4.1 INTRODUCTION TO THE FAMILY OF EXPONENTIAL FUNCTIONS

Growing at a Constant Percent Rate

Linear functions represent quantities that change at a constant rate. In this section we introduce functions that change at a constant *percent* rate, the *exponential functions*.

Salary Raises

Example 1 After graduation from college, you will probably be looking for a job. Suppose you are offered a job at a starting salary of \$40,000 per year. To strengthen the offer, the company promises annual raises of 6% per year for at least the first five years after you are hired. Let's compute your salary for the first few years.

If t represents the number of years since the beginning of your contract, then for $t = 0$, your salary is $\$40,000$. At the end of the first year, when $t = 1$, your salary increases by 6% so

$$\begin{aligned}\text{Salary when } t=1 &= \text{Original salary} + 6\% \text{ of Original salary}\\ &= 40{,}000 + 0.06 \cdot 40{,}000\\ &= 42{,}400 \text{ dollars.}\end{aligned}$$

After the second year, your salary again increases by 6%, so

$$\begin{aligned}\text{Salary when } t=2 &= \text{Former salary} + 6\% \text{ of Former salary}\\ &= 42{,}400 + 0.06 \cdot 42{,}400\\ &= 44{,}944 \text{ dollars.}\end{aligned}$$

Notice that your raise is higher in the second year than in the first since the second 6% increase applies both to the original \$40,000 salary and to the \$2400 raise given in the first year.

Salary calculations for four years have been rounded and recorded in Table 4.1. At the end of the third and fourth years your salary again increases by 6%, and your raise is larger each year. Not only are you given the 6% increase on your original salary, but your raises earn raises as well.

Table 4.1 *Raise amounts and resulting salaries for a person earning 6% annual salary increases*

Year	Raise amount (\$)	Salary (\$)
0		40,000.00
1	2400.00	42,400.00
2	2544.00	44,944.00
3	2696.64	47,640.64
4	2858.44	50,499.08

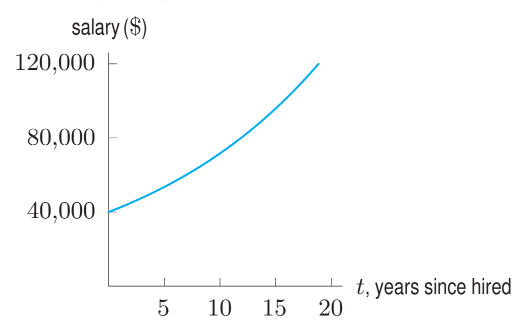


Figure 4.1: Salary over a 20-year period

Figure 4.1 shows salary over a 20-year period assuming that the annual increase remains 6%. Since the rate of change of your salary (in dollars per year) is not constant, the graph of this function is not a line. The salary increases at an increasing rate, giving the graph its upward curve. A function that increases at a constant percent rate is said to increase exponentially.

Population Growth

Exponential functions provide a reasonable model for many growing populations.

Example 2 During the 2000s, the population of Mexico increased at a constant annual percent rate of 1.2%.[1] Since the population grew by the same percent each year, it can be modeled by an exponential function.

Let's calculate the population of Mexico for the years after 2000. In 2000, the population was 100 million. The population grew by 1.2%, so

$$\begin{aligned}\text{Population in 2001} &= \text{Population in 2000 } + 1.2\% \text{ of Population in 2000}\\ &= 100 + 0.012(100)\\ &= 100 + 1.2 = 101.2 \text{ million.}\end{aligned}$$

Similarly,

$$\begin{aligned}\text{Population in 2002} &= \text{Population in 2001 } + 1.2\% \text{ of Population in 2001}\\ &= 101.2 + 0.012(101.2)\\ &= 101.2 + 1.2144 = 102.4144 \text{ million.}\end{aligned}$$

The calculations for years 2000 through 2007 have been rounded and recorded in Table 4.2. The population of Mexico increased by slightly more each year than it did the year before, because each year the increase is 1.2% of a larger number.

Table 4.2 *Calculated values for the population of Mexico*

Year	ΔP, increase in population	P, population (millions)
2000	—	100
2001	1.2	101.2
2002	1.21	102.41
2003	1.23	103.64
2004	1.25	104.89
2005	1.26	106.15
2006	1.27	107.42
2007	1.29	108.71

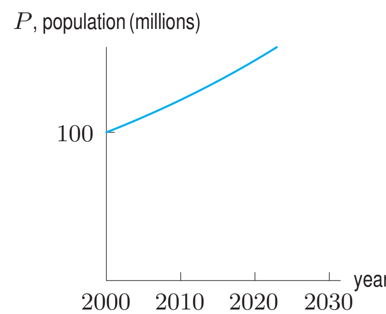


Figure 4.2: The projected population of Mexico, assuming 1.2% annual growth

Figure 4.2 gives a graph of the population of Mexico over a 30-year period, assuming a 1.2% annual growth rate. Notice that this graph curves upward like the graph in Figure 4.1.

Radioactive Decay

Exponential functions can also model decreasing quantities. A quantity which decreases at a constant percent rate is said to be decreasing exponentially.

Example 3 Carbon-14 is used to estimate the age of organic compounds. Over time, radioactive carbon-14 decays into a stable form. The decay rate is 11.4% every 1000 years. For example, if we begin with a 200-microgram (μg) sample of carbon-14 then

[1] http://www.census.gov/ipc/www/idb/country.php, accessed May 23, 2010.

$$\begin{aligned}\text{Amount remaining after 1000 years} &= \text{Initial amount} - 11.4\% \text{ of Initial amount}\\ &= 200 - 0.114 \cdot 200\\ &= 177.2\ \mu\text{g}.\end{aligned}$$

Similarly,

$$\begin{aligned}\text{Amount remaining after 2000 years} &= \text{Amount remaining after 1000 years} - 11.4\% \text{ of } \text{Amount remaining after 1000 years}\\ &= 177.2 - 0.114 \cdot 177.2 \approx 156.999\ \mu\text{g},\end{aligned}$$

and

$$\begin{aligned}\text{Amount remaining after 3000 years} &= \text{Amount remaining after 2000 years} - 11.4\% \text{ of } \text{Amount remaining after 2000 years}\\ &= 156.999 - 0.114 \cdot 156.999 \approx 139.101\ \mu\text{g}.\end{aligned}$$

These calculations are recorded in Table 4.3. During each 1000-year period, the amount of carbon-14 that decays is smaller than in the previous period. This is because we take 11.4% of a smaller quantity each time.

Table 4.3 *The amount of carbon-14 remaining over time*

Years elapsed	Amount decayed (μg)	Amount remaining (μg)
0	—	200.0
1000	22.8	177.2
2000	20.201	156.999
3000	17.898	139.101

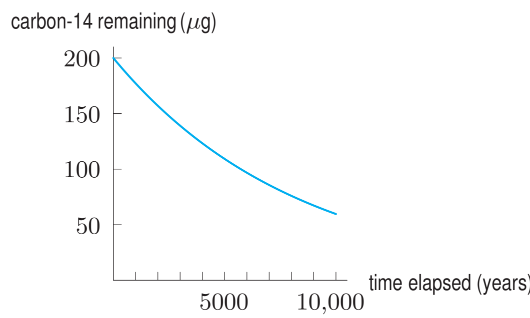


Figure 4.3: Amount of carbon-14 over 10,000 years

Figure 4.3 shows the amount of carbon-14 left from a 200 μg sample over 10,000 years. Because the amount decreases by a smaller amount over each successive time interval, the graph is not linear but bends upward, hence is concave up.

Growth Factors and Percent Growth Rates

The Growth Factor of an Increasing Exponential Function

The salary in Example 1 increases by 6% every year. We say that the annual percent growth rate is 6%. But there is another way to think about the growth of this salary. We know that each year,

$$\text{New salary} = \text{Old salary} + 6\% \text{ of Old salary}.$$

We can rewrite this as follows:

$$\text{New salary} = 100\% \text{ of Old salary} + 6\% \text{ of Old salary}.$$

So

$$\text{New salary} = 106\% \text{ of Old salary}.$$

Since $106\% = 1.06$, we have

$$\text{New salary} = 1.06 \cdot \text{Old salary}.$$

We call 1.06 the *growth factor*.

The Growth Factor of a Decreasing Exponential Function

In Example 3, the carbon-14 changes by -11.4% every 1000 years. The negative growth rate tells us that the quantity of carbon-14 decreases over time. We have

$$\text{New amount} = \text{Old amount} - 11.4\% \text{ of Old amount},$$

which can be rewritten as

$$\text{New amount} = 100\% \text{ of Old amount} - 11.4\% \text{ of Old amount}.$$

So,

$$\text{New amount} = 88.6\% \text{ of Old amount}.$$

Since $88.6\% = 0.886$, we have

$$\text{New amount} = 0.886 \cdot \text{Old amount}.$$

We call 0.886 the growth factor even though the amount of carbon-14 is decreasing. Although it may sound strange to refer to 0.886 as the growth factor, rather than the decay factor, we use "growth factor" to describe both increasing and decreasing quantities.

A General Formula for the Family of Exponential Functions

Because it grows at a constant percentage rate each year, the salary, S, in Example 1 is an example of an exponential function. We want a formula for S in terms of t, the number of years since being hired. Since the annual growth factor is 1.06, we know that for each year,

$$\text{New salary} = \text{Old salary} \cdot 1.06.$$

Thus, after one year, or when $t = 1$,

$$S = \underbrace{40{,}000}_{\text{Old salary}}(1.06).$$

Similarly, when $t = 2$,

$$S = \underbrace{40{,}000(1.06)}_{\text{Old salary}}(1.06) = 40{,}000(1.06)^2.$$

Here there are *two* factors of 1.06 because the salary has increased by 6% twice. When $t = 3$,

$$S = \underbrace{40,000(1.06)^2}_{\text{Old salary}}(1.06) = 40{,}000(1.06)^3$$

and continues in this pattern so that after t years have elapsed,

$$S = 40{,}000\underbrace{(1.06)(1.06)\dots(1.06)}_{t\text{ factors of }1.06} = 40{,}000(1.06)^t.$$

After t years the salary has increased by a factor of 1.06 a total of t times. Thus,

$$S = 40{,}000(1.06)^t.$$

These results, which are summarized in Table 4.4, are the same as in Table 4.1. Notice that in this formula we assume that t is an integer, $t \geq 0$, since the raises are given only once a year.

Table 4.4 *Salary after t years*

t (years)	S, salary (\$)
0	40,000
1	$40{,}000(1.06) = 42{,}400.00$
2	$40{,}000(1.06)^2 = 44{,}944.00$
3	$40{,}000(1.06)^3 = 47{,}640.64$
t	$40,000(1.06)^t$

This salary formula can be written as

$$S = \text{Initial salary} \cdot (\text{Growth factor})^t.$$

In general, we have:

An **exponential function** $Q = f(t)$ has the formula

$$f(t) = ab^t, \quad a \neq 0, b > 0,$$

where a is the initial value of Q (at $t = 0$) and b, the base, is the growth factor. The growth factor is given by

$$b = 1 + r$$

where r is the decimal representation of the percent rate of change.

- If there is exponential growth, then $r > 0$ and $b > 1$.
- If there is exponential decay, then $r < 0$ and $0 < b < 1$.

The constants a and b are called *parameters*.. The base b is restricted to positive values because if $b < 0$ then b^t is undefined for some exponents t, for example, $t = 1/2$.

Every function in the form $f(t) = ab^t$ with the input, t, in the exponent is an exponential function, provided $a \neq 0$. Note that if $b = 1$, then $f(t) = a \cdot 1^t = a$ and $f(t)$ is a constant, so when $b = 1$, the function is generally not considered exponential. Graphs showing exponential growth and decay are in Figures 4.4 and 4.5. Notice that in both cases the graph is concave up.

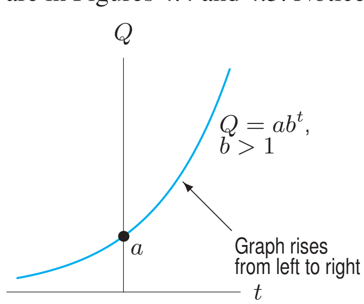


Figure 4.4: Exponential growth: $b > 1$

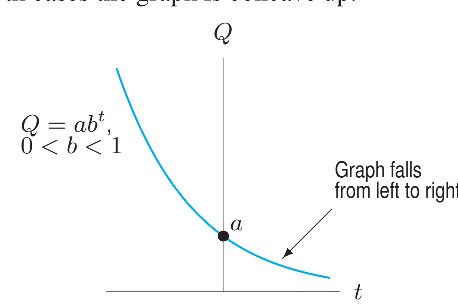


Figure 4.5: Exponential decay: $0 < b < 1$

Example 4 Use the formula $S = 40{,}000(1.06)^t$ to calculate your salary after 4 years, 12 years, and 40 years.

Solution After 4 years, $t = 4$, and we have

$$S = 40{,}000(1.06)^4 \approx \$50{,}499.08.$$

Notice that this agrees with Table 4.1 on page 130. After 12 years, $t = 12$, and we have

$$S = 40{,}000(1.06)^{12} \approx \$80{,}487.86.$$

After 12 years, the salary has more than doubled from the initial salary of $\$40,000$. When $t = 40$ we have

$$S = 40{,}000(1.06)^{40} \approx \$411{,}428.72.$$

Thus if you work for 40 years and consistently earn 6% annual raises, your salary will be over $\$400{,}000$ a year.

Example 5 Carbon-14 decays at a rate of 11.4% every 1000 years. Write a formula for the quantity, Q, of a 200-μg sample remaining as a function of time, t, in thousands of years.

Solution The growth factor of carbon-14 over 1000 years is $1 - 0.114 = 0.886$. Originally, there are 200 μg, so the quantity remaining after t thousand years is given by

$$Q = 200(0.886)^t.$$

Example 6 Using Example 2 on page 131, find a formula for P, the population of Mexico (in millions), in year t where $t = 0$ represents the year 2000.

Solution In 2000, the population of Mexico was 100 million, and it was growing at a constant 1.2% annual rate. The growth factor is $b = 1 + 0.012 = 1.012$, and $a = 100$, so

$$P = 100(1.012)^t.$$

Because the growth factor may change eventually, this formula may not give accurate results for large values of t.

Example 7 What does the formula $P = 100(1.012)^t$ predict when $t = 0$? When $t = -5$? What do these values tell you about the population of Mexico?

Solution If $t = 0$, then , since $(1.012)^0 = 1$, we have

$$P = 100(1.012)^0 = 100.$$

This makes sense because $t = 0$ stands for 2000, and in 2000 the population was 100 million. When $t = -5$ we have

$$P = 100(1.012)^{-5} \approx 94.210.$$

To make sense of this number, we must interpret the year $t = -5$ as five years before 2000; that is, as the year 1995. If the population of Mexico had been growing at a 1.2% annual rate from 1995 onward, then it was 94.21 million in 1995.

Example 8 On August 2, 1988, a US District Court judge imposed a fine on the city of Yonkers, New York, for defying a federal court order involving housing desegregation.[2] The fine started at \$100 for the first day and was to double daily until the city chose to obey the court order.

(a) What was the daily percent growth rate of the fine?
(b) Find a formula for the fine as a function of t, the number of days since August 2, 1988.
(c) If Yonkers waited 30 days before obeying the court order, what would the fine have been?

[2] *The Boston Globe,* August 27, 1988.

Solution

(a) Since the fine increased each day by a factor of 2, the fine grew exponentially with growth factor $b = 2$. To find the percent growth rate, we set $b = 1 + r = 2$, from which we find $r = 1$, or 100%. Thus the daily percent growth rate is 100%. This makes sense because when a quantity increases by 100%, it doubles in size.

(b) If t is the number of days since August 2, the formula for the fine, P in dollars, is

$$P = 100 \cdot 2^t.$$

(c) After 30 days, the fine is $P = 100 \cdot 2^{30} \approx 1.074 \cdot 10^{11}$ dollars, or $107,374,182,400.

Exercises and Problems for Section 4.1

Skill Refresher

In Exercises S1–S2, express the percentages in decimal form.

S1. 6%

S2. 0.6%

In Exercises S3–S4, express the decimals as a percent.

S3. 0.0012

S4. 1.23

Exercises

Are the functions in Exercises 1–9 exponential? If so, write the function in the form $f(t) = ab^t$.

1. $g(w) = 2\left(2^{-w}\right)$

2. $m(t) = (2 \cdot 3^t)^3$

3. $f(x) = \dfrac{3^{2x}}{4}$

4. $G(t) = 3(t)^t$

5. $q(r) = \dfrac{-4}{3^r}$

6. $j(x) = 2^x 3^x$

7. $Q(t) = 8^{t/3}$

8. $K(x) = \dfrac{2^x}{3 \cdot 3^x}$

9. $p(r) = 2^r + 3^r$

What is the growth factor in Exercises 10–13? Assume time is measured in the units given.

10. Water usage is increasing by 3% per year.

11. A city grows by 28% per decade.

12. A diamond mine is depleted by 1% per day.

13. A forest shrinks 80% per century.

In Exercises 14–17, give the starting value a, the growth factor b, and the growth rate r if $Q = ab^t = a(1 + r)^t$.

14. $Q = 1750(1.593)^t$

15. $Q = 34.3(0.788)^t$

16. $Q = 79.2(1.002)^t$

17. $Q = 0.0022(2.31)^{-3t}$

Problems

18. The populations, P, of six towns with time t in years are given by

(i) $P = 1000(1.08)^t$ (ii) $P = 600(1.12)^t$

(iii) $P = 2500(0.9)^t$ (iv) $P = 1200(1.185)^t$

(v) $P = 800(0.78)^t$ (vi) $P = 2000(0.99)^t$

(a) Which towns are growing in size? Which are shrinking?

(b) Which town is growing the fastest? What is the annual percent growth rate for that town?

(c) Which town is shrinking the fastest? What is the annual percent "decay" rate for that town?

(d) Which town has the largest initial population (at $t = 0$)? Which town has the smallest?

19. The value, V, of a $100,000 investment that earns 3% annual interest is given by $V = f(t)$ where t is in years. How much is the investment worth in 3 years?

20. An investment decreases by 5% per year for 4 years. By what total percent does it decrease?

21. Without a calculator, match each of the formulas to one of the graphs in Figure 4.6.

(a) $y = 0.8^t$ **(b)** $y = 5(3)^t$

(c) $y = -6(1.03)^t$ **(d)** $y = 15(3)^{-t}$

(e) $y = -4(0.98)^t$ **(f)** $y = 82(0.8)^{-t}$

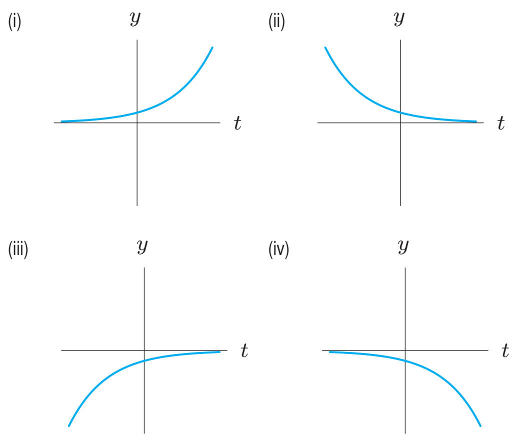


Figure 4.6

In Problems 22–27, the initial value (at year $t = 0$) and the percent change per year of a quantity Q are given.

(a) Write a formula for Q as a function of t.

(b) What is the value of Q when $t = 10$?

22. Initial amount 2000; increasing by 5% per year

23. Initial amount 35; decreasing by 8% per year

24. Initial amount 112.8; decreasing by 23.4% per year

25. Initial amount 5.35; increasing by 0.8% per year

26. Initial amount 5; increasing by 100% per year

27. Initial amount 0.2; decreasing by 0.5% per year

28. Figure 4.7 is the graph of $f(x) = 4 \cdot b^x$. Find the slope of the line segment PQ in terms of b.

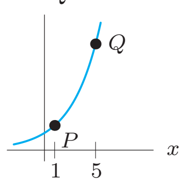


Figure 4.7

29. In 2010, the population of a country was 70 million and growing at a rate of 1.9% per year. Assuming the percentage growth rate remains constant, express the population, P, as a function of t, the number of years after 2010.

30. The mass, Q, of a sample of tritium (a radioactive isotope of hydrogen), decays at a rate of 5.626% per year. Write a function giving the mass of a 726-gram sample after a time, t, in years. Graph this decay function.

31. In 2010 the number of people infected by a virus was P_0. Due to a new vaccine, the number of infected people has decreased by 20% each year since 2010. In other words, only 80% as many people are infected each year as were infected the year before. Find a formula for $P = f(n)$, the number of infected people n years after 2010. Graph $f(n)$. Explain, in terms of the virus, why the graph has the shape it does.

32. Every year, a lake becomes more polluted, and 2% fewer organisms can live in it. If in 2011 there are one million organisms, write an equation relating O, the number of organisms, to time, t, in years since 2011.

33. **(a)** The annual inflation rate is 3.5% per year. If a movie ticket costs \$7.50, find a formula for p, the price of the ticket t years from today, assuming that movie tickets keep up with inflation.

(b) According to your formula, how much will movie tickets cost in 20 years?

34. The value \$$V$ of an investment in year t is given by $V = 2500(1.0325)^t$. Describe the investment in words.

35. A typical cup of coffee contains about 100 mg of caffeine and every hour approximately 16% of the amount of caffeine in the body is metabolized and eliminated.

(a) Write C, the amount of caffeine in the body in mg, as a function of t, the number of hours since the coffee was consumed.

(b) How much caffeine is in the body after 5 hours?

36. Grinnell Glacier in Glacier National Park in the US covered about 142 acres in 2007 and was shrinking at a rate of about 4.4% per year.[3]

(a) Write a formula for the size, S, of the Grinnell Glacier, in acres, as a function of years t since 2007.

(b) Use the model to predict the size of the glacier in the year 2015.

(c) According to the model, how many acres of ice were lost from the glacier between 2007 and 2010?

37. In January 2005, the population of California was 36.8 million and growing at an annual rate of 1.3%. Assume that growth continues at the same rate.

(a) By how much will the population increase between 2005 and 2030? Between 2030 and 2055?

(b) Explain how you can tell before doing the calculations which of the two answers in part (a) is larger.

38. A cold yam is placed in a hot oven. Newton's Law of Heating tells us that the difference between the oven's temperature and the yam's temperature decays exponentially with time. The yam's temperature is initially 0°F,

[3]"Warming climate shrinking Glacier Park's glaciers", www.usatoday.com, October 15, 2007.

the oven's temperature is 300°F, and the temperature difference decreases by 3% per minute. Find a formula for $Y(t)$, the yam's temperature at time t.

39. The population of India was about 1.15 billion people in 2010 and was growing at a rate of about 1.35% per year.

(a) Write a formula for the population, P, of India, in billions, as a function of years t since 2010.
(b) If the growth rate stays constant, predict the population of India in the year 2015 and the year 2020.
(c) Find the rate of change of India's population, in million people per year, during the year 2010.
(d) Find the rate of change of India's population, in people per minute, during the year 2010.

40. Polluted water is passed through a series of filters. Each filter removes 85% of the remaining impurities. Initially, the untreated water contains impurities at a level of 420 parts per million (ppm). Find a formula for L, the remaining level of impurities, after the water has been passed through a series of n filters.

41. In 2007, world solar photovoltaic (PV) market installations totaled 2826 megawatts and were growing at a rate of 62% per year. In the same year, Japan's PV market installations totaled 230 megawatts and were declining at a rate of 23% per year.[4]

(a) Use the information given to predict PV market installations in the world and in Japan in the year 2012.
(b) What percent of world PV market installations were in Japan in 2007? In 2012?

Write the exponential function $y = 5(0.5)^{t/3}$ in the forms given in Problems 42–43. Give the values of all constants.

42. $y = ab^t$

43. $y = a \cdot 4^{kt}$

44. Write the following exponential function in standard form, $f(t) = ab^t$, giving the values of a and b:

$$f(t) = \frac{60}{5 \cdot 2^{t/11.2}}.$$

45. In the year 2009, a total of 13.4 million passengers took a cruise vacation.[5] The global cruise industry has been growing at a rate of approximately 5% per year for the last five years.

(a) Write a formula to approximate the number, N, of cruise passengers (in millions) t years after 2009.
(b) How many cruise passengers are predicted in the year 2015? Approximately how many passengers went on a cruise in the year 2005?

46. The amount (in milligrams) of a drug in the body t hours after taking a pill is given by $A(t) = 25(0.85)^t$.

(a) What is the initial dose given?
(b) What percent of the drug leaves the body each hour?
(c) What is the amount of drug left after 10 hours?
(d) After how many hours is there less than 1 milligram left in the body?

47. Every year, teams from 64 colleges qualify to compete in the NCAA basketball playoffs. For each round, every team is paired with an opponent. A team is eliminated from the tournament once it loses a round. So, at the end of a round, only one half the number of teams move on to the next round. Let $N(r)$ be the number of teams remaining in competition after r rounds of the tournament have been played.

(a) Find a formula for $N(r)$ and graph $y = N(r)$.
(b) In 2009 the North Carolina Tar Heels defeated the Michigan State Spartans 89-72. How many rounds did #1 seed North Carolina have to go through to win the championship game?

48. The UN Food and Agriculture Organization estimates that 2.17% of the world's natural forests existing in 1990 were gone by the end of the decade. In 1990, the world's forest cover stood at 4077 million hectares.[6]

(a) How many million hectares of natural forests were lost during the 1990s?
(b) How many million hectares of natural forests existed in the year 2000?
(c) Write an exponential formula approximating the number of million hectares of natural forest in the world t years after 1990.
(d) What was the annual percent decay rate during the 1990s?
(e) During the years 2000 through 2005, the world's natural forests decreased by approximately 0.18% per year. Approximately how many million hectares of natural forests existed in the year 2005?

49. Forty percent of a radioactive substance decays in five years. By what percent does the substance decay each year?

50. The population of a small town increases by a growth factor of 1.134 over a two-year period.

(a) By what percent does the town increase in size during the two-year period?
(b) If the town grows by the same percent each year, what is its annual percent growth rate?

[4] www.solarbuzz.com/marketbuzz2008. Accessed February, 2010.
[5] http://www.f-cca.com, accessed May 25, 2010.
[6] www.fao.org/forestry/fra/fra/2005/en/, accessed January 2, 2010.

51. The *Home* section of many Sunday newspapers includes a mortgage table similar to Table 4.5.[7] The table gives the monthly payment per \$1000 borrowed for loans at various interest rates and time periods. Determine the monthly payment on a

(a) \$60,000 mortgage at 4% for fifteen years.
(b) \$60,000 mortgage at 4% for thirty years.
(c) \$60,000 mortgage at 6% for fifteen years.
(d) Over the life of the loan, how much money would be saved on a 15-year mortgage of \$60,000 if the rate were 4% instead of 6%?
(e) Over the life of the loan, how much money would be saved on an 4% mortgage of \$60,000 if the term of the loan was fifteen years rather than thirty years?

Table 4.5

Interest rate (%)	15-year loan	20-year loan	25-year loan	30-year loan
4.00	7.40	6.06	5.28	4.77
4.50	7.65	6.33	5.56	5.07
5.00	7.91	6.60	5.85	5.37
5.50	8.17	6.88	6.14	5.68
6.00	8.44	7.16	6.44	6.00
6.50	8.71	7.46	6.75	6.32
7.00	8.99	7.75	7.07	6.65
7.50	9.27	8.06	7.39	6.99
8.00	9.56	8.36	7.72	7.34

52. You owe \$2000 on a credit card. The card charges 1.5% monthly interest on your balance, and requires a minimum monthly payment of 2.5% of your balance. All transactions (payments and interest charges) are recorded at the end of the month. You make only the minimum required payment every month and incur no additional debt.

(a) Complete Table 4.6 for a twelve-month period.
(b) What is your unpaid balance after one year has passed? At that time, how much of your debt have you paid off? How much money in interest charges have you paid your creditors?

Table 4.6

Month	Balance	Interest	Minimum payment
0	\$2000.00	\$30.00	\$50.00
1	\$1980.00	\$29.70	\$49.50
2	\$1960.20		
⋮			

53. A one-page letter is folded into thirds to go into an envelope. If it were possible to repeat this kind of tri-fold 20 times, how many miles thick would the letter be? (A stack of 150 pieces of stationery is one inch thick; 1 mile = 5280 feet.)

Problems 54–55 concern ISO A-series paper, commonly used in many countries. $A0$ is the largest sheet, $A1$ next largest, and so on, with $A4$ being somewhat similar to the 8.5 by 11 inch paper standard in the US. The width in millimeters (mm) of a sheet of A_n paper in this series is given by the formula[8]

$$f(n) = 1000 \cdot 2^{-\frac{1}{4}-\frac{n}{2}}.$$

54. Show that this is an exponential function by writing it in standard form, $f(n) = ab^n$, . What do the values of a and b tell you about A-series paper?

55. Evaluate and simplify the ratio $f(n+2)/f(n)$. What does your answer tell you about A-series paper?

Use Figure 4.8 in Problems 56–59.

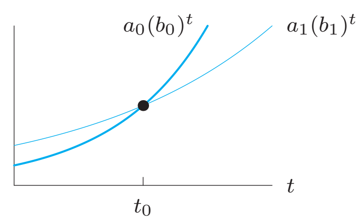


Figure 4.8

56. Which is greater, a_0 or a_1?

57. Which is greater, b_0 or b_1?

58. What happens to t_0 if a_0 is increased while the other quantities remain fixed?

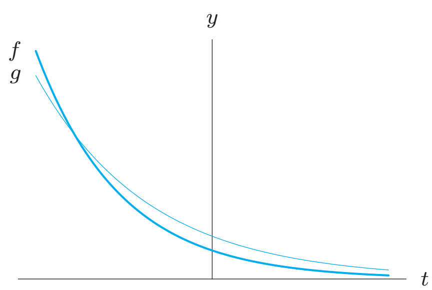
59. What happens to t_0 if b_1 is decreased while the other quantities remain fixed?

[7] http://www.drcalculator.com, accessed January 1, 2010.
[8] Actual paper sizes are rounded to the nearest mm, so this formula is only approximate. See /www.cl.cam.ac.uk/~mgk25/iso-paper.html, accessed February 24, 2008.

60. The figure shows graphs of two functions:

$$f(t) = a_0 \cdot 2^{-t/\tau_0}$$
$$g(t) = a_1 \cdot 2^{-t/\tau_1}.$$

Which is larger, a_0 or a_1? τ_0 or τ_1?



61. Let P be the number of students in a school district, N be the size of the tax base (in households), and r be the average annual tax rate (in \$/household).

(a) Find a formula for R, the total tax revenue, in terms of N and r.
(b) Find a formula for A, the average revenue per student.
(c) Suppose the tax base goes up by 2% and the tax rate is raised by 3%. Find formulas for the new tax base and tax rate in terms of N and r.
(d) Using your answer to part (c), find a formula for the new total tax revenue in terms of R. By what percent did R increase?
(e) Over the time period in part (c), the student population rises by 8%. Find a formula for the new average revenue in terms of A. Did the average revenue rise or fall? By how much?

4.2 COMPARING EXPONENTIAL AND LINEAR FUNCTIONS

The exponential function $Q = ab^t$ represents a quantity changing at a constant percent rate. In this section we compare exponential and linear models and we fit exponential models to data from tables and graphs.

Identifying Linear and Exponential Functions From a Table

Table 4.7 gives values of a linear and an exponential function. Notice that the value of x changes by equal steps of $\Delta x = 5$. The function f could be linear because the difference between consecutive values of $f(x)$ is constant: $f(x)$ increases by 15 each time x increases by 5.

Table 4.7 *Two functions, one linear and one exponential*

x	20	25	30	35	40	45
$f(x)$	30	45	60	75	90	105
$g(x)$	1000	1200	1440	1728	2073.6	2488.32

On the other hand, the difference between consecutive values of $g(x)$ is *not* constant:

$$1200 - 1000 = 200$$
$$1440 - 1200 = 240$$
$$1728 - 1440 = 288.$$

Thus, g is not linear. However, the *ratio* of consecutive values of $g(x)$ is constant:

$$\frac{1200}{1000} = 1.2, \quad \frac{1440}{1200} = 1.2, \quad \frac{1728}{1440} = 1.2,$$

and so on. Note that $1200 = 1.2(1000)$, $1440 = 1.2(1200)$, $1728 = 1.2(1440)$. Thus, each time x increases by 5, the value of $g(x)$ increases by a factor of 1.2. This pattern of constant ratios is indicative of exponential functions. In general:

For a table of data that gives y as a function of x and in which Δx is constant:

- If the *difference* of consecutive y-values is constant, the table could represent a linear function.
- If the *ratio* of consecutive y-values is constant, the table could represent an exponential function.

Finding a Formula for an Exponential Function

To find a formula for the exponential function in Table 4.7, we must determine the values of a and b in the formula $g(x) = ab^x$. The table tells us that $ab^{20} = 1000$ and that $ab^{25} = 1200$. Taking the ratio gives

$$\frac{ab^{25}}{ab^{20}} = \frac{1200}{1000} = 1.2.$$

Notice that the value of a cancels in this ratio, so

$$\frac{ab^{25}}{ab^{20}} = b^5 = 1.2.$$

We solve for b by raising each side to the $(1/5)^{\text{th}}$ power:

$$(b^5)^{1/5} = b = 1.2^{1/5} \approx 1.03714.$$

Now that we have the value of b, we can solve for a. Since $g(20) = ab^{20} = 1000$, we have

$$\begin{aligned} a(1.03714)^{20} &= 1000 \\ a &= \frac{1000}{1.03714^{20}} \approx 482.253. \end{aligned}$$

Thus, a formula for g is $g(x) = 482.253(1.037)^x$. (Note: We could have used $g(25)$ or any other value from the table to find a.)

Modeling Linear and Exponential Growth Using Two Data Points

If we are given two data points, we can fit either a line or an exponential function to the points. The following example compares the predictions made by a linear model and an exponential model fitted to the same data.

Example 1 At time $t = 0$ years, a species of turtle is released into a wetland. When $t = 4$ years, a biologist estimates there are 300 turtles in the wetland. Three years later, the biologist estimates there are 450 turtles. Let P represent the size of the turtle population in year t.

(a) Find a formula for $P = f(t)$ assuming linear growth. Interpret the slope and P-intercept of your formula in terms of the turtle population.
(b) Now find a formula for $P = g(t)$ assuming exponential growth. Interpret the parameters of your formula in terms of the turtle population.
(c) In year $t = 12$, the biologist estimates that there are 900 turtles in the wetland. What does this indicate about the two population models?

Solution

(a) Assuming linear growth, we have $P = f(t) = b + mt$, and

$$m = \frac{\Delta P}{\Delta t} = \frac{450 - 300}{7 - 4} = \frac{150}{3} = 50.$$

Calculating b gives

$$\begin{aligned} 300 &= b + 50 \cdot 4 \\ b &= 100, \end{aligned}$$

so $P = f(t) = 100 + 50t$. This formula tells us that 100 turtles were originally released into the wetland and that the number of turtles increases at the constant rate of 50 turtles per year.

(b) Assuming exponential growth, we have $P = g(t) = ab^t$. The values of a and b are calculated from the ratio

$$\frac{ab^7}{ab^4} = \frac{450}{300},$$

so

$$b^3 = 1.5.$$

Thus,

$$b = (1.5)^{1/3} \approx 1.145.$$

Using the fact that $g(4) = ab^4 = 300$ to find a gives

$$\begin{aligned} a(1.145)^4 &= 300 \\ a &= \frac{300}{1.145^4} \approx 175, \quad \text{Rounding to the nearest whole turtle} \end{aligned}$$

so $P = g(t) = 175(1.145)^t$. This formula tells us that 175 turtles were originally released into the wetland and the number increases at about 14.5% per year.

(c) In year $t = 12$, there are approximately 900 turtles. The linear function from part (a) predicts

$$P = 100 + 50 \cdot 12 = 700 \text{ turtles}.$$

The exponential formula from part (b), however, predicts

$$P = 175(1.145)^{12} \approx 889 \text{ turtles}.$$

The fact that 889 is closer to the observed value of 900 turtles suggests that, during the first 12 years, exponential growth is a better model of the turtle population than linear growth. The two models are graphed in Figure 4.9.

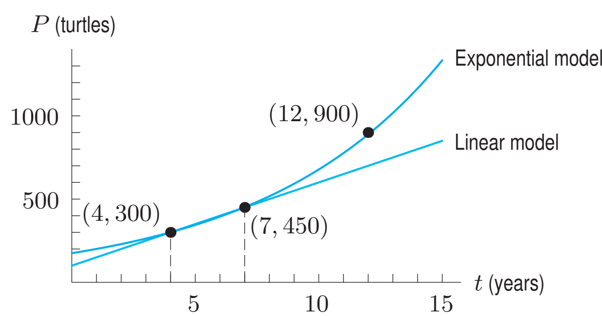


Figure 4.9: Comparison of the linear and exponential models of the turtle population

Similarities and Differences Between Linear and Exponential Functions

In some ways the general formulas for linear and exponential functions are similar. If y is a linear function of x and x is a positive integer, we can write $y = b + mx$ as

$$y = b + \underbrace{m + m + m + \ldots + m}_{x \text{ times}}.$$

Similarly, if y is an exponential function of x, so that $y = a \cdot b^x$ and x is a positive integer, we can write

$$y = a \cdot \underbrace{b \cdot b \cdot b \cdot \ \ldots \ \cdot b}_{x \text{ times}}.$$

So linear functions involve repeated sums whereas exponential functions involve repeated products. In both cases, x determines the number of repetitions.

There are other similarities between the formulas for linear and exponential functions. In the formula $y = b + mx$, b tells us the y-intercept and m tells us how the function is changing. In the formula $y = a \cdot b^x$, a tells us the y-intercept and b tells us how the function is changing.

Example 2 The following tables contain values from an exponential or linear function. For each table, decide if the function is linear or exponential, and find a possible formula for the function.

(a)

x	$f(x)$
0	65
1	75
2	85
3	95
4	105

(b)

x	$g(x)$
0	400
1	600
2	900
3	1350
4	2025

Solution

(a) The function values increase by 10 as x increases by 1, so this is a linear function with slope $m = 10$. Since $f(0) = 65$, the vertical intercept is 65. A possible formula is

$$f(x) = 65 + 10x.$$

(b) The function is not linear, since $g(x)$ increases by different amounts as x increases by 1. To determine whether g might be exponential, we look at ratios of consecutive values:

$$\frac{600}{400} = 1.5, \quad \frac{900}{600} = 1.5, \quad \frac{1350}{900} = 1.5, \quad \frac{2025}{1350} = 1.5.$$

Each time x increases by 1, the value of $g(x)$ increases by a factor of 1.5. This is an exponential function with growth factor 1.5. Since $g(0) = 400$, the vertical intercept is 400. A possible formula is

$$g(x) = 400(1.5)^x.$$

Exponential Growth Will Always Outpace Linear Growth in the Long Run

Figure 4.9 shows the graphs of the linear and exponential models for the turtle population from Example 1. The graphs highlight the fact that, although these two graphs remain fairly close for the first ten or so years, the exponential model predicts explosive growth later on.

It can be shown that an exponentially increasing quantity will, in the long run, always outpace a linearly increasing quantity. This fact led the 19^{th}-century clergyman and economist Thomas Malthus to make some rather gloomy predictions, which are illustrated in the next example.

Example 3 The population of a country is initially 2 million people and is increasing at 4% per year. The country's annual food supply is initially adequate for 4 million people and is increasing at a constant rate adequate for an additional 0.5 million people per year.

(a) Based on these assumptions, in approximately what year will this country first experience shortages of food?
(b) If the country doubled its initial food supply, would shortages still occur? If so, when? (Assume the other conditions do not change).
(c) If the country doubled the rate at which its food supply increases, in addition to doubling its initial food supply, would shortages still occur? If so, when? (Again, assume the other conditions do not change.)

Solution Let P represent the country's population (in millions) and N the number of people the country can feed (in millions). The population increases at a constant percent rate, so it can be modeled by an exponential function. The initial population is $a = 2$ million people and the annual growth factor is $b = 1 + 0.04 = 1.04$, so a formula for the population is

$$P = 2(1.04)^t.$$

In contrast, the food supply increases by a constant amount each year and is therefore modeled by a linear function. The initial food supply is adequate for $b = 4$ million people and the growth rate is $m = 0.5$ million per year, so the number of people that can be fed is

$$N = 4 + 0.5t.$$

(a) Figure 4.10(a) gives the graphs of P and N over a 105-year span. For many years, the food supply is far in excess of the country's needs. However, after about 78 years the population has begun to grow so rapidly that it catches up to the food supply and then outstrips it. After that time, the country will suffer from shortages.
(b) If the country can initially feed eight million people rather than four, the formula for N is

$$N = 8 + 0.5t.$$

However, as we see from Figure 4.10(b), this measure only buys the country three or four extra years with an adequate food supply. After 81 years, the population is growing so rapidly that the head start given to the food supply makes little difference.
(c) If the country doubles the rate at which its food supply increases, from 0.5 million per year to 1.0 million per year, the formula for N is

$$N = 8 + 1.0t.$$

Unfortunately the country still runs out of food eventually. Judging from Figure 4.10(c), this happens in about 102 years.

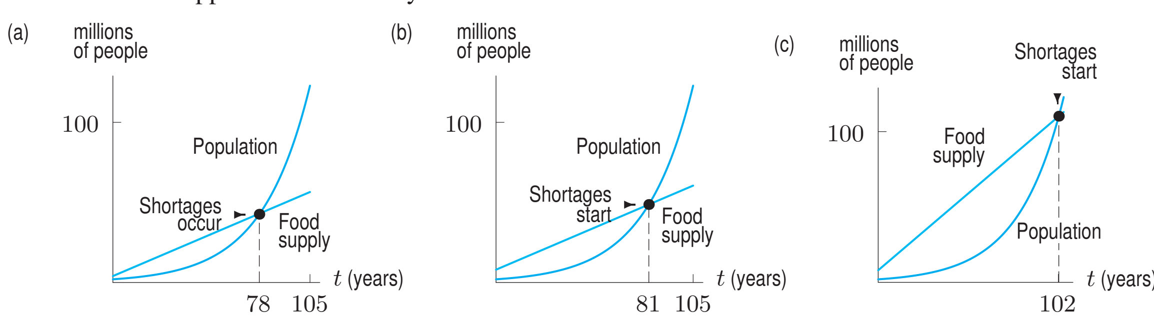


Figure 4.10: These graphs illustrate the fact that an exponentially growing population eventually outstrips a linearly growing food supply

Malthus believed that populations increase exponentially while food production increases linearly. The last example explains his gloomy predictions: Malthus believed that any population eventually outstrips its food supply, leading to famine and war.

Exercises and Problems for Section 4.2

Skill Refresher

In Exercises S1–S4, write each of the following with single positive exponents.

S1. $b^4 \cdot b^6$

S2. $8g^3 \cdot (-4g)^2$

S3. $\dfrac{18a^{10}b^6}{6a^3b^{-4}}$

S4. $\dfrac{(2a^3b^2)^3}{(4ab^{-4})^2}$

In Exercises S5–S6, evaluate the functions for $t = 0$ and $t = 3$.

S5. $f(t) = 5.6(1.043)^t$

S6. $g(t) = 12{,}837(0.84)^t$

In Exercises S7–S10, solve for x.

S7. $4x^3 = 20$

S8. $\dfrac{5x^3}{x^5} = 125$

S9. $\dfrac{4x^8}{3x^3} = 7$

S10. $\sqrt{4x^3} = 5$

Exercises

1. Write a formula for the price p of a gallon of gas in t days if the price is \$2.50 on day $t = 0$ and the price is:

(a) Increasing by \$0.03 per day.
(b) Decreasing by \$0.07 per day.
(c) Increasing by 2% per day.
(d) Decreasing by 4% per day.

2. A population has size 5000 at time $t = 0$, with t in years.

(a) If the population decreases by 100 people per year, find a formula for the population, P, at time t.
(b) If the population decreases by 8% per year, find a formula for the population, P, at time t.

3. The following formulas give the populations (in 1000s) of four different cities, A, B, C, and D, where t is in years. Which are changing exponentially? Describe in words how each of these populations is changing over time. Graph those that are exponential.

$$P_A = 200 + 1.3t, \quad P_B = 270(1.021)^t,$$
$$P_C = 150(1.045)^t, \quad P_D = 600(0.978)^t.$$

4. In an environment with unlimited resources and no predators, a population tends to grow by the same percentage each year. Should a linear or exponential function be used to model such a population? Why?

5. Find $g(t) = ab^t$ if $g(10) = 50$ and $g(30) = 25$.

6. Find a formula for $f(x)$, an exponential function such that $f(-8) = 200$ and $f(30) = 580$.

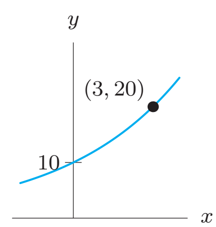
7. Suppose that $f(x)$ is exponential and that $f(-3) = 54$ and $f(2) = \frac{2}{9}$. Find a formula for $f(x)$.

8. Find a formula for $f(x)$, an exponential function such that $f(2) = 1/27$ and $f(-1) = 27$.

9. Find the equation of an exponential curve through the points $(-1, 2)$, $(1, 0.3)$.

For Exercises 10–15, find a formula for the exponential function.

10.



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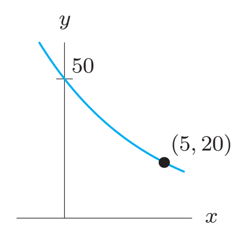

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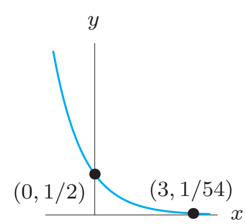

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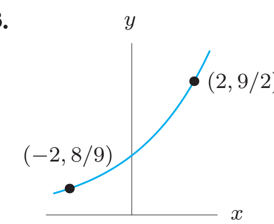

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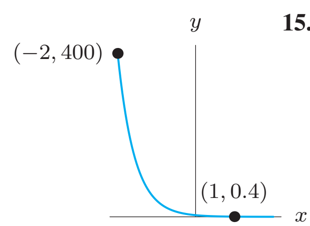

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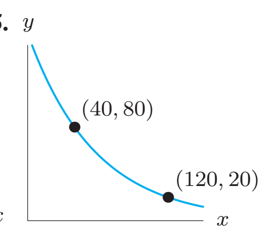


16. **(a)** Make a table of values for $P = f(t) = 1000(1.2)^t$, for $t = 0, 1, 2, 3, 4, 5$.

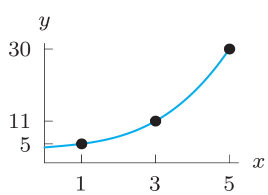
(b) Calculate each of the ratios of successive terms:

$$\frac{f(1)}{f(0)}, \quad \frac{f(2)}{f(1)}, \quad \frac{f(3)}{f(2)}, \quad \frac{f(4)}{f(3)}, \quad \frac{f(5)}{f(4)}.$$

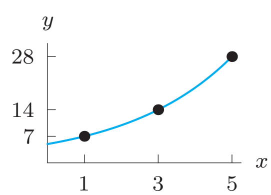
(c) Discuss the results. Why is this outcome to be expected?

In Exercises 17–18, find exponential functions for the graphs shown in the figures, or explain why this is not possible.

17.



18.



19. Find a possible formula for the exponential function g given that the points $(2.3, 0.4)$ and $(3.5, 0.1)$ are on its graph.

20. Find a possible formula for the exponential function f given $f(-5) = 22$ and $f(17) = 46$.

21. Which (if any) of the functions in the table could be linear? Which could be exponential? (Note that table values may reflect rounding.)

x	0	5	10	15	20
$f(x)$	95.4	85.9	77.3	69.6	62.6
$g(x)$	44.8	40.9	36.8	32.5	28.0
$h(x)$	37.3	36.6	35.9	35.2	34.5

The tables in Exercises 22–25 contain values from an exponential or a linear function. In each problem:

(a) Decide if the function is linear or exponential.

(b) Find a possible formula for each function and graph it.

22.

x	$f(x)$
0	12.5
1	13.75
2	15.125
3	16.638
4	18.301

23.

x	$g(x)$
0	0
1	2
2	4
3	6
4	8

24.

x	$h(x)$
0	14
1	12.6
2	11.34
3	10.206
4	9.185

25.

x	$i(x)$
0	18
1	14
2	10
3	6
4	2

Problems

26. Graphs of a linear and an exponential function are shown in Figure 4.11. Find formulas for each of the functions.

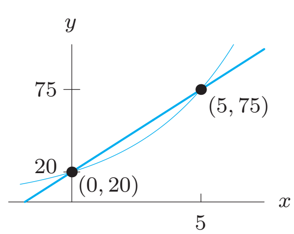


Figure 4.11

27. Let $p(x) = 2 + x$ and $q(x) = 2^x$. Estimate the values of x such that $p(x) < q(x)$.

28. If $f(x) = 12 + 20x$ and $g(x) = \frac{1}{2} \cdot 3^x$, for what values of x is $g(x) < f(x)$?

29. Find formulas for the exponential functions in Figure 4.12.

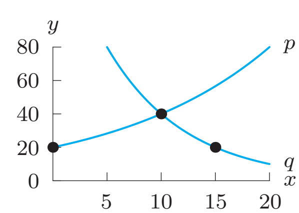


Figure 4.12

Decide whether the functions in Problems 30–32 could be approximately linear, approximately exponential, or are neither. For those that could be nearly linear or nearly exponential, find a formula.

30.

t	3	10	14
$Q(t)$	7.51	8.7	9.39

31.

t	5	9	15
$R(t)$	2.32	2.61	3.12

32.

t	5	12	16
$S(t)$	4.35	6.72	10.02

33. Figure 4.13 shows the balance, P, in a bank account.

(a) Find a possible formula for $P = f(t)$ assuming the balance grows exponentially.
(b) What was the initial balance?
(c) What annual interest rate does the account pay?

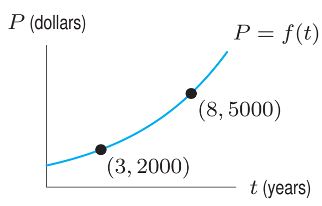


Figure 4.13

34. Suppose the city of Yonkers is offered two alternative fines by the judge. (See Example 8 on page 135.)

Penalty A: \$1 million on August 2 and the fine increases by \$10 million each day thereafter.

Penalty B: 1¢ on August 2 and the fine doubles each day thereafter.

(a) If the city of Yonkers plans to defy the court order until the end of the month (August 31), compare the fines incurred under Penalty A and Penalty B.
(b) If t represents the number of days after August 2, express the fine incurred as a function of t under

(i) Penalty A (ii) Penalty B

(c) Assume your formulas in part (b) holds for $t \geq 0$, is there a time such that the fines incurred under both penalties are equal? If so, estimate that time.

35. A 1987 treaty to protect the ozone layer produced dramatic declines in global production, P, of chlorofluorocarbons (CFCs). See Figure 4.14.[9] Find a formula for P as an exponential function of the number of years, t, since 1989. What was the annual percent decay rate?

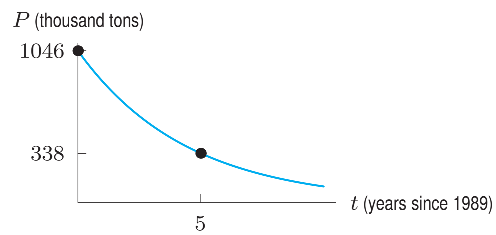


Figure 4.14

36. Figure 4.15 gives the voltage, $V(t)$, across a circuit element at time t seconds. For $t < 0$, the voltage is a constant 80 volts; for $t \geq 0$, the voltage decays exponentially.

(a) Find a piecewise formula for $V(t)$.
(b) At what value of t will the voltage reach 0.1?

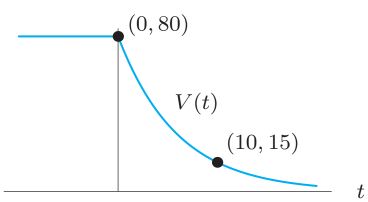


Figure 4.15

37. Short track 500m speed skating became a Winter Olympic event in 1994, and Chae Ji-Hoon of Korea won the event that year with a time of 43.45 seconds. In 2006, Apolo Ohno of the US won the event with a time of 41.94 seconds. Find a formula for the predicted winning time in the 500m speed skating event as a function of the number of years since 1994, and predict the winning time in 2018, if we assume the decrease in time is

(a) Linear **(b)** Exponential

[9]These numbers reflect the volume of the major CFCs multiplied by their respective ozone-depleting potentials (ODPs), as reported by the U. N. Environmental Programme Ozone Secretariat. See hq.uncp.org/ozone, accessed November, 2001.

38. There were 178.8 million licensed drivers in the US in 1989 and 187.2 million in 1999.[10] Find a formula for the number, N, of licensed drivers in the US as a function of t, the number of years since 1989, assuming growth is

(a) Linear **(b)** Exponential

39. In year $t = 0$ a lake is estimated to have about 3500 trout in it. Ten years later, at $t = 10$, the population of trout is believed to be about 1700.

(a) Write a formula for the size of the population P as a function of year t if we assume the decrease is linear. What is the rate of change, in fish per year, of the function over the ten-year period?
(b) Write a formula for the size of the population P as a function of year t if we assume the decrease is exponential. What is the percent rate of change, in percent per year, of the function over the ten-year period?
(c) Graph the two functions on the same coordinate system. Indicate the points at $t = 0$ and $t = 10$.

40. Cocoa production[11] is shown in Table 4.8 for the world and the Ivory Coast, in millions of tons, as a function of the number of years since 2000. In each case, determine if production is better modeled with a linear or an exponential function and find a formula for the function.

Table 4.8

Years since 2000	0	1	2	3	4
World cocoa production	3.1	3.875	4.844	6.055	7.568
Ivory Coast cocoa production	1.3	1.34	1.38	1.42	1.46

41. The average gain in life expectancy at birth in the US has remained almost constant for 150 years, at an increase of 3 months in life expectancy per year.[12] Assume that this rate of increase continues. Life expectancy in the US in 2009 was 78.1 years.

(a) Is life expectancy increasing linearly or exponentially?
(b) Find a formula for life expectancy, L, in years, in the US at birth as a function of the number of years t since 2000.
(c) What is the predicted life expectancy for babies born in 2050?

42. In terms of the initial population P_0, what is the value of the population at the end of 10 years, given each of the following assumptions? Graph each population against time.

(a) A population decreases linearly and the decrease is 10% in the first year.
(b) A population decreases exponentially at the rate of 10% a year.

43. The number of asthma sufferers in the world was about 84 million in 1990 and 300 million in 2009.[13] Let N represent the number of asthma sufferers (in millions) worldwide t years after 1990.

(a) Write N as a linear function of t. What is the slope? What does it tell you about asthma sufferers?
(b) Write N as an exponential function of t. What is the growth factor? What does it tell you about asthma sufferers?
(c) How many asthma sufferers are predicted worldwide in the year 2020 with the linear model? With the exponential model?

44. In 2000, the population of a town was 20,000, and it grew by 4.14% that year. By 2010, the town's population had reached 30,000.

(a) Can this population be best described by a linear or an exponential model, or neither? Explain.
(b) If possible, find a formula for $P(t)$, this population t years after 2000.

45. In 2000, the population of a town was 18,500 and it grew by 250 people by the end of the year. By 2010, its population had reached 22,500.

(a) Can this population be best described by a linear or an exponential model, or neither? Explain.
(b) If possible, find a formula for $P(t)$, the population t years after 2000.

4.3 GRAPHS OF EXPONENTIAL FUNCTIONS

As with linear functions, an understanding of the significance of the parameters a and b in the formula $Q = ab^t$ helps us analyze and compare exponential functions.

[10] *The World Almanac* 2002 (New York: World Almanac Education Group, Inc., 2002), p. 228.
[11] Adapted from www.treecrops.org/crops/cocoaoutlook, accessed November, 2009.
[12] "The Science of Living Longer", Time Magazine, February 22, 2010.
[13] www.healthylifestyle-solutions.blogspot.com, accessed January 2, 2010.

Graphs of the Exponential Family: The Effect of the Parameter a

In the formula $Q = ab^t$, the value of a tells us where the graph crosses the Q-axis, since a is the value of Q when $t = 0$. In Figure 4.16 each graph has the same value of b but different values of a and thus different vertical intercepts.

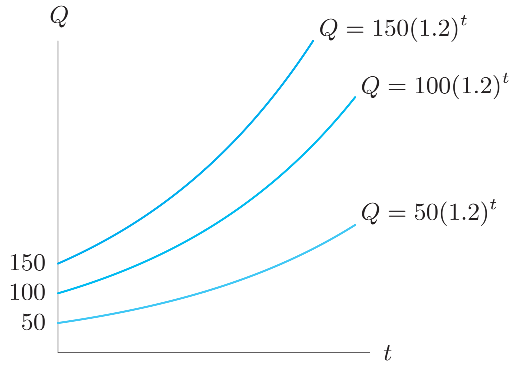


Figure 4.16: Graphs of $Q = a(1.2)^t$ for $a = 50$, 100, and 150

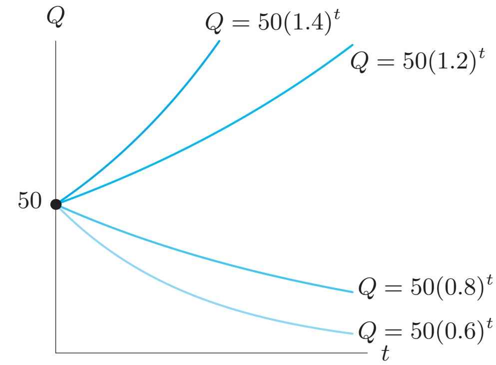


Figure 4.17: Graphs of $Q = 50b^t$ for $b = 0.6, 0.8, 1.2$ and 1.4

Graphs of the Exponential Family: The Effect of the Parameter b

The growth factor, b, is called the *base* of an exponential function. Provided a is positive, if $b > 1$, the graph climbs when read from left to right, and if $0 < b < 1$, the graph falls when read from left to right.

Figure 4.17 shows how the value of b affects the steepness of the graph of $Q = ab^t$. Each graph has a different value of b but the same value of a (and thus the same Q-intercept). For $b > 1$, the greater the value of b, the more rapidly the graph rises. For $0 < b < 1$, the smaller the value of b, the more rapidly the graph falls. In every case, however, the graph is concave up.

Horizontal Asymptotes

The t-axis is a *horizontal asymptote* for the graph of $Q = ab^t$, because Q approaches 0 as t gets large, either positively or negatively. For exponential decay, such as $Q = f(t) = a(0.6)^t$ in Figure 4.17, the value of Q approaches 0 as t gets large and positive. We write

$$Q \to 0 \quad \text{as} \quad t \to \infty.$$

This means that Q is as close to 0 as we like for all sufficiently large values of t. We say that the *limit* of $f(t)$ as t goes to infinity is 0, and we write

$$\lim_{t\to\infty} f(t) = 0.$$

For exponential growth, the value of Q approaches zero as t grows more negative. (See Figure 4.29 on page 159.) In this case, we write

$$Q \to 0 \quad \text{as} \quad t \to -\infty.$$

This means that Q is as close to 0 as we like for all sufficiently large negative values of t. Using limit notation, we write

$$\lim_{t\to-\infty} f(t) = 0.$$

We make the following definition:

> The horizontal line $y = k$ is a **horizontal asymptote** of a function, f, if the function values get arbitrarily close to k as x gets large (either positively or negatively or both). We describe this behavior using the notation
>
> $$f(x) \to k \qquad \text{as} \qquad x \to \infty$$
>
> or
>
> $$f(x) \to k \qquad \text{as} \qquad x \to -\infty.$$
>
> Alternatively, using limit notation, we write
>
> $$\lim_{x\to\infty} f(x) = k \quad \text{or} \quad \lim_{x\to-\infty} f(x) = k.$$

Example 1 A capacitor is the part of an electrical circuit that stores electric charge. The quantity of charge stored decreases exponentially with time. Stereo amplifiers provide a familiar example: When an amplifier is turned off, the display lights fade slowly because it takes time for the capacitors to discharge. (Thus, it can be unsafe to open a stereo or a computer immediately after it is turned off.)

If t is the number of seconds after the circuit is switched off, suppose that the quantity of stored charge (in micro-coulombs) is given by

$$Q = 200(0.9)^t, \quad t \geq 0.$$

(a) Describe in words how the stored charge changes over time.
(b) What quantity of charge remains after 10 seconds? 20 seconds? 30 seconds? 1 minute? 2 minutes? 3 minutes?
(c) Graph the charge over the first minute. What does the horizontal asymptote of the graph tell you about the charge?

Solution (a) The charge is initially 200 micro-coulombs. Since $b = 1 + r = 0.9$, we have $r = -0.10$, which means that the charge level decreases by 10% each second.
(b) Table 4.9 gives the value of Q at $t = 0, 10, 20, 30, 60, 120$, and 180. Notice that as t increases, Q gets closer and closer to, but does not quite reach, zero. The charge stored by the capacitor is getting smaller, but never completely vanishes.
(c) Figure 4.18 shows Q over a 60-second interval. The horizontal asymptote at $Q = 0$ corresponds to the fact that the charge gets very small as t increases. After 60 seconds, for all practical purposes, the charge is zero.

Table 4.9 *Charge (in micro-coulombs) stored by a capacitor over time*

t (seconds)	Q, charge level
0	200
10	69.736
20	24.315
30	8.478
60	0.359
120	0.000646
180	0.00000116

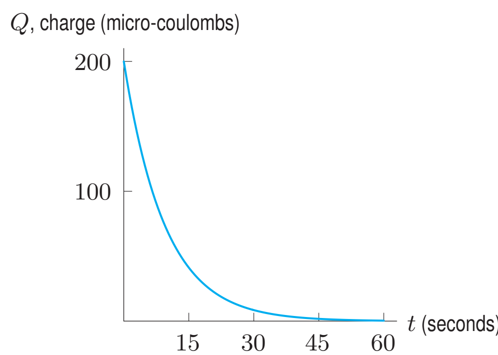


Figure 4.18: The charge stored by a capacitor over one minute

Solving Exponential Equations Graphically

We are often interested in solving equations involving exponential functions. In the following examples, we do this graphically. In Section 5.1, we will see how to solve equations using logarithms.

Example 2 In Example 8 on page 135, the fine, P, imposed on the city of Yonkers is given by $P = 100 \cdot 2^t$ where t is the number of days after August 2. In 1988, the annual budget of the city was \$337 million. If the city chose to disobey the court order, at what point would the fine have wiped out the entire annual budget?

Solution We need to find the day on which the fine reaches \$337 million. That is, we must solve the equation

$$100 \cdot 2^t = 337{,}000{,}000.$$

Using a computer or graphing calculator we can graph $P = 100 \cdot 2^t$ to find the point at which the fine reaches 337 million. From Figure 4.19, we see that this occurs between $t = 21$ and $t = 22$.

At day $t = 21$, August 23, the fine is:

$$P = 100 \cdot 2^{21} = 209{,}715{,}200$$

or just over \$200 million. On day $t = 22$, the fine is

$$P = 100 \cdot 2^{22} = 419{,}430{,}400$$

or almost \$420 million—quite a bit more than the city's entire annual budget!

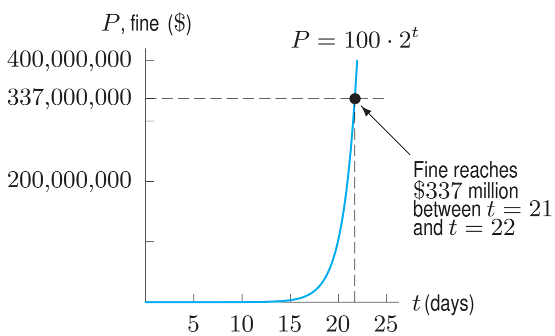


Figure 4.19: The fine imposed on Yonkers exceeds \$337 million after 22 days

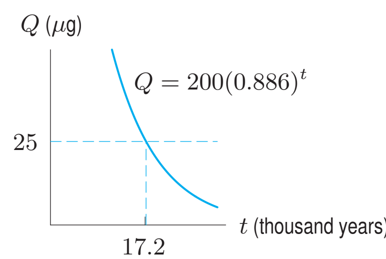


Figure 4.20: Solving the equation $200(0.886)^t = 25$

Example 3 A 200-μg sample of carbon-14 decays according to the formula

$$Q = 200(0.886)^t$$

where t is in thousands of years. Estimate when there is 25 μg of carbon-14 left.

Solution We must solve the equation

$$200(0.886)^t = 25.$$

At the moment, we cannot find a formula for the solution to this equation. However, we can estimate the solution graphically. Figure 4.20 shows a graph of $Q = 200(0.886)^t$ and the line $Q = 25$. The amount of carbon-14 decays to 25 micrograms at $t \approx 17.180$. Since t is measured in thousands of years, this means in about 17,180 years.

Finding an Exponential Function for Data

The data in Table 4.10 gives population data for the Houston Metro Area since 1900. (2010 is estimated.) In Section 1.6, we saw how to fit a linear function to data, but the data points shown in Figure 4.21 suggests that it may make more sense to fit the data using an exponential function.

Table 4.10 *Population (in thousands) of Houston Metro Area, t years after 1900*

t	N	t	N
0	184	60	1583
10	236	70	2183
20	332	80	3122
30	528	90	3733
40	737	100	4672
50	1070	110	5937

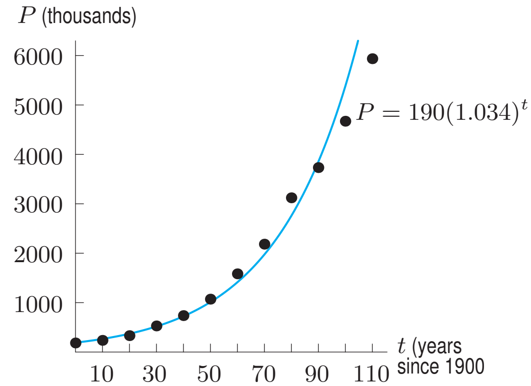


Figure 4.21: Houston population data with an exponential model

Using an exponential regression feature on a calculator or computer, the following exponential function was found:

$$P = 190(1.034)^t.$$

This equation fits the data nicely, with $a = 190$ being close to the initial data value of 184. The graph in Figure 4.21 shows that $b = 1.034$ is a suitable growth factor and the population was increasing at a rate of about 3.4% per year between 1900 and 2010.

Exercises and Problems for Section 4.3

Exercises

1. **(a)** Make a table of values for $f(x) = 2^x$ for $x = -3, -2, -1, 0, 1, 2, 3$.
(b) Graph $f(x)$. Describe the graph in words.

2. **(a)** Make a table of values for $f(x) = \left(\frac{1}{2}\right)^x$ for $x = -3, -2, -1, 0, 1, 2, 3$.
(b) Graph $f(x)$. Describe the graph in words.

3. The graphs of $f(x) = (1.1)^x$, $g(x) = (1.2)^x$, and $h(x) = (1.25)^x$ are in Figure 4.22. Explain how you can match these formulas and graphs without a calculator.

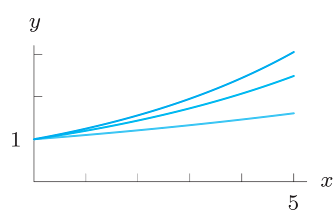


Figure 4.22

4. The graphs of $f(x) = (0.7)^x$, $g(x) = (0.8)^x$, and $h(x) = (0.85)^x$ are in Figure 4.23. Explain how you can match these formulas and graphs without a calculator.

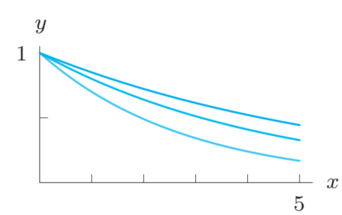


Figure 4.23

In Problems 5–10, will the graphs of the two functions cross in the first quadrant? Explain your reasoning without using a calculator.

5. $f(x) = 10(1.03)^x$; $\quad g(x) = 2(1.05)^x$

6. $f(x) = 250(1.2)^x$; $\quad g(x) = 300(1.3)^x$

7. $f(x) = 10(1.05)^x$; $\quad g(x) = 9(0.95)^x$

8. $f(x) = 5(0.8)^x$; $\quad g(x) = 2(1.05)^x$

9. $f(x) = 500(0.8)^x$; $\quad g(x) = 450(0.7)^x$

10. $f(x) = 1000(0.9)^x$; $\quad g(x) = 875(0.95)^x$

For Exercises 11–14, use Figure 4.24. Assume the equations for A, B, C, and D can all be written in the form $y = ab^t$.

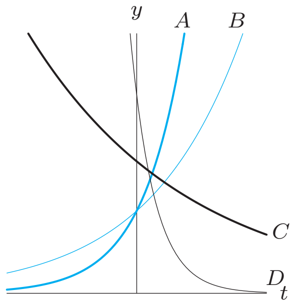


Figure 4.24

11. Which function has the largest value for a?

12. Which two functions have the same value for a?

13. Which function has the smallest value for b?

14. Which function has the largest value for b?

15. The volume of biodegradable material in a compost pile is shown in Figure 4.25, with time t measured in weeks. Use the graph to estimate:

(a) The volume after 5 weeks.

(b) The number of weeks until the volume is 20 ft^3.

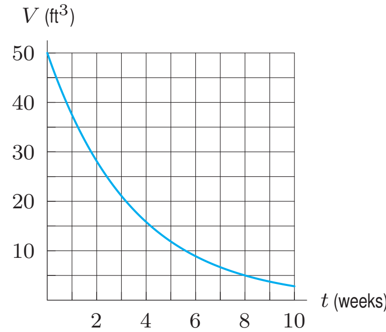


Figure 4.25

16. Solve $y = 46(1.1)^x$ graphically for x if $y = 91$.

17. Solve $p = 22(0.87)^q$ graphically for q if $p = 10$.

18. Solve $4m = 17(2.3)^w$ graphically for w if $m = 12$.

19. Solve $P/7 = (0.6)^t$ graphically for t if $P = 2$.

20. If $b > 1$, what is the horizontal asymptote of $y = ab^t$ as $t \to -\infty$?

21. If $0 < b < 1$, what is the horizontal asymptote of $y = ab^t$ as $t \to \infty$?

22. If the exponential function ab^x has the property that $\lim_{x\to\infty} ab^x = 0$, what can you say about the value of b?

Problems

In Problems 23–27, graph $f(x)$, a function defined for all real numbers and satisfying the condition.

23. $f(x) \to 3$ as $x \to -\infty$

24. $\lim\limits_{x\to\infty} f(x) = 5$

25. $\lim\limits_{x\to-\infty} f(x) = 2$ and $\lim\limits_{x\to\infty} f(x) = -1$

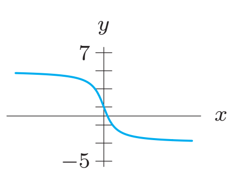
26. $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$

27. $f(x)$ has a horizontal asymptote of $y = 5$.

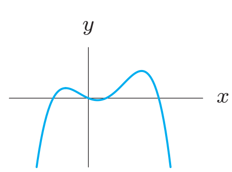
In Problems 28–29, assume that all important features are shown in the graph of $y = f(x)$. Estimate

(a) $\lim\limits_{x\to-\infty} f(x)$ $\qquad$ **(b)** $\lim\limits_{x\to\infty} f(x)$

28.



29.



30. Let $f(x) = 5 + 3(0.9)^x$. As $x \to \infty$, what happens to $f(x)$? Does this function have a horizontal asymptote, and if so, what is it? Justify your answer both analytically and graphically.

31. Consider the exponential functions graphed in Figure 4.26 and the six constants a, b, c, d, p, q.

(a) Which of these constants are definitely positive?

(b) Which of these constants are definitely between 0 and 1?

(c) Which of these constants could be between 0 and 1?

(d) Which two of these constants are definitely equal?

(e) Which one of the following pairs of constants could be equal?

a and p $\qquad$ b and d $\qquad$ b and q $\qquad$ d and q

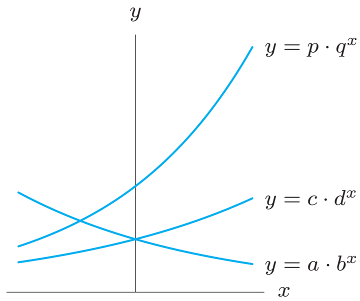


Figure 4.26

32. Set a window of $-4 \leq x \leq 4, -1 \leq y \leq 6$ and graph the following functions using several different values of a for each. Include some values of a with $a < 1$.

(a) $y = a2^x,\ 0 < a < 5.$
(b) $y = 2a^x,\ 0 < a < 5.$

33. For which value(s) of a and b is $y = ab^x$ an increasing function? A decreasing function? Concave up?

34. What are the domain and range of the exponential function $Q = ab^t$ where a and b are both positive constants?

Problems 35–36 use Figure 4.27, where y_0 is the y-coordinate of the point of intersection of the graphs. Describe what happens to y_0 if the following changes are made, assuming the other quantities remain the same.

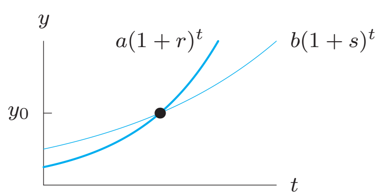


Figure 4.27

35. r is increased

36. a is increased

37. The city of Baltimore has been declining in population for the last fifty years.[14] In the year 2000, the population of Baltimore was 651 thousand and declining at a rate of 0.75% per year. If this trend continues:

(a) Give a formula for the population of Baltimore, P, in thousands, as a function of years, t, since 2000.
(b) What is the predicted population in 2010?
(c) To two decimal places, estimate t when the population is 550 thousand.

38. Let $P = f(t) = 1000(1.04)^t$ be the population of a community in year t.

(a) Evaluate $f(0)$ and $f(10)$. What do these expressions represent in terms of the population?
(b) Using a calculator or a computer, find appropriate viewing windows on which to graph the population for the first 10 years and for the first 50 years. Give the viewing windows you used and sketch the resulting graphs.
(c) If the percentage growth rate remains constant, approximately when will the population reach 2500 people?

39. Suppose you use your calculator to graph $y = 1.04^{5x}$. You correctly enter $y = 1.04\hat{\ }(5x)$ and see the graph in Figure 4.28. A friend graphed the function by entering $y = 1.04\hat{\ }5x$ and said, "The graph is a straight line, so I must have the wrong window." Explain why changing the window will not correct your friend's error.

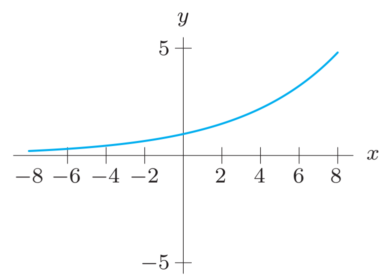


Figure 4.28

40. Suppose y, the number of cases of a disease, is reduced by 10% each year.

(a) If there are initially 10,000 cases, express y as a function of t, the number of years elapsed.
(b) How many cases will there be 5 years from now?
(c) How long does it take to reduce the number of cases to 1000?

41. The earth's atmospheric pressure, P, in terms of height above sea level is often modeled by an exponential decay function. The pressure at sea level is 1013 millibars and that the pressure decreases by 14% for every kilometer above sea level.

(a) What is the atmospheric pressure at 50 km?
(b) Estimate the altitude h at which the pressure equals 900 millibars.

[14] *The World Almanac and Book of Facts 2006* (New York), p. 480.

42. The population of a colony of rabbits grows exponentially. The colony begins with 10 rabbits; five years later there are 340 rabbits.

(a) Give a formula for the population of the colony of rabbits as a function of the time.
(b) Use a graph to estimate how long it takes for the population of the colony to reach 1000 rabbits.

43. Let f be a piecewise-defined function given by

$$f(x)=\begin{cases}2^x, & x<0\\ 0, & x=0\\ 1-\frac{1}{2}x, & x>0.\end{cases}$$

(a) Graph f for $-3 \le x \le 4$.
(b) The domain of $f(x)$ is all real numbers. What is its range?
(c) What are the intercepts of f?
(d) What happens to $f(x)$ as $x \to \infty$ and $x \to -\infty$?
(e) Over what intervals is f increasing? Decreasing?

44. Table 4.11 shows the concentration of theophylline, a common asthma drug, in the blood stream as a function of time after injection of a 300-mg initial dose.[15] It is claimed that this data set is consistent with an exponential decay model $C = ab^t$ where C is the concentration and t is the time.

(a) Estimate the values of a and b, using ratios to estimate b. How good is this model?
(b) Use a calculator or computer to find the exponential regression function for concentration as a function of time. Compare answers from parts (a) and (b).

Table 4.11

Time (hours)	0	1	3	5	7	9
Concentration (mg/l)	12.0	10.0	7.0	5.0	3.5	2.5

45. Table 4.12 shows global wind energy generating capacity, W (in megawatts), as a function of the number of years, t, since 1995.[16]

(a) Plot the data and explain why it is reasonable to approximate these data with an exponential function.
(b) Use a calculator or computer to fit an exponential function to these data.
(c) What annual percent growth rate does the exponential model show?

Table 4.12

t	0	1	2	3	4
W	4780	6070	7640	10,150	13,930
t	5	6	7	8	9
W	18,450	24,930	32,037	39,664	47,760

46. Three scientists, working independently of each other, arrive at the following formulas to model the spread of a species of mussel in a system of fresh water lakes:

$$f_1(x) = 3(1.2)^x,\ \ f_2(x) = 3(1.21)^x,\ \ f_3(x) = 3.01(1.2)^x,$$

where $f_n(x)$, $n = 1, 2, 3$, is the number of individual mussels (in 1000s) predicted by model number n to be living in the lake system after x months have elapsed.

(a) Graph these three functions for $0 \le x \le 60$, $0 \le y \le 40{,}000$.
(b) The graphs of these three models do not seem all that different from each other. But do the three functions make significantly different predictions about the future mussel population? To answer this, graph the difference function, $f_2(x) - f_1(x)$, of the population sizes predicted by models 1 and 2, as well as the difference functions, $f_3(x) - f_1(x)$ and $f_3(x) - f_2(x)$. (Use the same window as in part (a).)
(c) Based on your graphs in part (b), discuss the assertion that all three models are in good agreement as far as long-range predictions of mussel population are concerned. What conclusions can you draw about exponential functions in general?

4.4 APPLICATIONS TO COMPOUND INTEREST

What is the difference between a bank account that pays 12% interest once per year and one that pays 1% interest every month? Imagine we deposit \$1000 into the first account. Then, after 1 year, we have (assuming no other deposits or withdrawals)

$$\$1000(1.12) = \$1120.$$

[15] Based on D. N. Burghes, I. Huntley, and J. McDonald, *Applying Mathematics* (Ellis Horwood, 1982).
[16] The Worldwatch Institute, *Vital Signs* 2005 (New York: W.W. Norton & Company, 2005), p. 35.

But if we deposit $1000 into the second account, then after 1 year, or 12 months, we have

$$\$1000\ \underbrace{(1.01)(1.01)\ldots(1.01)}_{\text{12 months of 1\% monthly interest}} = 1000(1.01)^{12} = \$1126.83.$$

Thus, we earn $6.83 more in the second account than in the first. To see why this happens, notice that the 1% interest we earn in January itself earns interest at a rate of 1% per month. Similarly, the 1% interest we earn in February earns interest, and so does the interest earned in March, April, May, and so on. The extra $6.83 comes from interest earned on interest. This effect is known as *compounding*. We say that the first account earns 12% interest *compounded annually* and the second account earns 12% interest *compounded monthly*.

Nominal Versus Effective Rate

The expression 12% compounded monthly means that interest is added twelve times per year and that $12\%/12 = 1\%$ of the current balance is added each time. We refer to the 12% as the *nominal rate* (nominal means "in name only"). When the interest is compounded more frequently than once a year, the account effectively earns more than the nominal rate. Thus, we distinguish between nominal rate and *effective annual rate*, or *effective rate*. The effective annual rate tells you how much interest the investment actually earns. In the US, the effective annual rate is sometimes called the APY (annual percentage yield).

Example 1 What are the nominal and effective annual rates of an account paying 12% interest, compounded annually? Compounded monthly?

Solution Since an account paying 12% annual interest, compounded annually, grows by exactly 12% in one year, we see that its nominal rate is the same as its effective rate: both are 12%.

The account paying 12% interest, compounded monthly, also has a nominal rate of 12%. On the other hand, since it pays 1% interest every month, after 12 months, its balance increases by a factor of

$$\underbrace{(1.01)(1.01)\ldots(1.01)}_{\text{12 months of 1\% monthly growth}} = 1.01^{12} \approx 1.1268250.$$

Thus, effectively, the account earns 12.683% interest in a year, so its effective rate is 12.683%.

Example 2 What is the effective annual rate of an account that pays interest at the nominal rate of 6% per year, compounded daily? Compounded hourly?

Solution Since there are 365 days in a year, daily compounding pays interest at the rate of

$$\frac{6\%}{365} = 0.0164384\% \text{ per day.}$$

Thus, the daily growth factor is

$$1 + \frac{0.06}{365} = 1.000164384.$$

If at the beginning of the year the account balance is P, after 365 days the balance is

$$P \cdot \underbrace{\left(1 + \frac{0.06}{365}\right)^{365}}_{\substack{\text{365 days of} \\ \text{0.0164384\% daily interest}}} = P \cdot (1.0618313).$$

Thus, this account earns interest at the effective annual rate of 6.18313%.

Notice that daily compounding results in a higher rate than yearly compounding (6.183% versus 6%), because with daily compounding the interest has the opportunity to earn interest.

If interest is compounded hourly, since there are $24 \cdot 365$ hours in a year, the balance at year's end is

$$P \cdot \left(1 + \frac{0.06}{24 \cdot 365}\right)^{24 \cdot 365} = P \cdot (1.0618363).$$

The effective rate is now 6.18363% instead of 6.18313%—that is, just slightly better than the rate of the account that compounds interest daily. The effective rate increases with the frequency of compounding.

In the previous examples, we computed the growth factor for one year. We now compute the growth factor over t years.

Example 3 At the beginning of the year you deposit P dollars in an account paying interest at the nominal rate of 4% per year compounded quarterly. By what factor does P grow in 3 years?

Solution Compounding quarterly pays interest at the rate of

$$\frac{4\%}{4} = 1\% \text{ per quarter.}$$

Thus, the quarterly growth factor is $1 + 0.01 = 1.01$. At the beginning of the year the account balance is P. After one year the balance is

$$P \cdot \underbrace{\left(1 + \frac{0.04}{4}\right)^4}_{\substack{\text{4 quarters(1 year) of} \\ \text{1\% quarterly interest}}} = P \cdot (1.0406).$$

So after three years the balance is

$$P \cdot \left(1 + \frac{0.04}{4}\right)^4 \cdot \left(1 + \frac{0.04}{4}\right)^4 \cdot \left(1 + \frac{0.04}{4}\right)^4 = P \cdot \underbrace{\left(1 + \frac{0.04}{4}\right)^{4 \cdot 3}}_{\substack{\text{3 years of} \\ \text{1\% quarterly interest}}} = P \cdot (1.126825).$$

After three years the initial account balance P grows by a factor of 1.126825.

To summarize:

If interest at an annual rate of r is compounded n times a year, then r/n times the current balance is added n times a year. Therefore, with an initial deposit of $\$P$, the balance t years later is

$$B = P \cdot \left(1 + \frac{r}{n}\right)^{nt}.$$

Note that r is the nominal rate; for example, $r = 0.05$ if the annual rate is 5%.

Exercises and Problems for Section 4.4

Exercises

1. An account pays interest at a nominal rate of 8% per year. Find the effective annual yield if interest is compounded

(a) Monthly **(b)** Weekly
(c) Daily

2. What is the effective annual yield if a deposit of \$1000 grows to \$3500 in 15 years?

3. An investment grows by 5% per year for 20 years. By what percent does it increase over the 20-year period?

4. An investment decreases by 50% over an 8-year period. At what effective annual percent rate does it decrease?

5. Suppose \$1000 is deposited into an account paying interest at a nominal rate of 8% per year. Find the balance three years later if the interest is compounded

(a) Monthly **(b)** Weekly
(c) Daily

6. A bank pays interest at the nominal rate of 1.3% per year. What is the effective annual rate if compounding is:

(a) Annual **(b)** Monthly

In Exercises 7–10, what is the balance after 1 year if an account containing \$500 earns the stated yearly nominal interest, compounded

(a) Annually **(b)** Weekly (52 weeks per year)

(c) Every minute (525,600 per year)

7. 1% **8.** 3% **9.** 5% **10.** 8%

In Exercises 11–14, what are the nominal and effective annual rates for an account paying the stated annual interest, compounded

(a) Annually? **(b)** Quarterly?
(c) Daily?

11. 1% **12.** 100% **13.** 3% **14.** 6%

Problems

15. An investment grows by 3% per year for 10 years. By what percent does it increase over the 10-year period?

16. An investment grows by 30% over a 5-year period. What is its effective annual percent growth rate?

17. An investment decreases by 60% over a 12-year period. At what effective annual percent rate does it decrease?

18. If you need \$25,000 six years from now, what is the minimum amount of money you need to deposit into a bank account that pays 5% annual interest, compounded:

(a) Annually **(b)** Monthly **(c)** Daily

(d) Your answers get smaller as the number of times of compounding increases. Why is this so?

19. A sum of \$850 is invested for 10 years and the interest is compounded quarterly. There is \$1000 in the account at the end of 10 years. What is the nominal annual rate?

20. If the balance, M, at time t in years, of a bank account that compounds its interest payments monthly is given by

$$M = M_0(1.07763)^t,$$

(a) What is the effective annual rate for this account?
(b) What is the nominal annual rate?

21. Suppose \$300 was deposited into one of five bank accounts and t is time in years. For each verbal description (i)–(v), state which formulas (a)–(e) could represent it.

(a) $B = 300(1.2)^t$ **(b)** $B = 300(1.12)^t$
(c) $B = 300(1.06)^{2t}$ **(d)** $B = 300(1.06)^{t/2}$
(e) $B = 300(1.03)^{4t}$

(i) This investment earned 12% annually, compounded annually.
(ii) This investment earned, on average, more than 1% each month.
(iii) This investment earned 12% annually, compounded semi-annually.
(iv) This investment earned, on average, less than 3% each quarter.
(v) This investment earned, on average, more than 6% every 6 months.

22. Without making any calculations, briefly describe in words what the following formulas tell you about the value of the investments they describe. Be specific. Note that units are in dollars and years.

(a) $V = 1500(1.077)^t$
(b) $V = 9500(0.945)^t$
(c) $V = 1000 \cdot 3^{t/5}$
(d) $V = 500\left(1 + \frac{0.04}{12}\right)^{12t}$

4.5 THE NUMBER e

An irrational number, introduced by Euler[17] in 1727, is so important that it is given a special name, e. Its value is approximately $e \approx 2.71828\ldots$. It is often used for the base, b, of the exponential function. Base e is called the *natural base*. This may seem mysterious, as what could possibly be natural about using an irrational base such as e? The answer is that the formulas of calculus are much simpler if e is used as the base for exponentials. Some of the remarkable properties of the number e are introduced in Problems 52–53 on page 167. Since $2 < e < 3$, the graph of $Q = e^t$ lies between the graphs of $Q = 3^t$ and $Q = 2^t$. See Figure 4.29.

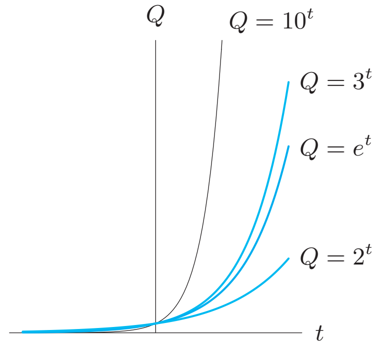


Figure 4.29: Graphs of exponential functions with various bases

Exponential Functions with Base e

Since any positive base b can be written as a power of e:

$$b = e^k,$$

any exponential function $Q = ab^t$ can be rewritten in terms of e:

$$Q = ab^t = a\left(e^k\right)^t = ae^{kt}.$$

If $b > 1$, then k is positive; if $0 < b < 1$, then k is negative. The constant k is called the *continuous growth rate*. In general:

> For the exponential function $Q = ab^t$, the **continuous growth rate**, k, is given by solving $e^k = b$. Then
>
> $$Q = ae^{kt}.$$
>
> If a is positive,
> - If $k > 0$, then Q is increasing.
> - If $k < 0$, then Q is decreasing.

The value of the continuous growth rate, k, may be given as a decimal or a percent. If t is in years, for example, then the units of k are given per year; if t is in minutes, then k is given per minute.

Example 1 Give the continuous growth rate of each of the following functions and graph each function:

$$P = 5e^{0.2t}, \qquad Q = 5e^{0.3t}, \qquad \text{and} \quad R = 5e^{-0.2t}.$$

[17] Leonhard Euler (1707-1783), a Swiss mathematician, introduced e, $f(x)$ notation, π, and i (for $\sqrt{-1}$).

Solution The function $P = 5e^{0.2t}$ has a continuous growth rate of 20%, and $Q = 5e^{0.3t}$ has a continuous 30% growth rate. The function $R = 5e^{-0.2t}$ has a continuous growth rate of -20%. The negative sign in the exponent tells us that R is decreasing instead of increasing.

Because $a = 5$ in all three formulas, all three functions cross the vertical axis at 5. Note that the graphs of these functions in Figure 4.30 have the same shape as the exponential functions in Section 4.3. They are concave up and have horizontal asymptotes of $y = 0$. (Note that $P \to 0$ and $Q \to 0$ as $t \to -\infty$, whereas $R \to 0$ as $t \to \infty$.)

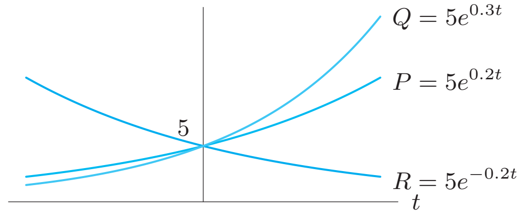


Figure 4.30: Exponential functions with different continuous growth rates

Example 2 A population increases from 7.3 million at a continuous rate of 2.2% per year. Write a formula for the population, and estimate graphically when the population reaches 10 million.

Solution We express the formula in base e since the continuous growth rate is given. If P is the population (in millions) in year t, then

$$P = 7.3e^{0.022t}.$$

See Figure 4.31. We see that $P = 10$ when $t \approx 14.3$. Thus, it takes about 14.3 years for the population to reach 10 million.

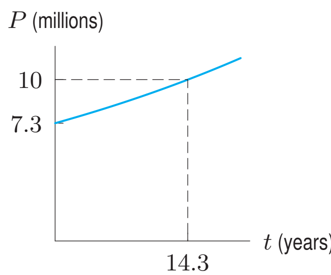


Figure 4.31: Estimating when the population reaches 10 million

Example 3 Caffeine leaves the body at a continuous rate of 17% per hour. How much caffeine is left in the body 8 hours after drinking a cup of coffee containing 100 mg of caffeine?

Solution If A is the amount of caffeine in the body t hours after drinking the coffee, then

$$A = 100e^{-0.17t}.$$

Note that the continuous growth rate is -17% since A is decreasing. After 8 hours, we have $A = 100e^{-0.17(8)} = 25.67$ mg.

The Difference Between Annual and Continuous Growth Rates

If $P = P_0(1.07)^t$, with t in years, we say that P is growing at an *annual* rate of 7%. If $P = P_0e^{0.07t}$, with t in years, we say that P is growing at a *continuous* rate of 7% per year. Since

$e^{0.07} = 1.0725\ldots$, we can rewrite $P_0e^{0.07t} = P_0(1.0725)^t$. In other words, a 7% continuous rate and a 7.25% annual rate generate the same increases in P. We say the two rates are equivalent.

We can check that $e^{0.0677} = 1.07\ldots$, so a 7% annual growth rate is equivalent to a 6.77% continuous growth rate. The continuous growth rate is always smaller than the equivalent annual rate.

The bank account example at the start of the previous section reminds us why a quantity growing at continuous rate of 7% per year increases faster than a quantity growing at an annual rate of 7%: In the continuous case, the interest earns more interest.

Connection: The Number e and Compound Interest

The number e has a surprising relationship with compound interest when the compounding period is made smaller and smaller.

If $\$1.00$ is invested in a bank account that pays 100% interest once a year, then, assuming no other deposits or withdrawals, after one year we have

$$\$1.00(1 + 100\%) = \$2.00.$$

Now we decrease the compounding period. Suppose $\$1.00$ is invested in a bank account that pays 100% nominal interest compounded n times a year; then

$$\text{Balance after one year} = \$1.00\left(1 + \frac{100\%}{n}\right)^n$$

As the frequency of compounding increases, the balance increases, because the interest earns more. How large can the balance grow? Table 4.13 shows the balance after one year as the interest is calculated more and more frequently. It appears that the balance approaches $e = 2.71828182\ldots$.

Assuming that the initial balance of $\$1$ is growing at a continuous rate of 100%, the balance after one year would be $1 \cdot e^1 = e$ dollars, so the surprising relationship arises from the fact that by increasing the compounding frequency we get the effect of continuous growth that we discussed earlier.

Table 4.13 *Balance after 1 year, 100% nominal interest, various compounding frequencies*

Frequency	Approximate balance
1 (annually)	\$2.00
2 (semi-annually)	\$2.25
4 (quarterly)	\$2.441406
12 (monthly)	\$2.613035
365 (daily)	\$2.714567
8760 (hourly)	\$2.718127
525,600 (each minute)	\$2.718279
31,536,000 (each second)	\$2.718282

In Example 2 of Section 4.4 we calculated the effective interest rates for two accounts with a 6% per year nominal interest rate, but different compounding periods. We see that the account with more frequent compounding earns a higher effective rate, though the increase is small. Compounding more and more frequently—every minute or every second or many times per second—increases the effective rate still further. However, there is again a limit to how much an account can earn by increasing the frequency of compounding.

Table 4.14 *Effect of increasing the frequency of compounding, 6% nominal interest*

Compounding frequency	Annual growth factor	Effective annual rate
Annually	1.0600000	6%
Monthly	1.0616778	6.16778%
Daily	1.0618313	6.18313%
Hourly	1.0618363	6.18363%
$\vdots$	$\vdots$	$\vdots$
Continuously	$e^{0.06} \approx 1.0618365$	6.18365%

Table 4.14 shows several compounding periods with their annual growth factors and effective annual rates. As the compounding periods become shorter, we discover that the growth factor approaches $e^{0.06}$. Using a calculator, we check that the final value for the annual growth factors in Table 4.14 is given by

$$e^{0.06} \approx 1.0618365,$$

If an account with a 6% nominal interest rate per year delivers an annual growth factor of $e^{0.06}$, we say that the interest has been *compounded continuously*. In general:

> If interest on an initial deposit of $\$P$ is *compounded continuously* at a nominal rate of r per year, the balance t years later can be calculated using the formula
>
> $$B = Pe^{rt}.$$
>
> For example, if the nominal rate is 6%, then $r = 0.06$.

It is important to realize that the functions $B = Pe^{0.06t}$ and $B = P(1.0618365)^t$ both give the balance in a bank account growing at a continuous rate of 6% per year. These formulas both represent the *same* exponential function—they just describe it in different ways.[18]

Example 4 In November 2005, the Wells Fargo Bank offered interest at a 2.323% continuous yearly rate.[19] Find the effective annual rate.

Solution Since $e^{0.02323} = 1.0235$, the effective annual rate is 2.35%. As expected, the effective annual rate is larger than the continuous yearly rate.

Example 5 Which is better: An account that pays 8% annual interest compounded quarterly or an account that pays 7.95% annual interest compounded continuously?

[18]Actually, this is not precisely true, because we rounded off when we found $b = 1.0618365$. However, we can find b to as many digits as we want, and to this extent the two formulas are the same.

[19]http://money.cnn.com/2005/11/30/debt/informa_rate, accessed November 30, 2005.

Solution The account that pays 8% interest compounded quarterly pays 2% interest 4 times a year. Thus, in one year the balance is

$$P(1.02)^4 \approx P(1.08243),$$

which means the effective annual rate is 8.243%.

The account that pays 7.95% interest compounded continuously has a year-end balance of

$$Pe^{0.0795} \approx P(1.08275),$$

so the effective annual rate is 8.275%. Thus, 7.95% compounded continuously pays more than 8% compounded quarterly.

Exercises and Problems for Section 4.5

Skill Refresher

In Exercises S1–S4, using a calculator approximate to 3 decimal places.

S1. $e^{0.07}$ **S2.** $10e^{-0.14}$ **S3.** $\dfrac{2}{\sqrt[3]{e}}$ **S4.** e^{3e}

In Exercises S5–S8, evaluate the functions for $t = 0$ and $t = 4$.

S5. $f(t) = 2.3e^{0.3t}$

S6. $g(t) = 4.2e^{-0.12t}$

S7. $h(t) = 153 + 8.6e^{0.43t}$

S8. $k(t) = 289 - 4.7e^{-0.0018t}$

In Exercises S9–S15, write the function in the form $f(t) = ae^{kt}$.

S9. $f(t) = \left(3e^{0.04t}\right)^3$ **S10.** $g(z) = 5e^{7z} \cdot e^{4z} \cdot 3e^{z}$

S11. $Q(t) = e^{7-3t}$ **S12.** $Q(t) = \sqrt{e^{3+6t}}$

S13. $m(x) = \dfrac{7e^{0.2x}}{\sqrt{3e^{x}}}$ **S14.** $P(t) = \left(2\sqrt[3]{e^{5t}}\right)^4$

S15. $H(r) = \dfrac{\left(e^{0.4r}\right)^2}{6e^{0.15r}}$

Exercises

1. Without a calculator, match the functions $y = e^x$, $y = 2e^x$, and $y = 3e^x$ to the graphs in Figure 4.32.

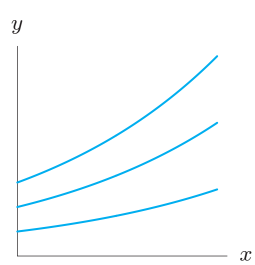


Figure 4.32

2. Without a calculator, match the functions $y = 2^x$, $y = 3^x$, and $y = e^x$ with the graphs in Figure 4.33.

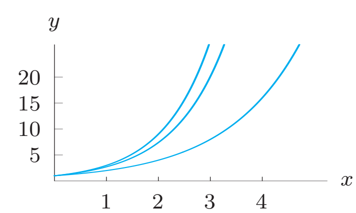


Figure 4.33

3. Without a calculator, match the functions (a)–(d) with the graphs (I)–(IV) in Figure 4.34.

(a) $e^{0.25t}$ **(b)** $(1.25)^t$ **(c)** $(1.2)^t$ **(d)** $e^{0.3t}$

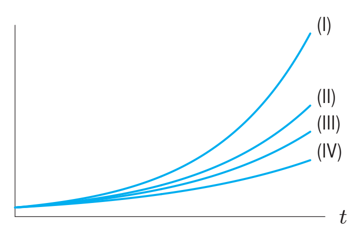


Figure 4.34

4. Without graphing on a calculator, match the functions (a)–(d) with the graphs (I)–(IV) in Figure 4.35.

(a) 1.5^x **(b)** $e^{0.45x}$ **(c)** $e^{0.47x}$ **(d)** $e^{0.5x}$

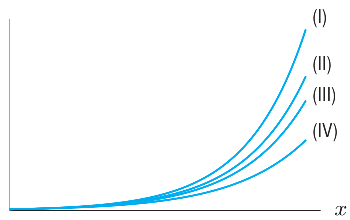


Figure 4.35

5. Without a calculator, match the functions $y = e^x$, $y = e^{-x}$, and $y = -e^x$ to the graphs in Figure 4.36.

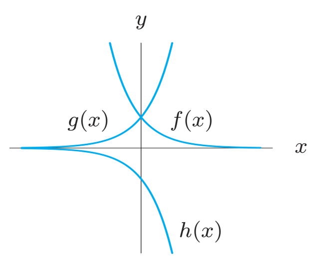


Figure 4.36

6. Without a calculator, match each formula to one of the graphs (I)–(IV) in Figure 4.37.

(a) $e^{-0.01t}$ **(b)** $e^{0.05t}$ **(c)** $e^{-0.10t}$ **(d)** $e^{0.20t}$

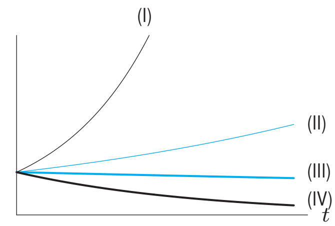


Figure 4.37

7. Without a calculator, match the functions (a)–(d) with the graphs (I)–(IV) in Figure 4.38.

(a) $y = e^x$ **(b)** $y = e^{-x}$

(c) $y = e^{-2x}$ **(d)** $y = e^{-3x}$

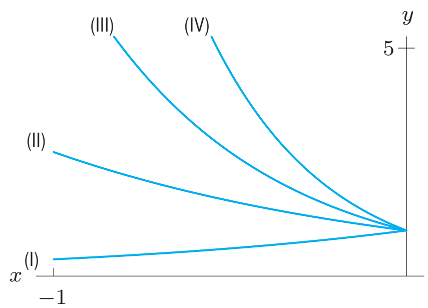


Figure 4.38

Problems

8. Without a calculator, arrange the following quantities in ascending order:

(a) $3^{2.2}, e^{2.2}, (\sqrt{2})^{2.2}$
(b) $3^{-2.2}, e^{-2.2}$

In Problems 9–12, find the limits.

9. $\lim_{x\to\infty} e^{-3x}$

10. $\lim_{t\to-\infty} 5e^{0.07t}$

11. $\lim_{t\to\infty} (2 - 3e^{-0.2t})$

12. $\lim_{t\to-\infty} 2e^{-0.1t+6}$

13. If $\lim_{t\to\infty} ae^{kt} = \infty$, what can you say about the values of a and k?

In Problems 14–19, a quantity Q is changing over time t.

(a) What is the quantity at time $t = 0$?

(b) Is the quantity increasing or decreasing over time?

(c) What is the percent per unit time growth or decay rate?

(d) Is the growth rate continuous?

14. $Q = 25e^{0.032t}$

15. $Q = 2.7(0.12)^t$

16. $Q = 158(1.137)^t$

17. $Q = 0.01e^{-0.2t}$

18. $Q = 50e^{1.05t}$

19. $Q = 2^t$

In Problems 20–22, an initial quantity Q_0 and a growth rate are given. Give a formula for quantity Q as a function of time t, and find the value of the quantity at $t = 10$, if we assume that the growth rate is:

(a) Not continuous **(b)** Continuous

20. $Q_0 = 100$; growth rate of 5% per unit time

21. $Q_0 = 8$; growth rate of 12% per unit time

22. $Q_0 = 500$; decay rate of -7% per unit time

23. A population of 3.2 million grows at a constant percentage rate.

(a) What is the population one century later if there is:
(i) An annual growth rate of 2%
(ii) A continuous growth rate of 2% per year

(b) Explain how you can tell which of the answers would be larger before doing the calculations.

24. The following formulas each describe the size of an animal population, P, in t years since the start of the study. Describe the growth of each population in words.

(a) $P = 200(1.028)^t$ **(b)** $P = 50e^{-0.17t}$
(c) $P = 1000(0.89)^t$ **(d)** $P = 600e^{0.20t}$
(e) $P = 2000 - 300t$ **(f)** $P = 600 + 50t$

25. A town has population 3000 people at year $t = 0$. Write a formula for the population, P, in year t if the town

(a) Grows by 200 people per year.
(b) Grows by 6% per year.
(c) Grows at a continuous rate of 6% per year.
(d) Shrinks by 50 people per year.
(e) Shrinks by 4% per year.
(f) Shrinks at a continuous rate of 4% per year.

26. A population is 25,000 in year $t = 0$ and grows at a continuous rate of 7.5% per year.

(a) Find a formula for $P(t)$, the population in year t.
(b) By what percent does the population increase each year? Why is this more than 7.5%?

27. A population grows from its initial level of 22,000 at a continuous growth rate of 7.1% per year.

(a) Find a formula for $P(t)$, the population in year t.
(b) By what percent does the population increase each year?

28. An investment of \$7000 earns interest at a continuous annual rate of 5.2%. What is the investment's value in 7 years?

29. Without making any calculations, describe in words what the following formulas tell you about the value of three different investments. Be specific. Note that units are in dollars and years.

(a) $V = 1000e^{0.115t}$ **(b)** $V = 1000 \cdot 2^{t/6}$
(c) $V = 1000(1.122)^t$

30. Calculate the amount of money in a bank account if \$2000 is deposited for 15 years at an interest rate of

(a) 5% annually
(b) 5% continuously per year

31. At time t in years, the value, V, of an investment of \$1000 is given by $V = 1000e^{0.02t}$. When is the investment worth \$3000?

32. How long does it take an investment to double if it grows according to the formula $V = 537e^{0.015t}$? Assume t is in years.

33. From time $t = 0$, with t in years, a \$1200 deposit in a bank account grows according to the formula

$$B = 1200e^{0.03t}.$$

(a) What is the balance in the account at the end of 100 years?
(b) When does the balance first go over \$50,000?

34. The same amount of money is deposited into two different bank accounts paying the same nominal rate, one compounded annually and the other compounded continuously. Which curve in Figure 4.39 corresponds to which compounding method? What is the initial deposit?

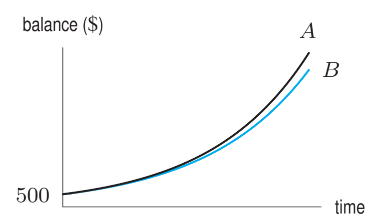


Figure 4.39

35. Find the effective annual yield and the continuous growth rate if $Q = 5500\,e^{0.19\,t}$.

36. If \$5000 is deposited in an account paying a nominal interest rate of 4% per year, how much is in the account 10 years later if interest is compounded

(a) Annually?
(b) Continuously?

37. Find the effective annual rate if \$1000 is deposited at 5% annual interest, compounded continuously.

38. An investment of \$500 earns interest at a 6.75% continuous annual rate. Give a formula $V = ab^t$ for the investment's value in year t, and state the values of a, b, and the annual growth rate r.

39. A bank account pays 6% annual interest. As the number of compounding periods increases, the effective interest rate earned also rises.

(a) Find the annual interest rate earned by the account if the interest is compounded:

(i) Quarterly (ii) Monthly
(iii) Weekly (iv) Daily

(b) Evaluate $e^{0.06}$, where $e = 2.71828\ldots$ Explain what your result tells you about the bank account.

40. Three different investments are given.

(a) Find the balance of each of the investments after a two-year period.
(b) Rank them from best to worst in terms of rate of return. Explain your reasoning.

- Investment A: \$875 deposited at 13.5% per year compounded daily for 2 years.
- Investment B: \$1000 deposited at 6.7% per year compounded continuously for 2 years.
- Investment C: \$1050 deposited at 4.5% per year compounded monthly for 2 years.

41. Rank the following three bank deposit options from best to worst.

- Bank A: 7% compounded daily
- Bank B: 7.1% compounded monthly
- Bank C: 7.05% compounded continuously

42. Which is better, an account paying 5.3% interest compounded continuously or an account paying 5.5% interest compounded annually? Justify your answer.

43. The GDP of Chile was 145.8 billion dollars in 2007 and was growing at a continuous rate of 5.1% per year.[20]

(a) Find a formula for G, the GDP of Chile in billion dollars, as a function of t, the number of years since 2007.
(b) By what percent does the GDP increase each year?
(c) Use your answer to part (b) to write a formula for G as a function of t using the form $G = ab^t$
(d) Graph your answers to part (a) and part (c) on the same axes. Explain what you see.

44. Which is larger after 5 years: an investment of \$1000 earning 5% per year compounded monthly or an investment of \$1100 earning 4% per year compounded continuously? Which is larger after 10 years? Justify your answers.

45. One bank pays 5% interest compounded annually and another bank pays 5% interest compounded continuously. Given a deposit of \$10,000, what is the difference in the balance between the two banks in 8 years?

46. An investment of \$1000 earns 8% interest compounded continuously. An investment of \$1500 earns 6% interest compounded annually. Figure 4.40 shows the balance of the two investments over time.

(a) Which graph goes with which investment?
(b) Use a graph of the two functions to estimate the time until the balances are equal.

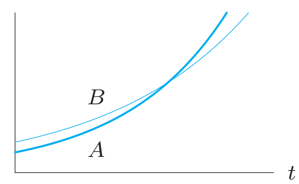


Figure 4.40

47. What can you say about the value of the constants a, k, b, l in Figure 4.41?

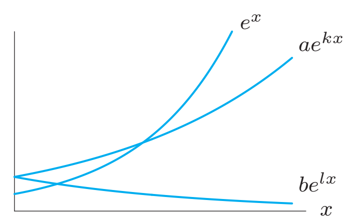


Figure 4.41

48. World poultry production was 94.7 million tons in the year 2009 and increasing at a continuous rate of 1.1% per year.[21] Assume that this growth rate continues.

(a) Write an exponential formula for world poultry production, P, in million tons, as a function of the number of years, t, since 2009.
(b) Use the formula to estimate world poultry production in the year 2015.
(c) Use a graph to estimate the year in which world poultry production goes over 110 million tons.

[20] www.nationmaster.com, accessed February 2010.
[21] www.sourcejuice.com/1266927/2009/10/19/world-poultry-production-growth-but-decline-trade, accessed Jan., 2010.

49. A radioactive substance decays at a continuous rate of 14% per year, and 50 mg of the substance is present in the year 2009.

(a) Write a formula for the amount present, A (in mg), t years after 2009.
(b) How much will be present in the year 2019?
(c) Estimate when the quantity drops below 5 mg.

50. The annual inflation rate, r, for a five-year period is given in Table 4.15.

(a) By what total percent did prices rise between the start of 2000 and the end of 2004?
(b) What is the average annual inflation rate for this time period?
(c) At the beginning of 2000, a shower curtain costs \$20. Make a prediction for the good's cost at the beginning of 2010, using the average inflation rate found in part (b).

Table 4.15

t	2000	2001	2002	2003	2004
r	3.4%	2.8%	1.6%	2.3%	2.7%

51. In the 1980s a northeastern bank experienced an unusual robbery. Each month an armored car delivered cash deposits from local branches to the main office, a trip requiring only one hour. One day, however, the delivery was six hours late. This delay turned out to be a scheme devised by an employee to defraud the bank. The armored car drivers had lent the money, a total of approximately \$200,000,000, to arms merchants, who then used it as collateral against the purchase of illegal weapons. The interest charged for this loan was 20% per year compounded continuously. How much was the fee for the six-hour period?

52. This problem explores the value of $(1+1/n)^n$ for integer values of n as n gets large.

(a) Use a calculator or computer to evaluate $(1+1/n)^n$, correct to seven decimal places, for $n = 1000$, 10,000, 100,000, and 1,000,000. Does $(1+1/n)^n$ appear to be an increasing or decreasing function of n?
(b) The limit of the sequence of values in part (a) is $e = 2.718281828\ldots$. What power of 10 is needed to give a value of e correct to 6 decimal places?
(c) What happens if you evaluate $(1 + 1/n)^n$ using much larger values of n? For example, try $n = 10^{16}$ on a calculator.

53. With more terms giving a better approximation, it can be shown that

$$e = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots.$$

(a) Use a calculator to sum the five terms shown.
(b) Find the sum of the first seven terms.
(c) Compare your sums with the calculator's displayed value for e (which you can find by entering $e\hat{}1$) and state the number of correct digits in the five- and seven-term sum.
(d) How many terms of the sum are needed in order to give a nine-decimal-digit approximation equal to the calculator's displayed value for e?

CHAPTER SUMMARY

- **Exponential Functions**
  Value of $f(t)$ changes at constant percent rate with respect to t.
- **General Formula for Exponential Functions**
  Exponential function: $f(t) = ab^t, \quad b > 0$.
  f increasing for $b > 1$, decreasing for $0 < b < 1$.
  Growth factor: $b = 1 + r$.
  Growth rate: r, percent change as a decimal.
- **Comparing Linear and Exponential Functions**
  An increasing exponential function eventually overtakes any linear function.
- **Graphs of Exponential Functions**
  Concavity; asymptotes; effect of parameters a and b. Solving exponential equations graphically; finding an exponential function for data.
- **The Number e**
  Continuous growth: $f(t) = ae^{kt}$.
  f is increasing for $k > 0$, decreasing for $k < 0$.
  Continuous growth rate: k.
- **Compound Interest**
  For compounding n times per year, balance,
  $$B = P\left(1 + \frac{r}{n}\right)^{nt}.$$
  For continuous compounding, $B = Pe^{kt}$.
  Nominal rate, r or k, versus effective rate earned over one year.
- **Horizontal Asymptotes and Limits to Infinity**

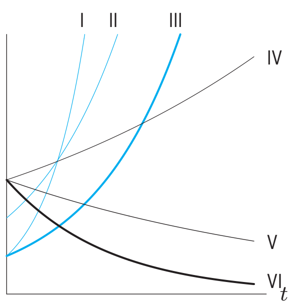
REVIEW EXERCISES AND PROBLEMS FOR CHAPTER FOUR

Exercises

In Exercises 1–6, you start with 500 items. How many do you have after the following change?

1. 10% increase
2. 100% increase
3. 1% decrease
4. 42% decrease
5. 42% increase followed by 42% decrease
6. 42% decrease followed by 42% increase
7. Find a formula for $P = f(t)$, the size of a population that begins in year $t = 0$ with 2200 members and decreases at a 3.2% annual rate.
8. Without a calculator or computer, match each exponential formula to one of the graphs I–VI.

   (a) $10(1.2)^t$ **(b)** $10(1.5)^t$ **(c)** $20(1.2)^t$
   (d) $30(0.85)^t$ **(e)** $30(0.95)^t$ **(f)** $30(1.05)^t$



9. A quantity increases from 10 to 12. By what percent has it increased? Now suppose that it had increased from 100 to 102. What is the percent increase in this case?
10. Determine whether the function whose values are in Table 4.16 could be exponential.

Table 4.16

x	1	2	4	5	8	9
$f(x)$	4096	1024	64	16	0.25	0.0625

In Problems 11–14, could the function be linear or exponential or is it neither? Write possible formulas for the linear or exponential functions.

11.

r	1	3	7	15	31
$p(r)$	13	19	31	55	103

12.

x	6	9	12	18	24
$q(x)$	100	110	121	146.41	177.16

13.

x	10	12	15	16	18
$f(x)$	1	2	4	8	16

14.

t	1	2	3	4	5
$g(t)$	512	256	128	64	32

Problems

15. In 2010, the cost of a train ticket from Boston to New York was \$95.[22] Assume that the price rises by 7% per year. Make a table showing the price of tickets each year until 2014.
16. Radioactive gallium-67 decays by 1.48% every hour; there are 100 milligrams initially.
    (a) Find a formula for the amount of gallium-67 remaining after t hours.
    (b) How many milligrams are left after 24 hours? After 1 week?
17. A bank pays interest at the nominal rate of 4.2% per year. What is the effective annual rate if compounding is:
    (a) Annual **(b)** Monthly **(c)** Continuous
18. Explain the difference between linear and exponential growth. That is, without writing down any formulas, describe how linear and exponential functions progress differently from one value to the next.

In Problems 19–24, find formulas for the exponential functions satisfying the given conditions.

19. $h(0) = 3$ and $h(1) = 15$
20. $f(3) = -3/8$ and $f(-2) = -12$

[22] http://www.amtrak.com, accessed May 23, 2010.

21. $g(1/2) = 4$ and $g(1/4) = 2\sqrt{2}$

22. $g(0) = 5$ and $g(-2) = 10$

23. $g(1.7) = 6$ and $g(2.5) = 4$

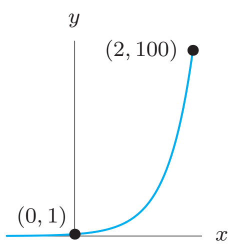
24. $f(1) = 4$ and $f(3) = d$

25. Suppose $f(-3) = 5/8$ and $f(2) = 20$. Find a formula for f assuming it is:

(a) Linear **(b)** Exponential

For Problems 26–31, find formulas for the exponential functions.

26.



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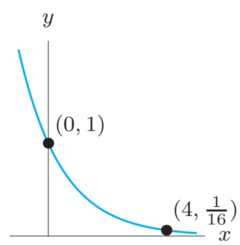

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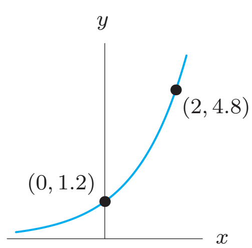

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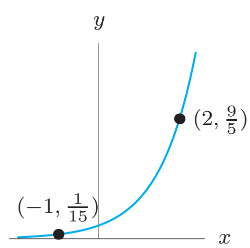

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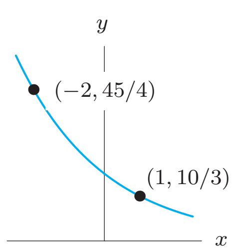
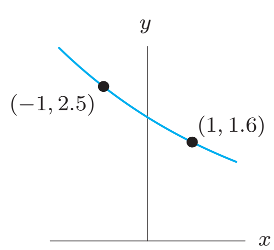


31.



32. Find possible formulas for the functions in Figure 4.42.

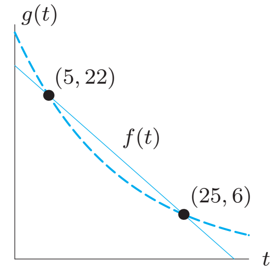


Figure 4.42

33. Let $P(t)$ be the population of a country, in millions, t years after 1990, with $P(7) = 3.21$ and $P(13) = 3.75$.

(a) Find a formula for $P(t)$ assuming it is linear. Describe in words the country's annual population growth given this assumption.

(b) Find a formula for $P(t)$ assuming it is exponential. Describe in words the country's annual population growth given this assumption.

34. A population has size 100 at time $t = 0$, with t in years.

(a) If the population grows by 10 people per year, find a formula for the population, P, at time t.

(b) If the population grows by 10% per year, find a formula for the population, P, at time t.

(c) Graph both functions on the same axes.

Find the limits in Problems 35–40.

35. $\lim\limits_{x\to\infty} 257(0.93)^x$

36. $\lim\limits_{t\to\infty} 5.3e^{-0.12t}$

37. $\lim\limits_{x\to-\infty} (15 - 5e^{3x})$

38. $\lim\limits_{t\to-\infty} (21(1.2)^t + 5.1)$

39. $\lim\limits_{x\to\infty} (7.2 - 2e^{3x})$

40. $\lim\limits_{x\to-\infty} (5e^{-7x} + 1.5)$

41. In 1940, there were about 10 brown tree snakes per square mile on the island of Guam, and in 2002, there were about 20,000 per square mile.[23] Find an exponential formula for the number, N, of brown tree snakes per square mile on Guam t years after 1940. What was, on average, the annual percent increase in the population during this period?

42. A 2010 Lexus LS costs \$64,680.[24] Assume that the car depreciates a total of 42% during its first 5 years.

(a) Suppose the depreciation is exponential. Find a formula for the value of the car at time t.

[23] *Science News*, Vol. 162, August 10, 2002, p. 85.
[24] http://www.lexus.com, accessed November, 2009.

(b) Suppose instead that the depreciation is linear. Find a formula for the value of the car at time t.
(c) If this were your car and you were trading it in after 4 years, which depreciation model would you prefer (exponential or linear)?

43. Table 4.17 gives the approximate number of cell phone subscribers, S, in the United States.[25]

(a) Explain how you know an exponential function fits the data. Find a formula for S in terms of t, the number of years since 2001.
(b) Interpret the growth rate in terms of cell phone subscribers.
(c) In 2008, there were 262.7 million subscribers. Does this fit the pattern?

Table 4.17

Year	2001	2002	2003	2004	2005	2006
Subscribers (m.)	128.4	140.8	158.7	182.1	207.9	233

44. Find the annual growth rates of a quantity which:

(a) Doubles in size every 7 years
(b) Triples in size every 11 years
(c) Grows by 3% per month
(d) Grows by 18% every 5 months

In Problems 45–48, graph $f(x)$, a function defined for all real numbers and satisfying the condition.

45. $f(x) \to 5$ as $x \to \infty$

46. $f(x) \to 3$ as $x \to \infty$ and $f(x) \to -2$ as $x \to -\infty$

47. $\lim_{x\to-\infty} f(x) = -4$

48. $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = -\infty$

49. The functions $f(x) = (\frac{1}{2})^x$ and $g(x) = 1/x$ are similar in that they both tend toward zero as x becomes large. Using a calculator, determine which function, f or g, approaches zero faster.

In Problems 50–51, graph the function to find horizontal asymptotes.

50. $f(x) = 8 - 2^x$

51. $f(x) = 3^{-x^2} + 2$

52. Without making any calculations, briefly describe the following investments, where t is in years. Be specific.

(a) $V = 2500e^{-0.0434t}$
(b) $V = 4000(1.005)^{12t}$
(c) $V = 8000 \cdot 2^{-t/14}$
(d) $V = 5000 + 250(t - 10)$

53. Without making any calculations, briefly describe in words what the following formulas tell you about the size of the animal populations they describe. Be specific. Note that t is in years.

(a) $P = 5200(1.118)^t$
(b) $P = 4600(1.01)^{12t}$
(c) $P = 3800\left(\frac{1}{2}\right)^{t/12}$
(d) $P = 8000e^{0.0778t}$
(e) $P = 1675 - 25(t - 30)$

54. Without a calculator, match each of the following formulas to one of the graphs in Figure 4.6.

(a) $y = 8.3e^{-t}$ **(b)** $y = 2.5e^t$ **(c)** $y = -4e^{-t}$

55. If t is in years, the formulas for dollar balances of two different bank accounts are:

$$f(t) = 1100(1.05)^t \quad \text{and} \quad g(t) = 1500e^{0.05t}.$$

(a) Describe in words the bank account modeled by f.
(b) Describe the account modeled by g. State the effective annual yield.

56. Accion is a non-profit microlending organization which makes small loans to entrepreneurs who do not qualify for bank loans.[26] A New York woman who sells clothes from a cart has the choice of a \$1000 loan from Accion to be repaid by \$1150 a year later and a \$1000 loan from a loan shark with an annual interest rate of 22%, compounded annually.

(a) What is the annual interest rate charged by Accion?
(b) To pay off the loan shark for a year's loan of \$1000, how much would the woman have to pay?
(c) Which loan is a better deal for the woman? Why?

57. Write $p(x) = \dfrac{7e^{6x} \cdot \sqrt{e} \cdot (2e^x)^{-1}}{10e^{4x}}$ in the form $p(x) = ae^x$. All constants should be expressed exactly.

In Problems 58–59, write each function in standard form. Note that one of them is linear and one exponential.

58. $r(v) = vj^w - 4tj^w + kvj^w$

59. $s(w) = vj^w - 4tj^w + kvj^w$

[25] www.infoplease.com/ipa/A0933563.html, accessed December 22, 2009.
[26] http://www.accionusa.org/, accessed November, 2005.

Problems 60–61 concern ISO A- and B-series paper, commonly used in many countries. The width in millimeters (mm) of a sheet of A_n paper in this series is given by the formula[27]

$$f(n) = 1000 \cdot 2^{-\frac{1}{4}-\frac{n}{2}}.$$

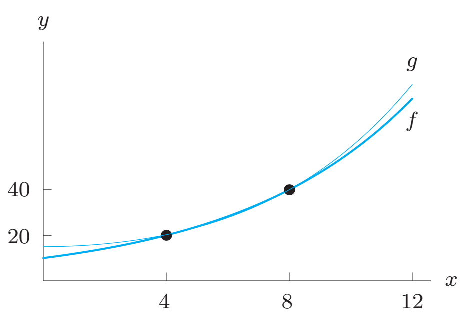
The width $g(n)$ of a sheet of B_n paper is the geometric mean of the widths of A_n paper and the next larger size, A_{n-1} paper. Since the geometric mean of two quantities p and q is the square root of their product, this means

$$g(n) = \sqrt{f(n) \cdot f(n-1)}.$$

60. Evaluate $g(1)$. What does your answer tell you about B-series paper?

61. Show that g is an exponential function by writing it in standard form.

62. The figure gives graphs of two functions, f and g. Explain why not both of these functions can be exponential.



63. Write a paragraph that compares the function $f(x) = a^x$, where $a > 1$, and $g(x) = b^x$, where $0 < b < 1$. Include graphs in your answer.

Problems 64–67 use Figure 4.43, which shows $f(x) = ab^x$ and $g(x) = cd^x$ on three different scales. Their point of intersection is marked.

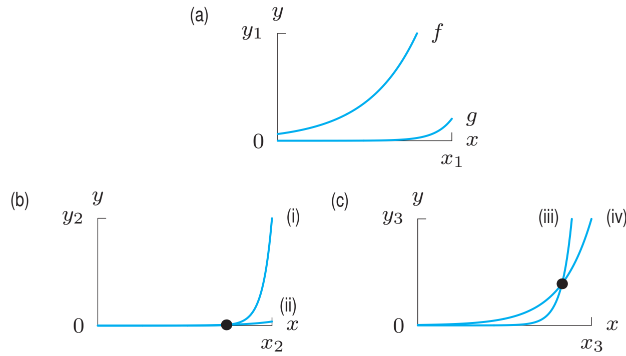


Figure 4.43

64. Which is larger, a or c?

65. Which is larger, b or d?

66. Rank in order from least to greatest: x_1, x_2, x_3.

67. Match f and g to the graphs labeled (i)–(iv) in (b) and (c).

68. On November 27, 1993, the *New York Times* reported that wildlife biologists have found a direct link between the increase in the human population in Florida and the decline of the local black bear population. From 1953 to 2009, the human population increased, on average, at a rate of 6% per year, while the black bear population decreased at a rate of 6% per year. In 1953 the black bear population was 11,000.

(a) The 2009 human population of Florida was 16 million. What was the human population in 1953?
(b) Find the black bear population for 2009.
(c) Had this trend continued,[28] when would the black bear population have numbered less than 100?

69. **(a)** Using a computer or calculator, graph $f(x) = 2^x$.
(b) Find the slope of the line tangent to f at $x = 0$ to an accuracy of two decimals. [Hint: Zoom in on the graph until it is indistinguishable from a line and estimate the slope using two points on the graph.]
(c) Find the slope of the line tangent to $g(x) = 3^x$ at $x = 0$ to an accuracy of two decimals.
(d) Find b (to two decimals) such that the line tangent to the function $h(x) = b^x$ at $x = 0$ has slope 1.

70. Sales of energy-efficient compact fluorescent lamps in China have been growing approximately exponentially. Table 4.18 shows the sales in millions.[29]

(a) Use a calculator or computer to find the exponential regression function for sales, S (in millions), as a function of the number of years, t, since 1994.

[27] Actual paper sizes are rounded to the nearest mm, so this formula is only approximate. See http://www.cl.cam.ac.uk/~mgk25/iso-paper.html, accessed February 24, 2008.
[28] Since 1993, the black bear population has in fact remained stable: www.myfwc.com/bear, accessed January 5, 2006.
[29] S. Nadel and Hong, "Market Data on Efficient Lighting," Right Light 6 Conference, Session 8, May, 2005.

(b) Plot the function with the data. Does it appear to fit the data well?
(c) What annual percent growth rate does the exponential model show?
(d) If this growth rate continues, what sales are predicted in the year 2010?

Table 4.18

Year	1994	1996	1998	2000	2002	2003
Sales (millions)	20	30	60	125	295	440

Find possible formulas for the functions in Problems 71–73.

71. V gives the value of an account that begins in year $t = 0$ with \$12,000 and earns 4.2% annual interest, compounded continuously.

72. The exponential function $p(t)$ given that $p(20) = 300$ and $p(50) = 40$.

73. The linear function $q(x)$ whose graph intersects the graph of $y = 5000e^{-x/40}$ at $x = 50$ and $x = 150$.

74. An investment worth $V = \$2500$ in year $t = 0$ earns 4.2% annual interest, compounded continuously. Find a formula for V in terms of t.

75. A population is represented by $P = 12{,}000e^{-0.122t}$. Give the values of a, k, b, and r, where $P = ae^{kt} = ab^t$. What do these values tell you about the population?

76. This problem uses a calculator or computer to explore graphically the value of $(1 + 1/x)^x$ as x gets large.

(a) Graph $y = (1 + 1/x)^x$ for $1 \le x \le 10$.
(b) Are the values of y in part (a) increasing or decreasing?
(c) Do the values of y in part (a) appear to approach a limiting value?
(d) Graph $y = (1 + 1/x)^x$, for $1 \le x \le 100$, and then for $1 \le x \le 1000$. Do the y values appear to approach a limiting value? If so, approximately what is it?
(e) Graph $y = (1 + 1/x)^x$ and $y = e$ on the same axes, for $1 \le x \le 10{,}000$. What does the graph suggest?
(f) By checking $x = 10{,}000$, $x = 20{,}000$, and so on, decide how large (as a multiple of 10,000) x should be to give a value of e correct to 4 decimal places.

77. Hong Kong shifted from British to Chinese rule in 1997. Figure 4.44 shows[30] the number of people who emigrated from Hong Kong during each of the years from 1980 to 1992.

(a) Find an exponential function that approximates the data.
(b) What does the model predict about the number of emigrants in 1997?
(c) Briefly explain why this model is or is not useful to predict emigration in the year 2010.

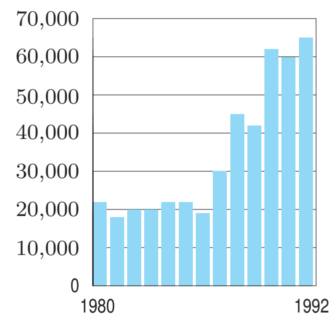


Figure 4.44

78. Before the AIDS epidemic, Botswana[31] had a rapidly growing population, as shown in Table 4.19. In 2005, the population started falling.

(a) Fit an exponential growth model, $P = ab^t$, to this data set, where P is the population in millions and t measures the years since 1975 in 5-year intervals—so $t = 1$ corresponds to 1980. Estimate a and b. Plot the data set and $P = ab^t$ on the same graph.
(b) Starting from 1975, how long does it take for the population of Botswana to double? When is the population of Botswana projected to exceed 214 million, the 1975 population of the US?

Table 4.19

Year	1975	1980	1985	1990
Population (millions)	0.755	0.901	1.078	1.285

Problems 79–80 use Figure 4.45, where t_0 is the t-coordinate of the point of intersection of the graphs. Describe what happens to t_0 if the following changes are made, assuming the other quantities remain the same.

[30] Adapted from the *New York Times*, July 5, 1995.
[31] N. Keyfitz, *World Population Growth and Aging* (Chicago: University of Chicago Press), 1990.

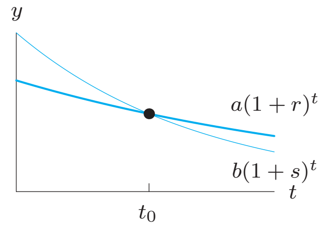


Figure 4.45

79. b is decreased

80. r is increased

81. It is a well-documented fact that the earning power of men is higher than that of women.[32] Table 4.20 gives the median income of year-round full-time workers in the US in dollars.

(a) Plot the data and connect the points.
(b) Let t be the year. Construct two functions of the form $W(t) = ae^{b(t-1950)}$, one each for the men's and women's earning power data.
(c) Graph the two functions from 1950 to 2000 and again from 2000 to 2080.
(d) Do the graphs in part (c) predict women's salaries will catch up with men's? If so, when?
(e) Comment on your predictions in part (d).

Table 4.20

Year	1950	1960	1970	1980	1990	2000
Female	953	1261	2237	4920	10,070	16,063
Male	2570	4080	6670	12,530	20,293	28,343

82. According to a letter to the *New York Times* on April 10, 1993, "... the probability of [a driver's] involvement in a single-car accident increases exponentially with increasing levels of blood alcohol." The letter goes on to state that when a driver's blood-alcohol content (BAC) is 0.15, the risk of such an accident is about 25 times greater than for a nondrinker.

(a) Let p_0 be a nondrinker's probability of being involved in a single-car accident. Let $f(x)$ be the probability of an accident for a driver whose blood alcohol level is x. Find a formula for $f(x)$. (This only makes sense for some values of x.)
(b) At the time of the letter, the legal definition of intoxication was a BAC of 0.1 or higher. According to your formula for $f(x)$, how many times more likely to be involved in a single-car accident was a driver at the legal limit than a nondrinker?
(c) Suppose that new legislation is proposed to change the definition of legal intoxication. The new definition states that a person is legally intoxicated when their likelihood of involvement in a single-car accident is three times that of a non-drinker. To what BAC would the new definition of legal intoxication correspond?

Pure water is not perfectly clear—it has a bluish cast—because it absorbs light differently at different wavelengths. The table gives values of the *absorption coefficient* $\mu(\lambda)$, in units of cm^{-1}, as a function of the wavelength λ in nanometers or nm.[33] If light of wavelength λ nanometers (nm) travels through l cm of water, the percent transmitted (without being absorbed) is given by

$$T_\lambda(l) = e^{-\mu(\lambda)\cdot l}.$$

Note that here, T_λ is the name of the function. Answer Problems 83–84.

color	λ	$\mu(\lambda)$	color	λ	$\mu(\lambda)$
violet	400	0.000066	yellow	570	0.000695
indigo	445	0.000075	orange	590	0.001351
blue	475	0.000114	red	650	0.003400
green	510	0.000325	—	—	—

83. What percent of red light will be transmitted after passing through 200 cm of water?

84. What percent of blue light will be transmitted over the same distance?

[32] *The World Almanac and Book of Facts 2006*, p. 84.
[33] Note that aside from absorption, optical scattering also plays a role. See http://omlc.ogi.edu/spectra/water/data/pope97.dat, http://eosweb.larc.nasa.gov/EDDOCS/Wavelengths_for_Colors.html, and http://en.wikipedia.org/wiki/Color_of_water, accessed February 29, 2008.

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–32 true or false? Give an explanation for your answer.

1. Exponential functions are functions that increase or decrease at a constant percent rate.

2. The independent variable in an exponential function is always found in the exponent.

3. If $y = 40(1.05)^t$ then y is an exponential function of t.

4. The following table shows a function that could be exponential.

x	1	2	4	5	6
y	1	2	4	7	11

5. If your salary, S, grows by 4% each year, then $S = S_0(0.04)^t$ where t is in years.

6. If $f(t) = 4(2)^t$ then $f(2) = 64$.

7. If $f(t) = 3(\frac{2}{5})^t$ then f is a decreasing function.

8. If $Q = f(t) = 1000(0.5)^t$ then when $Q = 125, t = 3$.

9. If $Q = f(t) = ab^t$ then a is the initial value of Q.

10. If we are given two data points, we can find a linear function and an exponential function that go through these points.

11. A population that has 1000 members and decreases at 10% per year can be modeled as $P = 1000(0.10)^t$.

12. A positive increasing exponential function always becomes larger than any increasing linear function in the long run.

13. A possible formula for an exponential function that passes through the point $(0, 1)$ and the point $(2, 10)$ is $y = 4.5t + 1$.

14. If a population increases by 50% each year, then in two years it increases by 100%.

15. In the formula $Q = ab^t$, the value of a tells us where the graph crosses the Q-axis.

16. In the formula $Q = ab^t$, if $a > 1$, the graph always rises as we read from left to right.

17. The symbol e represents a constant whose value is approximately 2.71828.

18. If $f(x) \to k$ as $x \to \infty$ we say that the line $y = k$ is a horizontal asymptote.

19. Exponential graphs are always concave up.

20. If there are 110 grams of a substance initially and its decay rate is 3% per minute, then the amount after t minutes is $Q = 110(0.03)^t$ grams.

21. If a population had 200 members at time zero and was growing at 4% per year, then the population size after t years can be expressed as $P = 200(1.04)^t$.

22. If $P = 5e^{0.2t}$, we say the continuous growth rate of the function is 2%.

23. If $P = 4e^{-0.90t}$, we say the continuous growth rate of the function is 10%.

24. If $Q = 3e^{0.2t}$, then when $t = 5$, $Q = 3$.

25. If $Q = Q_0e^{kt}$, with Q_0 positive and k negative, then Q is decreasing.

26. If an investment earns 5% compounded monthly, its effective rate will be more than 5%.

27. If a \$500 investment earns 6% per year, compounded quarterly, we can find the balance after three years by evaluating the formula $B = 500(1 + \frac{6}{4})^{3 \cdot 4}$.

28. If interest on a \$2000 investment is compounded continuously at 3% per year, the balance after five years is found by evaluating the formula $B = 2000e^{(0.03)(5)}$.

29. Investing \$10,000 for 20 years at 5% earns more if interest is compounded quarterly than if it is compounded annually.

30. Investing $\$P$ for T years always earns more if interest is compounded continuously than if it is compounded annually.

31. There is no limit to the amount a twenty-year \$10,000 investment at 5% interest can earn if the number of times the interest is compounded becomes greater and greater.

32. If you put \$1000 into an account that earns 5.5% compounded continuously, then it takes about 18 years for the investment to grow to \$2000.

SKILLS REFRESHER FOR CHAPTER 4: EXPONENTS

We list the definition and properties that are used to manipulate exponents.

Definition of Zero, Negative, and Fractional Exponents

If m and n are positive integers:[34]

- $a^0 = 1$
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{1/n} = \sqrt[n]{a}$, the n^{th} root of a
- $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Properties of Exponents

- $a^m \cdot a^n = a^{m+n}$ For example, $2^4 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^7$.
- $\dfrac{a^m}{a^n} = a^{m-n}, a \neq 0$ For example, $\dfrac{2^4}{2^3} = \dfrac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} = 2^1$.
- $(a^m)^n = a^{mn}$ For example, $(2^3)^2 = 2^3 \cdot 2^3 = 2^6$.
- $(ab)^n = a^n b^n$
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$

Be aware of the following notational conventions:

$$ab^n = a(b^n), \qquad \text{but } ab^n \neq (ab)^n,$$
$$-b^n = -(b^n), \qquad \text{but } -b^n \neq (-b)^n,$$
$$-ab^n = (-a)(b^n).$$

For example, $-2^4 = -(2^4) = -16$, but $(-2)^4 = (-2)(-2)(-2)(-2) = +16$. Also, be sure to realize that for $n \neq 1$,

$$(a+b)^n \neq a^n + b^n \qquad \text{Power of a sum} \neq \text{Sum of powers.}$$

Example 1 Evaluate without a calculator:

(a) $(27)^{2/3}$ (b) $(4)^{-3/2}$ (c) $8^{1/3} - 1^{1/3}$

Solution

(a) We have $(27)^{2/3} = \sqrt[3]{27^2} = \sqrt[3]{729} = 9$, or, equivalently, $(27)^{2/3} = \left(27^{1/3}\right)^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$.

(b) We have $(4)^{-3/2} = (2)^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$.

(c) We have $8^{1/3} - 1^{1/3} = 2 - 1 = 1$.

[34] We assume that the base is restricted to the values for which the power is defined.