6 defective sets and 14 good ones. If a sample of 3 sets is chosen, what is the probability that the sample contains:
a. all defective sets? $\frac{1}{57}$ **b.** at least one defective set? $\frac{194}{285}$

C **21.** A standard deck of 52 cards is shuffled and the cards are dealt face up one at a time until an ace appears.
a. What is the probability that the first ace appears on the third card?
b. Show that the probability of getting the first ace on or before the ninth card is greater than 50%.

COMMUNICATION: Reading

Vocabulary in Mathematics

Like most disciplines, mathematics has a special vocabulary. Many of the terms found in a book like this are used only in mathematics. A few examples are *polynomial, logarithm, asymptote,* and *cosine*. Other terms, however, are used in other disciplines or in everyday discourse with meanings different from their mathematical meanings.

1. Consider the word *experiment*. Explain the difference between its meaning in probability and its meaning in science.
2. Select one of the words *degree, function, radical, polar, series,* and look it up in a dictionary. Write down two or three meanings for the word, including its mathematical meaning. For each meaning, write a sentence that uses the word with that meaning.

Application

In connection with Exercises 19 and 20, you might want to discuss defective pacemakers and defective space telescopes to point up how levels of acceptability for defective products can vary greatly.

Additional Answers Written Exercises

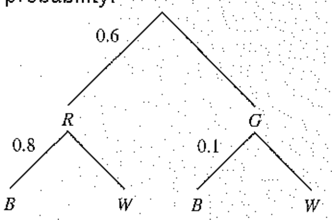
19. $\frac{8279}{23,765} \approx 0.348$

21. a. 0.0681
b. P(1st ace on or before 9th card) = P(ace on 1st) + P(ace on 2nd) + $\cdots$ + P(ace on 9th) = 0.0769 + 0.0724 + 0.0681 + 0.0639 + 0.0599 + 0.0561 + 0.0524 + 0.0489 + 0.0456 = 0.5442 > 50%

Teaching Notes, p. 596D

Warm-Up Exercises

For Exercises 1-6, use the tree diagram shown. Find each probability.



1. $P(G)$ 0.4
2. $P(W \mid R)$ 0.2
3. $P(W \mid G)$ 0.9
4. $P(B)$ (Hint: $P(B) = P(RB) + P(GB)$) 0.52
5. $P(W)$ 0.48
6. Show that $P(RB) + P(RW) + P(GB) + P(GW) = 1$.
   $0.48 + 0.12 + 0.04 + 0.36 = 1$

Motivating the Section

Use a simple example, such as life expectancy, to introduce this section. Suppose, for example, that in a given population, 60% of the females live to the age of 75 and beyond, while only 42% of the males live 75+ years. If the population is 52% female and 48% male, then conditional probability can be used to determine the probability that a 75-year-old person is female. ($P(75) = P(\text{F and } 75) + P(\text{M and } 75) \approx 0.51$. Then $P(\text{F} \mid 75) \approx 0.61$.)

Applications of Probability

16-5 Working with Conditional Probability

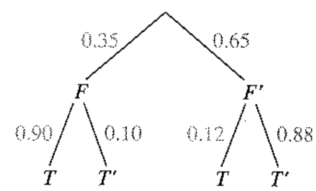
Objective *To solve problems involving conditional probability.*

In this section we will consider the probability of a certain cause when a certain effect is observed. For example, a flu can cause symptoms such as high fever and sore throat, but these can be symptoms of other disorders besides flu. Example 1 examines the probability that a person who has a fever (an effect) also has the flu (a possible cause of the fever).

Example During a flu epidemic, 35% of a school's students have the flu. Of those with the flu, 90% have high temperatures. However, a high temperature is also possible for people without the flu; in fact, the school nurse estimates that 12% of those without the flu have high temperatures.

a. Incorporate the facts given above into a tree diagram.
b. About what percent of the student body have a high temperature?
c. If a student has a high temperature, what is the probability that the student has the flu?

Solution **a.** In the tree diagram at the right, F and F' represent the events "has the flu" and "does not have the flu," and T and T' represent "has a high temperature" and "does not have a high temperature." The probabilities in red come directly from the description in the example. The probabilities in blue are deduced from the fact that all the branches from any given point of a tree must have probabilities that total 1.



b. To find $P(T)$, the probability that a student has a high temperature, we add the probabilities of the two paths leading to a T:

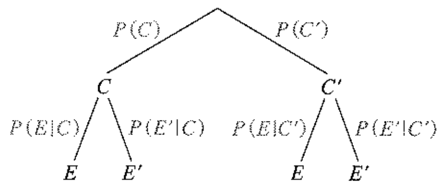
$$\begin{aligned} P(T) &= P(F \text{ and } T) + P(F' \text{ and } T) \\ &= 0.35 \times 0.90 + 0.65 \times 0.12 \\ &= 0.315 + 0.078 = 0.393 \end{aligned}$$

Thus, 39.3% of the student body have a high temperature.

c. Since a high temperature already exists, we consider only the *portion* of the students who have high temperatures. Part (b) has shown that this portion, 0.393, is the sum of 0.315 (those with high temperatures and the flu) and 0.078 (those with high temperatures and no flu). Thus:

$$P(F \mid T) = \frac{P(F \text{ and } T)}{P(T)} = \frac{0.315}{0.393} \approx 0.802$$

The technique used in the preceding example can be generalized. Suppose we have a situation in which there is a cause C and an effect E. Also suppose we know the probability of the occurrence of C and the probabilities of the occurrence of E given the occurrence or nonoccurrence of C. (These three probabilities are shown in red in the tree diagram above.) The problem is to find the probability that C has occurred given that E has.



Let us first symbolically list what we know and can immediately deduce:

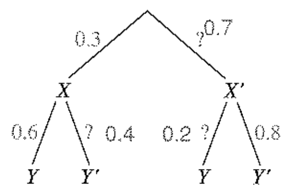
1. $P(C)$ and therefore $P(C') = 1 - P(C)$
2. $P(E|C)$ and therefore $P(E' \mid C) = 1 - P(E \mid C)$
3. $P(E \mid C')$ and therefore $P(E' \mid C') = 1 - P(E \mid C')$

Then, to find $P(C \mid E)$, we take the following two steps:

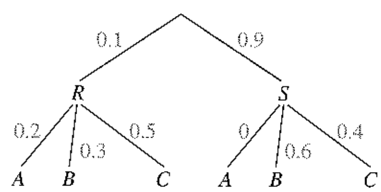
Step 1. Find $P(E) = P(C \text{ and } E) + P(C' \text{ and } E)$
$= P(C) \cdot P(E \mid C) + P(C') \cdot P(E \mid C')$

Step 2. Find $P(C \mid E) = \dfrac{P(C \text{ and } E)}{P(E)}$. Each of these is calculated in Step 1.

CLASS EXERCISES



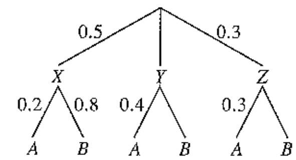
1. **a.** Copy the tree diagram at the right and find the missing probabilities.
   b. The question mark at the lower left refers to $P(Y' \mid X)$. To what probabilities do the other two question marks refer? $P(Y \mid X')$, $P(X')$
   c. Find $P(X \text{ and } Y)$. 0.18
   d. Find $P(X' \text{ and } Y)$. 0.14
   e. Find $P(Y)$. 0.32
   f. Find $P(X|Y)$. 0.5625



2. Use the tree diagram at the right to find each probability.
   a. $P(A|R)$ 0.2
   b. $P(A|S)$ 0
   c. $P(R \text{ and } C)$ 0.05
   d. $P(S \text{ and } C)$ 0.36
   e. $P(C)$ 0.41
   f. $P(R|C)$ 0.122
3. ***Reading*** Study the example in this section. Given the results of parts (b) and (c) of the example, show two different ways to find the probability that a student who has a high temperature does *not* have the flu.

Additional Examples

1. Use the probability tree to find the specified probabilities.



a. $P(Y)$ **b.** $P(B|Y)$
c. $P(B|Z)$ **d.** $P(X \cap A)$
e. $P(Y \cap A)$ **f.** $P(Z \cap A)$
g. $P(A)$ **h.** $P(B)$
i. $P(X|A)$ **j.** $P(Z|A)$

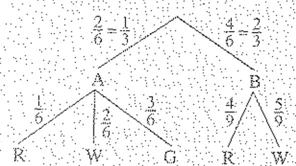
a. $P(Y) = 1 - 0.5 - 0.3 = 0.2$
b. $P(B|Y) = 1 - 0.4 = 0.6$
c. $P(B|Z) = 1 - 0.3 = 0.7$
d. $P(X \cap A) = P(X) \cdot P(A \mid X) = 0.5(0.2) = 0.1$
e. $P(Y \cap A) = P(Y) \cdot P(A|Y) = 0.2(0.4) = 0.08$
f. $P(Z \cap A) = P(Z) \cdot P(A|Z) = 0.3(0.3) = 0.09$
g. $P(A) = P(X \cap A) + P(Y \cap A) + P(Z \cap A) = 0.1 + 0.08 + 0.09 = 0.27$
h. $P(B) = 1 - P(A) = 1 - 0.27 = 0.73$
i. $P(X|A) = \dfrac{P(X \cap A)}{P(A)} = \dfrac{0.1}{0.27} \approx 0.37$
j. $P(Z|A) = \dfrac{P(Z \cap A)}{P(A)} = \dfrac{0.09}{0.27} \approx 0.33$

(*Continues on next page*)

Additional Examples cont.

2. Jar A contains 1 red, 2 white, and 3 green balls. Jar B contains 4 red and 5 white balls. A die is rolled. If a "1" or a "6" comes up, a ball is randomly picked from Jar A. Otherwise, a ball is randomly picked from Jar B.
   a. Draw a tree diagram that shows the given facts.
   b. Find the probability of picking a white ball.
   c. If a white ball is picked, what is the probability that it came from Jar A?

a.



b. $P(W) = P(A \cap W) + P(B \cap W) = P(A) \cdot P(W|A) + P(B) \cdot P(W|B) = \frac{1}{3} \cdot \frac{2}{6} + \frac{2}{3} \cdot \frac{5}{9} = \frac{13}{27}$

c. $P(A|W) = \frac{P(A \cap W)}{P(W)} = \frac{1}{9} \div \frac{13}{27} = \frac{3}{13}$
(Notice that $P(A|G) = 1$ and $P(B|G) = 0$.)

Additional Answers Written Exercises

4. 0.152; 0.848; 0

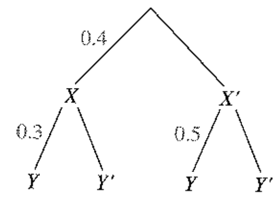
Suggested Assignments

Discrete Math
Day 1: 626/1–9 odd
Day 2: 628/11–17 odd

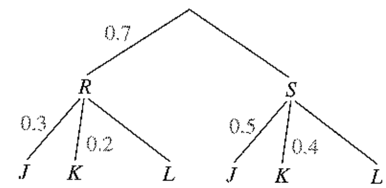
WRITTEN EXERCISES

A

1. Use the tree diagram at the left below to find each probability.
   a. $P(X')$ 0.6 **b.** $P(Y' \mid X)$ 0.7 **c.** $P(Y' \mid X')$ 0.5 **d.** $P(X \text{ and } Y)$ 0.12
   e. $P(X' \text{ and } Y)$ 0.3 **f.** $P(Y)$ 0.42 **g.** $P(X \mid Y)$ 0.286 **h.** $P(X' \mid Y)$ 0.714



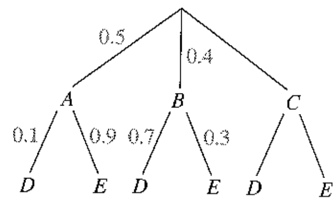
Ex. 1



Ex. 2

2. Use the tree diagram at the right above to find each probability.
   a. $P(S)$ 0.3 **b.** $P(L \mid R)$ 0.5 **c.** $P(L \mid S)$ 0.1 **d.** $P(J \text{ and } R)$ 0.21
   e. $P(J \text{ and } S)$ 0.15 **f.** $P(J)$ 0.36 **g.** $P(R \mid J)$ 0.583 **h.** $P(S \mid J)$ 0.417

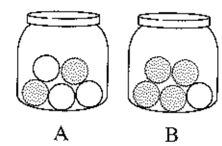
3. Use the tree diagram at the right to find each sum of probabilities.
   a. $P(A \text{ and } D) + P(A \text{ and } E)$ 0.5
   b. $P(B \text{ and } D) + P(B \text{ and } E)$ 0.4
   c. $P(C \text{ and } D) + P(C \text{ and } E)$ 0.1



Exs. 3, 4

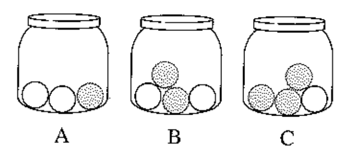
4. Refer to the tree in Exercise 3 and suppose that event D cannot possibly happen if event C happens. Find $P(A \mid D)$, $P(B \mid D)$, and $P(C \mid D)$.

5. Jar A contains 2 red balls and 3 white balls. Jar B contains 4 red balls and 1 white ball. A coin is tossed. If it shows "heads," a ball is randomly picked from Jar A; if it shows "tails," a ball is randomly picked from Jar B.



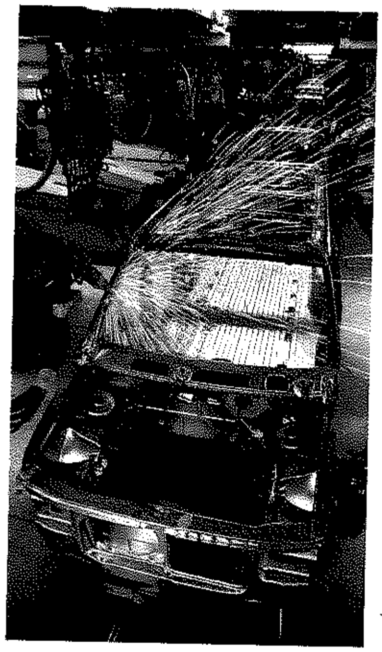
   a. Draw a tree diagram showing the probabilities of each jar and then the probabilities of picking a red ball or a white ball.
   b. Find the probability of picking a red ball. 0.6
   c. If a red ball is picked, find the probability that it came from Jar A. 0.333

6. Jars A, B, and C contain red and white balls as shown. A die is rolled. If an even number comes up, a ball is randomly picked from Jar A. If a "1" or a "3" comes up, a ball is randomly picked from Jar B. If a "5" comes up, a ball is randomly picked from Jar C.



   a. Incorporate the facts given above into a tree diagram.
   b. Find the probability of picking a red ball. $\frac{11}{24} \approx 0.458$
   c. If a red ball is picked, what is the probability that it came from Jar A? from Jar B? from Jar C? $\frac{4}{11} \approx 0.364$; $\frac{4}{11} \approx 0.364$; $\frac{3}{11} \approx 0.273$

7. **Manufacturing** Machine A produces 60% of the ball bearings manufactured by a factory and Machine B produces the rest. Five percent of Machine A's bearings fail to have the required precision, and two percent of Machine B's bearings fail.
   a. Incorporate the facts given above into a tree diagram.
   b. What percent of the bearings fail to have the required precision? 3.8%
   c. If a bearing is inspected and fails to have the required precision, what is the probability that it was produced by Machine A? 0.789



8. **Manufacturing** Five percent of the welds on an automobile assembly line are defective. The defective welds are found using an X-ray machine. The machine correctly rejects 92% of the defective welds and correctly accepts all of the good welds.
   a. Incorporate the facts given above into a tree diagram.
   b. What percent of the welds are accepted by the machine? 95.4%
   c. Find the probability that an accepted weld is defective. 0.004
9. **Insurance** An auto insurance company charges younger drivers a higher premium than it does older drivers because younger drivers as a group tend to have more accidents. The company has 3 age groups: Group A includes those under 25 years old, 22% of all its policyholders. Group B includes those 25–39 years old, 43% of all of its policyholders. Group C includes those 40 years old or older. Company records show that in any given one-year period, 11% of its Group A policyholders have an accident. The percentages for groups B and C are 3% and 2%, respectively.
   a. What percent of the company's policyholders are expected to have an accident during the next 12 months? 4.41%
   b. Suppose Mr. X has just had a car accident. If he is one of the company's policyholders, what is the probability that he is under 25? 0.549
10. **Insurance** Suppose the insurance company of Exercise 9 not only classifies drivers by age, but in the case of drivers under 25 years old, it also notes whether they have had a driver's education course. One quarter of its policyholders under 25 have had driver's education and 5% of these have an accident in a one-year period. Of those under 25 who have not had driver's education, 13% have an accident within a one year period. A 20-year-old woman takes out a policy with this company and within one year she has an accident. What is the probability that she did *not* have a driver's education course? 0.886

Supplementary Materials

Alternative Assessment, 51–52

Problem Solving

As an extension of Exercise 8, have students suppose that each week the assembly line makes 1000 welds and suppose further that a defective weld that is undetected costs the company $75.00. Then ask students to answer the following questions.

1. How much will the company lose on bad welds each week? ($300)
2. Suppose, to save money, the company decides not to use the X-ray machine. Now how much will the company lose on bad welds each week? ($3750)
3. Finally, suppose each X-ray costs $*C*. For what values of *C* is the use of X-rays a wise financial decision? (Use the machine if each X-ray costs less than $3.45.)

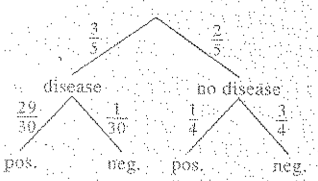
Exercise Note

Suggest that students make tree diagrams to help them solve Exercises 13–16.

Additional Answers Written Exercises

11. a. Answers may vary. A false negative can be serious because someone with the disease will not receive treatment. On the other hand, a false positive can be serious because someone who is healthy might receive expensive, potentially damaging treatment that isn't needed.

b.



c. 0.853
d. 0.938

B **11. Medicine** A medical research lab proposes a screening test for a disease. In order to try out this test, it is given to 100 people, 60 of whom are known to have the disease and 40 of whom are known not to have the disease. A positive test indicates the disease and a negative test indicates no disease. Unfortunately, such medical tests can produce two kinds of errors:

(1) A *false negative* test: For the 60 people who do have the disease, this screening test indicates that 2 do *not* have it.

(2) A *false positive* test: For the 40 people who do not have the disease, this screening test indicates that 10 *do* have it.

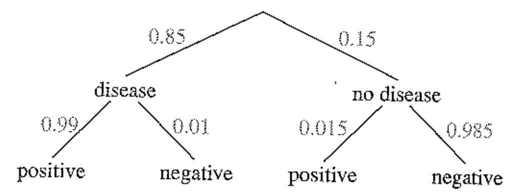
a. Which of the false tests do you think is more serious? Why?

b. Incorporate the facts given above into a tree diagram. (Be sure to convert the given integers into probabilities.)

c. Suppose the test is given to a person not in the original group of 100 people. It is not known whether this person has the disease, but the test result is positive. What is the probability that the person really does have the disease?

d. Suppose the test is given to a person whose disease status is unknown. If the test result is negative, what is the probability that the person does *not* have the disease?

12. Medicine Part (c) of Exercise 11 indicates that about 85% of those who test positive really do have the disease, so that 15% of those who test positive do not have it. This 15% error may seem high, but people with a positive screening test are usually given a more thorough diagnostic test. Even the diagnostic test can yield errors but they are much less likely than the screening test, as the diagram shows.



a. What is the probability that the diagnostic test gives:

(1) a false negative result? 0.0085

(2) a false positive result? 0.00225

b. What is the probability that:

(1) the diagnostic test gives the correct result? 0.989

(2) a person with a positive diagnostic test has the disease? 0.997

(3) a person with a negative diagnostic test does not have the disease? 0.946

13. The children of a math professor play two games that use dice. In one game, two dice are rolled and the sum of the numbers on the dice is called out. In the other game, a single die is rolled and its number is called out. The professor hears the children in another room call out the number 2, and knowing that they play the two games about equally often, the professor is able to calculate the probability they are playing the two-dice game. What is this probability?

14. Solve Exercise 13 if the children call out: **13.** $\frac{1}{7} \approx 0.143$

a. the number 4 $\frac{1}{3}$ **b.** the number 7 1 **c.** the number 1 0

15. Suppose you have two pairs of dice. One pair is fair, but each die of the other pair is weighted so that a six comes up with probability $\frac{1}{4}$ instead of the usual $\frac{1}{6}$. If you randomly choose one pair of dice, roll them, and obtain two sixes, what is the probability that you rolled the weighted dice? $\frac{9}{13} \approx 0.692$

16. Suppose you have 2 coins. One of them is fair, but the other has two "heads." You choose one coin at random, flip it n times, and get "heads" each time.

a. Find the probability that the coin is two-headed for $n = 1$, $n = 2$, and $n = 10$.

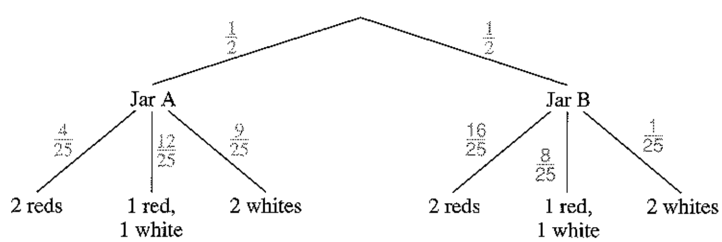
b. In terms of n, find the probability that the coin is two-headed.

C **17.** Refer to the two jars pictured in Exercise 5. Suppose that someone randomly picks one of the jars, but you don't know which one. Before guessing which jar the "mystery jar" is, you have your choice of doing either of the following experiments.

Experiment 1: Pick 2 balls from the "mystery jar" and note their colors. Replace the first ball before choosing the second.

Experiment 2: Pick 2 balls from the "mystery jar" and note their colors. Do *not* replace the first ball before choosing the second.

The tree diagram for Experiment 1 looks like this:



a. Copy and complete the diagram.

b. Suppose you performed Experiment 1 and got 2 red balls. Then you would no doubt guess that they came from Jar B. Which jar would you guess if you got 2 white balls? 1 red and 1 white ball? What is the probability that you would guess the correct jar if you performed the first experiment?

c. Make a tree diagram for Experiment 2.

d. If you performed Experiment 2, which jar would you guess if you got 2 red balls? 2 white balls? 1 red and 1 white ball? What is the probability that you would guess correctly if you performed the second experiment?

e. Which experiment gives you the better chance of guessing correctly?

f. Suppose a third experiment allows you to pick just one ball and then guess the jar. How likely are you to guess correctly in this experiment? Compare your answer with the probability of being correct in the first or second experiment. Are you surprised?

Additional Answers Written Exercises

16. a. $\frac{2}{3}, \frac{4}{5}, \frac{1024}{1025}$

b. $\frac{2^n}{2^n + 1}$

17. b. Jar A; Jar A; 0.74

d. Jar B; Jar A; Jar A; 0.75

e. The second

f. 0.7. The probability of being correct when choosing just one ball is no more than 0.05 from the probability of being correct using two balls.

Teaching Notes, p. 596D

Warm-Up Exercises

In Exercises 1 and 2, find **u** · **v**.

1. **u** = (3, 5, −2) and **v** = $\left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$ $\frac{7}{5}$
2. **u** = (20, 10, −5) and **v** = (0.1, 0.3, 0.6) 2.0

In Exercises 3 and 4, find the probabilities.

3. Two dice are rolled and the sum of the numbers is given below.

$P(6)$	?	$\frac{5}{36}$
$P(7)$	?	$\frac{6}{36}$
$P(8)$	?	$\frac{5}{36}$
$P(9)$	?	$\frac{4}{36}$
$P(10)$	?	$\frac{3}{36}$
$P(11)$	?	$\frac{2}{36}$
$P(12)$	?	$\frac{1}{36}$

4. A jar contains 6 red marbles and 2 white marbles. Two marbles are drawn without replacement.

P(2 red)	?	$\frac{15}{28}$
P(2 white)	?	$\frac{1}{28}$
P(1 red and 1 white)	?	$\frac{3}{7}$

16-6 Expected Value

Objective *To find expected value in situations involving gains and losses and to determine whether a game is fair.*

In this section we will consider probabilistic situations involving gains or losses. For example:

1. What can you expect to win or lose in various games of chance?
2. What is the value of a \$1 ticket in a \$1 million lottery?
3. Should someone who is 18 years old pay \$160 for collision damage insurance on his or her \$1500 car?

Let us begin with the first question. Consider a simple game in which a die is rolled and you win points from, or lose points to, another player as follows.

Event	Die shows 1, 2, or 3	Die shows 4 or 5	Die shows 6
Gain or loss	+10 points	−13 points	−1 point
Probability	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

To calculate the *expected value* of this game, you must multiply each gain or loss by its probability and then add the products.

$$\text{Expected value} = 10 \cdot \frac{3}{6} + (-13) \cdot \frac{2}{6} + (-1) \cdot \frac{1}{6}$$
$$= +\frac{3}{6} = 0.5 \text{ point}$$

The expected value of 0.5 point means that if you were to play this game many times, your *average gain per game* would be 0.5 point. The other player would expect to lose 0.5 point per game.

In general, if a given situation involves various payoffs (gains or losses of points, money, time, and so on), then its **expected value** is calculated as follows.

Payoff	x_1	x_2	x_3	$\cdots$	x_n
Probability	$P(x_1)$	$P(x_2)$	$P(x_3)$	$\cdots$	$P(x_n)$

$$\text{Expected value} = x_1 \cdot P(x_1) + x_2 \cdot P(x_2) + x_3 \cdot P(x_3) + \cdots + x_n \cdot P(x_n)$$

If the expected value of a game is 0, then the game is called a **fair game**. The dice game previously discussed is not a fair game because the expected value is 0.5 point and not 0. An example of a fair game is a coin toss in which you win a point if the coin comes up "heads" and you lose a point if the coin comes up "tails." The expected value is:

$$(1 \text{ point}) \times \frac{1}{2} + (-1 \text{ point}) \times \frac{1}{2} = 0$$

Example 1 If the sum of two rolled dice is 8 or more, you win \$2; if not, you lose \$1.

a. Show that this is not a fair game.

b. To have a fair game, the \$2 winnings should instead be what amount?

Solution Make a table of payoffs. For the probabilities, use the table on page 601.

Event	Sum ≥ 8	Sum < 8
Payoff	\$2	−\$1
Probability	$\frac{15}{36}$	$\frac{21}{36}$

a. Expected value of game $= 2\left(\frac{15}{36}\right) + (-1)\left(\frac{21}{36}\right) = +\frac{1}{4} = \$.25$

b. To have a fair game, let x be the winnings for a sum greater than or equal to 8 and find the value of x that produces an expected value of 0.

$$\text{Expected value} = x\left(\frac{15}{36}\right) + (-1)\left(\frac{21}{36}\right) = 0$$

$$15x - 21 = 0$$

$$x = \$1.40$$

Example 2 In a certain state's lottery, six numbers are randomly chosen without repetition from the numbers 1 to 40. If you correctly pick all 6 numbers, only 5 of the 6, or only 4 of the 6, then you win \$1 million, \$1000, or \$100, respectively. What is the value of a \$1 lottery ticket?

Solution First we must find the probabilities of winning:

$$P(\text{all 6 correct}) = \frac{{}_6C_6}{{}_{40}C_6} \approx 0.00000026$$

$$P(\text{5 of 6 correct}) = \frac{{}_6C_5 \cdot {}_{34}C_1}{{}_{40}C_6} \approx 0.000053$$

$$P(\text{4 of 6 correct}) = \frac{{}_6C_4 \cdot {}_{34}C_2}{{}_{40}C_6} \approx 0.0022$$

Then we multiply these probabilities by their associated gains, temporarily ignoring the cost of the \$1 ticket:

Expected gain

$\approx 1{,}000{,}000(0.00000026) + 1000(0.000053) + 100(0.0022)$

$= 0.26 + 0.053 + 0.22 \approx \$.53$

When we consider the \$1 cost of the ticket, the expected value of the ticket is about $\$0.53 - \$1 = -\$0.47$.

Motivating the Section

To introduce this section, sketch on the board a spinner divided into quarters. Mark two of the regions "Win \$1," mark one of them "Win \$5," and mark the last region "Lose \$15." Tell students that the spinner will be spun 40 times; ask how often they expect to land on "Win \$5" (10 times) and then write $10 \times \$5 = \$50$ on the board. Ask how often they expect to land on "Win \$1" (20 times) and write $20 \times \$1 = \$20$ on the board. Finally, ask how often they expect to land on "Lose \$15" (10 times) and write $10 \times (-\$15) = -\$150$ on the board. Add the expected winnings and losses to obtain −\$80. Explain that this is an average loss of \$80 in 40 spins, or \$80/40 = \$2.00 per spin. Distributing the division as shown below should help students see the reasoning behind multiplying payoffs by the corresponding probability:

$$\frac{\$5 \times 10 + \$1 \times 20 + (-\$15) \times 10}{40}$$

$= \$5 \times \frac{10}{40} + \$1 \times \frac{20}{40} + (-\$15) \times \frac{10}{40} =$

$\$5(0.25) + \$1(0.50) + (-\$15)(0.25) = -\$2.00$

Example Note

After discussing Example 2, you might wish to ask students why they think anyone would buy a ticket worth −\$0.47. Common replies might be "Someone has to win" or "It's entertaining."

Additional Examples

1. Two coins are tossed. If both land heads up, then player A wins \$4 from player B. If exactly one coin lands heads up, then B wins \$1 from A. If both land tails up, then B wins \$2 from A. Show that this is a fair game.

$P(\text{HH}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(\text{TT}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(\text{one head}) = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$

Heads	2	1	0
Payoff to A	\$4	$-\$1$	$-\$2$
Probability	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Expected value of game $= \$4\left(\frac{1}{4}\right) + (-\$1)\left(\frac{1}{2}\right) + (-\$2)\left(\frac{1}{4}\right) = 0$

Thus, the game is fair.

2. Three cards are drawn at random, without replacement, from a standard deck. Find the expected value for the occurrence of hearts.

Hearts	Probability
0	$\frac{_{39}C_3}{_{52}C_3} = \frac{703}{1700}$
1	$\frac{_{39}C_2 \cdot {_{13}C_1}}{_{52}C_3} = \frac{741}{1700}$
2	$\frac{_{39}C_1 \cdot {_{13}C_2}}{_{52}C_3} = \frac{234}{1700}$
3	$\frac{_{13}C_3}{_{52}C_3} = \frac{22}{1700}$

Example 3 An 18-year-old student must decide whether to spend \$160 for one year's car collision damage insurance. The insurance carries a \$100 deductible, which means that when the student files a damage claim, the student must pay \$100 of the damage amount, with the insurance company paying the rest (up to the value of the car). Because the car is only worth \$1500, the student consults with an insurance agent who draws up a table of possible damage amounts and their probabilities based on the driving records for 18-year-olds in the region.

Event	Accident costing \$1500	Accident costing \$1000	Accident costing \$500	No accident
Payoff	\$1400	\$900	\$400	\$0
Probability	0.05	0.02	0.03	0.90

What is the expected value of this insurance?

Solution We temporarily ignore the \$160 cost and calculate the expected payoff from the table above.

$$\begin{aligned}\text{Expected payoff} &= 1400(0.05) + 900(0.02) + 400(0.03) + 0(0.90)\\ &= 70 + 18 + 12 + 0\\ &= \$100\end{aligned}$$

Now we consider the \$160 cost and find that:

$$\text{Expected value} = 100 - 160 = -\$60$$

While an insurance policy with an expected value of $-\$60$ may not seem like a "fair game" between the student and the insurance company, the company uses the \$60 to pay for operating expenses, salaries, and profit.

CLASS EXERCISES

Find the expected payoff.

1.

Payoff	5	10
Probability	0.6	0.4

7

2.

Payoff	3	1	−1
Probability	0.5	0.1	0.4

1.2

3. Players A and B play a game in which a die is rolled and A wins 2 points from B if a 5 or 6 appears. Otherwise, B wins 1 point from A. Decide if this is a fair game. Yes

4. ***Reading*** The game in Example 1 can be made fair by changing the winnings from \$2 to \$1.40 (see part (b)). Find a different way to make the game fair.

5. Suppose you toss 3 coins and win payoffs as shown in the table below. Complete the table and find your expected payoff.

Number of "heads"	3	2	1	0
Payoff	\$5	\$3	\$1	−\$9
Probability	?	?	?	?

$\frac{1}{8}$; $\frac{3}{8}$; $\frac{3}{8}$; $\frac{1}{8}$; \$1

6. ***Discussion*** Discuss why the student in Example 3 may want the car collision damage insurance even though its expected value to the student is negative.

WRITTEN EXERCISES

Find the expected payoff.

A 1.

Payoff	9	7	−5
Probability	0.1	0.3	0.6

0

2.

Payoff	6	3	−5
Probability	0.2	0.1	0.7

−2

3.

Payoff	60	52	50
Probability	0.4	0.5	0.1

55

4.

Payoff	13	−7	−12
Probability	0.4	0.2	0.4

−1

For Exercises 5–8, decide if each game is a fair game. If not, state which player has the advantage.

5. A die is rolled. If the number that shows is odd, player A wins \$1 from player B. If it is a 6, A wins \$2 from B. Otherwise B wins \$3 from A. Not fair; B
6. A box contains 2 red balls and 1 white ball. Two balls are randomly chosen without replacement. If both are red, player A wins \$5 from player B. Otherwise B wins \$2 from A. Not fair; A
7. Two dice are rolled. If the sum is 6, 7, or 8, player A wins \$5 from player B. Otherwise B wins \$4 from A. Fair
8. Two dice are rolled. If the sum of the numbers showing on the dice is odd, player A wins \$1 from player B. If both dice show the same number, A wins \$3 from B. Otherwise B wins \$3 from A. Fair
9. Suppose you play a game in which you make a bet and then draw a card from a well-shuffled deck that includes the standard 52 cards as well as 2 jokers. If you draw a joker, you keep your bet and win \$5; if you draw a face card, you keep your bet and win \$2; and if you draw any other card, you lose your bet. What is your expected gain or loss on this game if your bet is \$1? An 11¢ loss
10. Suppose you have \$10 to bet on the game described in Exercise 9. Is your expected gain or loss any different if you bet the whole \$10 on one game rather than betting \$1 at a time on 10 successive games?

Additional Examples cont.

The expected value of hearts occurring is:

$$0\left(\frac{703}{1700}\right) + 1\left(\frac{741}{1700}\right) + 2\left(\frac{234}{1700}\right) + 3\left(\frac{22}{1700}\right) = \frac{1275}{1700} = 0.75$$

Additional Answers Class Exercises

4. Answers may vary. For example, if the sum is greater than 7, win \$1; if the sum equals 7, no money is exchanged; and if the sum is less than 7, lose \$1.

6. The student may not have \$1500 available to replace the car. If the student has \$160 for insurance, he or she would be able to replace the car.

Additional Answers Written Exercises

10. The expected loss for one \$10 game is \$6.78. The expected loss for ten \$1 games is \$1.10.

Suggested Assignments

Discrete Math
Day 1: 633/1–13 odd
Day 2: 634/14, 15–23 odd

Supplementary Materials

Alternative Assessment, 52
Student Resource Guide, 152–154

Review Note

Students may need to review notation from Chapters 12 and 13 in order to answer Exercises 15 and 16.

Additional Answers Written Exercises

15. $\sum_{i=1}^{n} x_i P(x_i)$

16. Let $\mathbf{u} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{v} = (P(x_1), P(x_2), \ldots, P(x_n))$; then the expected value equals $\mathbf{u} \cdot \mathbf{v}$.

B **11.** Test Taking On a multiple-choice test, a student is given five possible answers for each question. The student receives 1 point for a correct answer and loses $\frac{1}{4}$ point for an incorrect answer. If the student has no idea of the correct answer for a particular question and merely guesses, what is the student's expected gain or loss on the question? 0

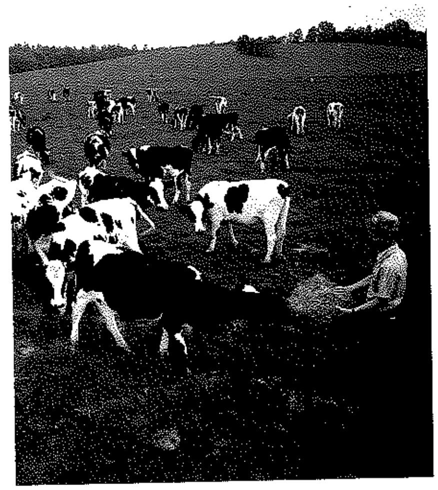
12. Test Taking Suppose you are taking the multiple-choice test described in Exercise 11. Suppose also that on one of the questions you can eliminate two of the five answers as being wrong. If you guess at one of the remaining three answers, what is your expected gain or loss on the question? $\frac{1}{6}$

13. A box contains 3 red balls and 2 green balls. Two balls are randomly chosen without replacement. If both are green, you win \$2. If just one is green, you win \$1. Otherwise you lose \$1. What is your expected gain or loss? 50¢ gain

14. In the carnival game "chuck-a-luck," you pick a number from 1 to 6 and roll 3 dice in succession. If your number comes up all 3 times, you win \$3; if your number comes up twice, you win \$2; if it comes up once, you win \$1; otherwise you lose \$1. What is your expected gain or loss? (*Hint*: Make a table like the one in Class Exercise 5.) 8¢ loss

15. Rewrite the definition of expected value (page 630) using sigma notation.

16. Rewrite the definition of expected value (page 630) as a dot product of two vectors.



17. Farming A dairy farmer estimates that next year the farm's cows will produce about 25,000 gallons of milk. Because of variation in the market price of milk and the cost of feeding the cows, the profit per gallon may vary with the probabilities given in the table below. Estimate the profit on the 25,000 gallons.

\$20,850

Gain per gallon	\$1.10	\$.90	\$.70	\$.40	\$.00	−\$.10
Probability	0.30	0.38	0.20	0.06	0.04	0.02

18. Insurance At many airports, a person can pay only \$1 for a \$100,000 life insurance policy covering the duration of the flight. In other words, the insurance company pays \$100,000 if the insured person dies from a possible flight crash; otherwise the company gains \$1 (before expenses). Suppose that past records indicate 0.45 deaths per million passengers. How much can the company expect to gain on one policy? on 100,000 policies? 95.5¢; \$95,500

19. **Business** A construction company wants to submit a bid for remodeling a school. The research and planning needed to make the bid cost \$4000. If the bid is accepted, the company would make \$26,000. Would you advise the company to spend the \$4000 if the bid has only a 20% probability of being accepted? Explain your reasoning.

20. **Consumer Economics** Suppose the warranty period for your family's new television is about to expire and you are debating about whether to buy a one-year maintenance contract for \$35. If you buy the contract, all repairs for one year are free. Consumer information shows that 12% of the televisions like yours require an annual repair that costs \$140 on the average. Would you advise buying the maintenance contract? Explain your reasoning.

21. A lottery has one \$1000 prize, five \$100 prizes, and twenty \$10 prizes. What is the expected gain from buying one of the 2000 tickets sold for \$1 each? 15¢ loss

22. In a state lottery, five numbers are randomly chosen from the numbers 1 to 30. If you pick all 5 numbers, you win \$100,000; and if you pick 4 of the 5 numbers, you win \$100. What is the value of a \$1 lottery ticket? −21¢

23. Players A and B are playing a game in which A wins a point every time a coin lands "heads" and B wins a point every time the coin lands "tails." (No points are lost by either player.) The first person to reach 3 points wins \$100. If A currently has 2 points and B has 1 point, what is A's expected gain? (*Hint*: Make a tree diagram showing the ways in which the game can be finished. From the diagram, determine the probability that A wins.) \$75

24. Suppose you play a game in which you make a bet, toss a coin, and either win an amount equal to your bet or lose your bet depending on whether you correctly call "heads" or "tails." Also suppose you begin with a \$1 bet and double your bet on each toss until you win once and leave the game or until you have lost \$15. What can you expect to win with this betting strategy? \$0

C 25. A die is rolled repeatedly until a "1" appears.

a. Complete the table.

Number of rolls until "1" appears	1	2	3	4	5	$\cdots$	n	$\cdots$
Probability	$\frac{1}{6}$	$\frac{5}{6}\cdot\frac{1}{6}$	$\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}$	?	?	$\cdots$	?	$\cdots$

b. Express the expected number of rolls as the sum of an infinite series. Then factor $\frac{1}{6}$ from the sum.

c. It can be proved (most easily with calculus) that

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Use this to show that the expected number of rolls until a "1" appears is 6.

Additional Answers Written Exercises

19. Answers may vary. The expected value is positive, but the company stands an 80% chance of losing \$4000. Perhaps the company should bid on other jobs with lower costs of bidding and higher probabilities of being accepted.

20. Answers may vary. No, because after considering the cost of the policy, the expected value is negative.

25. a. $\left(\frac{5}{6}\right)^3\cdot\frac{1}{6}$; $\left(\frac{5}{6}\right)^4\cdot\frac{1}{6}$; $\left(\frac{5}{6}\right)^{n-1}\cdot\frac{1}{6}$

b. $\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}\cdot\frac{1}{6} = \frac{1}{6}\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}$

c. $\frac{1}{6}\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1} = \frac{1}{6}\cdot\frac{1}{\left(1-\frac{5}{6}\right)^2} = \frac{1}{6}\cdot\frac{1}{\left(\frac{1}{6}\right)^2} = 6$

Supplementary Materials

Tests, 50–51
Alternative Assessment, 70
Student Resource Guide, 155–156

Additional Answers Chapter Test

1. Empirical probability is determined by observing what previously happened. Theoretical probability is determined by reasoning about the events.

Chapter Summary

1. The probability of an event is a number (between 0 and 1, inclusive) that indicates the likelihood of the event's occurrence. Probabilities can be determined either empirically or theoretically.
2. An *experiment* is any action having various outcomes that occur unpredictably. A set S is a *sample space* of an experiment if each outcome of the experiment corresponds to exactly one element of S. Any subset of S is an *event*.
3. If the sample space of an experiment consists of n equally likely outcomes, m of which correspond to event A, then the probability of A is $P(A) = \frac{m}{n}$. The event "not A" occurs whenever event A does not, and $P(\text{not } A) = 1 - P(A)$.
4. Events A and B are *mutually exclusive* if they cannot occur simultaneously.
5. For any two events A and B, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, or, if A and B are mutually exclusive, $P(A \text{ or } B) = P(A) + P(B)$.
6. For two events A and B, the probability that B occurs on the condition that A occurs is denoted $P(B|A)$ and is called a *conditional probability.*
7. Events A and B are *independent* if the occurrence of one does not affect the probability of the other's occurrence, that is, $P(B|A) = P(B)$.
8. For any two events A and B, $P(A \text{ and } B) = P(A) \cdot P(B|A)$, or, if A and B are independent, $P(A \text{ and } B) = P(A) \cdot P(B)$.
9. The binomial probability theorem gives the probabilities of 0, 1, 2, . . . , n occurrences of the event A in n repeated independent trials.
10. If a given situation involves payoffs $x_1, x_2, \ldots, x_n$ with corresponding probabilities $P(x_1), P(x_2), \ldots, P(x_n)$, then the situation's *expected value* is:

$$x_1 \cdot P(x_1) + x_2 \cdot P(x_2) + x_3 \cdot P(x_3) + \cdots + x_n \cdot P(x_n)$$

Key vocabulary and ideas

probability (p. 597)
empirical probability (p. 597)
theoretical probability (p. 597, p. 598)
experiment, sample space, event (p. 598)
probability of event A, or $P(A)$ (p. 598)
$P(A \text{ or } B)$ (p. 599)
mutually exclusive events (p. 599)
$P(\text{not } A)$ (p. 599)
odds (p. 601)
conditional probability (p. 607)
independent events (p. 607)
$P(A \text{ and } B)$ (p. 608)
binomial probability theorem (p. 614)
Monte Carlo simulation (p. 617)
expected value (p. 630)
fair game (p. 630)

Chapter Test

1. ***Writing*** Write a paragraph in which you discuss the difference between, and give examples of, empirical and theoretical probability. 16-1

2. An experiment consists of randomly drawing a card from a standard deck and rolling a die. Find the probability that:
   a. the card is a spade and the die shows a 5 $\frac{1}{24}$
   b. the card is *not* a spade and the die shows a 5 $\frac{1}{8}$
   c. the card is *not* a spade and the die does *not* show a 5 $\frac{5}{8}$
3. A fish bowl contains slips of paper numbered 1 through 9. Two slips are drawn, one after the other and without replacement. Find the probability of the each of the following events. 16-2
   a. Both numbers are odd. $\frac{5}{18}$
   b. The first number drawn is 6. $\frac{1}{9}$
   c. The first number drawn is greater than 6 and the second number is less than 4. $\frac{1}{8}$
4. For a certain brand of marigold seeds, the seeds sprout on average 9 out of 10 times. If 5 seeds are selected at random and planted, find the probability that: 16-3
   a. all 5 seeds sprout 0.59
   b. at least 3 of the 5 seeds sprout 0.991
5. A Central School PTA committee is to consist of 3 teachers and 3 parents, who are to be chosen at random from the 24 teachers and 145 parents involved in the PTA. If half of the teachers and 99 parents are women, find the probability that the committee has: 16-4
   a. only female members 0.034
   b. only 1 parent and only 1 teacher who are women 0.081
6. Ninety-five percent of the sneakers manufactured by a shoe company have no defects. In order to find the 5% that do have defects, inspectors carefully look over every pair of sneakers. Still, the inspectors sometimes make mistakes because 8% of the defective pairs pass inspection and 1% of the good pairs fail the inspection test. 16-5
   a. Incorporate the facts given above into a tree diagram.
   b. What percent of the pairs of sneakers pass inspection? 94.5%
   c. If a pair of sneakers passes inspection, what is the probability that it has a defect? 0.004
7. Suppose you play a game in which 5 coins are tossed simultaneously. If 1, 2, 3, or 4 "heads" occur, you win $1 for each "head." If all "heads" or all "tails" occur, however, you lose $20. 16-6
   a. Copy and complete the following table.

Number of "heads"	0	1	2	3	4	5
Payoff	−$20	?	?	?	?	?
Probability	$\frac{1}{32}$	?	?	?	?	?

$1 $2 $3 $4 −$20

$\frac{5}{32}$ $\frac{10}{32}$ $\frac{10}{32}$ $\frac{5}{32}$ $\frac{1}{32}$

   b. What is the game's expected payoff? $1.09

Additional Answers Chapter Test

6. a.