VOTING METHODS

Topics in Contemporary Mathematics

MA 103

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Introduction: The Mathematics of Voting

Content:

- Preference Ballots and Preference Schedules
- Voting methods including,

    1). The Plurality Method

    2). The Borda Count Method

    3). The Plurality-with-Elimination Method (Instant Runoff Voting)

    4). The Method of Pairwise Comparisons
- Rankings

Assigning Labels to Candidates

Given that there is an election with four candidates:

Candidate's name	Label
Matt	A
Nick	B
Joe	C
Sam	D

We have assigned each candidate a label just for simplicity.

Preference Ballots

A ballot in which voters rank each of the candidates in order of preference is called a preference ballot.

A ballot in which ties are not allowed is called a linear ballot.

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st A	1st C	1st A	1st D	1st A	1st C	1st D	1st A	1st C	1st B
2nd B	2nd B	2nd B	2nd C	2nd B	2nd D	2nd C	2nd B	2nd B	2nd D
3rd C	3rd D	3rd C	3rd B	3rd C	3rd B	3rd B	3rd C	3rd D	3rd C
4th D	4th A	4th D	4th A	4th D	4th A	4th A	4th D	4th A	4th A

Preference Schedule

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st A	1st C	1st A	1st D	1st A	1st C	1st D	1st A	1st C	1st B
2nd B	2nd B	2nd B	2nd C	2nd B	2nd D	2nd C	2nd B	2nd B	2nd D
3rd C	3rd D	3rd C	3rd B	3rd C	3rd B	3rd B	3rd C	3rd D	3rd C
4th D	4th A	4th D	4th A	4th D	4th A	4th A	4th D	4th A	4th A

Figure: An illustration of 10 ballots using the linear preference ballot format for voting.

The ballot can also be organized using a preference schedule by grouping together identical ballots.

Table: Preference schedule for the election

Number of voters	4	2	2	1	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Transitivity and Elimination of Candidates

In a preference ballot, the voter's preferences are transitive - that is if a voter prefers candidate A over candidate B and prefers B over candidate C then the voter prefers A over C.

This means that if we want to see which candidate someone would vote for in a two person election all we need to check is which candidate is placed higher on the ballot.

Transitivity and Elimination of Candidates

The relative preferences of a voter is not affected by the elimination of one or more of the candidates.

Example:

Ballot
1^{st} D
2^{nd} B
3^{rd} C
4^{th} A

If candidate B drops out before the ballot is submitted, how would this voter rank the remaining candidate?

Answer →

Ballot
1^{st} D
2^{nd} C
3^{rd} A

Example:

Consider the following preference schedule for an election;

Number of voters	5	3	5	3	2	3
1st choice	A	A	C	D	D	B
2nd choice	B	D	E	C	C	E
3rd choice	C	B	D	B	B	A
4th choice	D	C	A	E	A	C
5th choice	E	E	B	A	E	D

(a) How many people voted in this election?

(b) How Many first-place votes are needed for a majority?

(c) If it came down to a choice between candidate A and D, which one would get more votes?

Example:

Consider the following preference schedule for an election;

Number of voters	5	3	5	3	2	3
1st choice	A	A	C	D	D	B
2nd choice	B	D	E	C	C	E
3rd choice	C	B	D	B	B	A
4th choice	D	C	A	E	A	C
5th choice	E	E	B	A	E	D

(a) How many people voted in this election? 5+3+5+3+2+3=21.

(b) How Many first-place votes are needed for a majority? At least 11 (i.e more than half)

(c) If it came down to a choice between candidate A and D, which one would get more votes? Candidate A is favored over D by 5 + 3 + 3 = 11 of the voters.

The Plurality Method

In plurality method, the candidate with the most first-place vote (called the plurality candidate) wins.

Thus in plurality method, voters don't need to rank the candidates.

The only information needed is the voters first choice.

The Plurality Method Cont’d

Plurality method is an extension of the principle of majority rule, which states that in an election between two candidates one with the majority (more than half) of votes wins.

The candidate with the majority of first-place votes is called the majority candidate.

With two candidates a plurality candidate is also a majority candidate.

With three or more candidates there is no guarantee that there is going to be a majority candidate.

Plurality Method cont'd

Example:

Candidate	Number of 1st choice votes
A	4
B	1
C	3
D	2

Under the plurality method, the winner is candidate A.

The majority criterion

> If a candidate has a majority of the first-place votes, then that candidate should be the winner of the election.

Does the plurality method satisfy the majority criterion?

Yes it does.

A principal weakness of the plurality method is that there is no head-to-head comparison.

Table: Preference schedule for an election

Number of voters	51	50	5
1st choice	A	B	C
2nd choice	B	E	B
3rd choice	C	D	E
4th choice	D	C	D
5th choice	E	A	A

Under the plurality method, the winner of the election is candidate A.

Notice that there are 55 voters that have candidate A as their last choice

By contrast, candidate B has 50 first-place votes and 56 second-place votes.

When candidate B is compared with either candidate D and E on a head-to-head basis it gets all 106 votes

Common sense tells us that candidate B is a far better choice to represent the wishes of the voters.

The Condorcet criterion

If a candidate is preferred by the voters over each of the other candidates in a head-to-head comparison, then that candidate should be the winner of the election.

The Borda Count Method

In this method each place on a ballot is assigned points.

If we have an election with N candidates we will give 1 point for last place, 2 points for second to last, . . . , and N points for first place. The candidate with the highest total number of points is the winner.

We will call such a candidate the Borda winner.

Borda Count Method

Example:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Table: The Borda points for the election

Number of voters	14	10	8	4	1
1st choice: 4 points	A: 56 pts	C: 40 pts	D: 32 pts	B: 16 pts	C: 4pts
2nd choice: 3 points	B: 42 pts	B: 30 pts	C: 24 pts	D: 12 pts	D: 3pts
3rd choice: 2 points	C: 28 pts	D: 20 pts	B: 16 pts	C: 8 pts	B: 2 pts
4th choice: 1 point	D: 14 pts	A: 10 pts	A: 8 pts	A: 4 pts	A: 1 pt

Candidate	Total number of points
A	56 +10 + 8+ 4 + 1 = 79
B	42 + 30 + 16 + 16 + 2 = 106
C	28 + 40 + 24 + 8 + 4 = 104
D	14 + 20 + 32 + 12 + 3 = 81

The Borda winner is candidate B.

Borda Count Method

What is wrong with this method? Let's look at another example.

Example:

Table: Preference schedule

Number of voters	6	2	3
1st choice	A	B	C
2nd choice	B	C	D
3rd choice	C	D	B
4th choice	D	A	A

Table: The Borda points for the election

Number of voters	6	2	3
1st choice: 4 points	A: 24 pts	C: 8 pts	D: 12 pts
2nd choice: 3 points	B: 18 pts	B: 6 pts	C: 9 pts
3rd choice: 2 points	C: 12 pts	D: 4 pts	B: 6 pts
4th choice: 1 point	D: 6 pts	A: 2 pts	A: 3 pts

Candidate	Total number of points
A	24 +2 + 3 = 29
B	18 + 6 + 6 = 30
C	12+ 8 + 9 = 29
D	6 + 4 + 12 = 22

The Borda winner is B.

Observe that Borda count method violates the majority criterion and the Condorcet criterion.

The Plurality-with-Elimination Method

The idea is to eliminate the candidates with the fewest first-place votes one at a time until one of them gets a majority;

The Plurality-with-Elimination Method

Round 1. Count the first-place votes for each. If a candidate has a majority of first-place votes, then that candidate is the winner. Otherwise, eliminate the candidate (or candidates if there is a tie) with the fewest first-place votes.

Round 2. Eliminate the candidate(s) from the preference schedule and recount the first-place votes. If a candidate has a majority of first-place votes, then declare that candidate the winner. Otherwise, eliminate the candidate(s) with the fewest first-place votes.

Round 3, 4, . . . Repeat the process, each time eliminating one or more candidates until there is a candidate with a majority of the first-place votes. That candidate is the winner of the election.

The Plurality-with-Elimination Method

Example:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Round 1:

Candidate	A	B	C	D
Number of first-place votes	14	4	11	8

B has the fewest first-place votes and is eliminated. B's votes will go to D.

Round 2:

Candidate	A	B	C	D
Number of first-place votes	14		11	12

In this round C has the fewest first-place votes and is eliminated. C's votes will go to D

Round 3:

Candidate	A	B	C	D
Number of first-place votes	14			23

The winner of the election, with 23 first-place votes, is D.

What's Wrong with the Plurality-with-Elimination Method?

Example:

The following straw poll was obtained before an election;

Number of voters	7	8	10	4
1st choice	A	B	C	A
2nd choice	B	C	A	C
3rd choice	C	A	B	B

Based on the straw poll, C is going to win (i.e by applying the plurality-with-elimination method).

If four voters who indicated A as their first-choice are disappointed and decide to switch their votes and vote for C first and A second in the election, then the official election result is as follows:

Table: Official election result

Number of voters	7	8	14
1st choice	A	B	C
2nd choice	B	C	A
3rd choice	C	A	B

Upon applying the plurality-with-elimination method to the official election result, B becomes the winner.

Observe that C lost the election because it got additional first-place votes in the official election.

The monotonicity criterion

If candidate X is a winner of an election and, in a reelection, the only changes in the ballots are changes that favor X (and only X), then X should remain a winner of the election.

Based on the last example we know that the plurality-with-elimination method violates the monotonicity criterion.

Plurality-with-elimination method also violates the Condorcet criterion

The Method of Pairwise Comparisons

In this method, every candidate is matched head-to-head against every other candidate.

Unlike the other three methods, this method satisfies all three of the fairness criteria, i.e the majority criterion, the Condorcet criterion and the monotonicity criterion.

Example

Using the method of pairwise comparisons, find the winner of the election given by the following preference schedule.

Number of voters	8	7	6	2	1
1st choice	A	D	D	C	E
2nd choice	B	B	B	A	A
3rd choice	C	A	E	B	D
4th choice	D	C	C	D	B
5th choice	E	E	A	E	C

Answer:

A versus B: 11 votes to 13 votes (B wins). B gets 1 point.
A versus C: 16 votes to 8 votes (A wins). A gets 1 point.
A versus D: 11 votes to 13 votes (D wins). D gets 1 point.
A versus E: 17 votes to 7 votes (A wins). A gets 1 point.
B versus C: 22 votes to 2 votes (B wins). B gets 1 point.
B versus D: 10 votes to 14 votes (D wins). D gets 1 point.
B versus E: 23 votes to 1 vote (B wins). B gets 1 point.
C versus D: 10 votes to 14 votes (D wins). D gets 1 point.
C versus E: 17 votes to 7 votes (C wins). C gets 1 point.
D versus E: 23 votes to 1 vote (D wins). D gets 1 point.

The final tally is: A = 2 points, B = 3 points, C = 1 points, D = 4 points , E = 0 points.
Thus, candidate D is the winner.

What's Wrong with the Method of Pairwise Comparison?

Example

Using the method of pairwise comparisons, find the winner of the election given by the following preference schedule.

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

Answer

A vs B: 7 to 15 votes, B gets 1 point
A vs C: 16 to 6 votes, A gets 1 point
A vs D: 13 to 9 votes A gets 1 point
A vs E: 18 to 4 votes A gets 1 point
B vs C: 10 to 12 votes C gets 1 point
B vs D: 11 to 11 votes B & D get 1/2 point
B vs E: 14 to 8 votes B gets 1 point
C vs D: 12 to 10 votes C gets 1 point
C vs E: 10 to 12 votes E gets 1 point
D vs E: 18 votes to 4 votes D gets 1 point

Example Cont'd

The final tally is A = 3 points, B = 2 1/2 points, C = 2 points, D = 1 1/2 points, E = 1 point.

Based on this, A is the winner.

Suppose C withdraws from the election and is therefore eliminated.

Then we have only four players and six pairwise comparisons to consider.

Example Cont'd

The results are as follows:

A vs B: 7 to 15 votes B gets 1 point

A vs D: 13 to 9 votes A gets 1 point

A vs E: 18 to 4 votes A gets 1 point

B vs D: 11 to 11 votes B & D get 1/2 point

B vs E: 14 to 8 votes B gets 1 point

D vs E: 18 to 4 votes D gets 1 point

In this new scenario: A =2 points, B = 2 1/2 points, D = 1 1/2 points, and E = 0 points and the winner is candidate B.

In other words, when C is not in the running, then the number-one pick is B.
How can the presence or absence of C in the candidate pool be relevant to this decision.

Thus the method violates a fourth fairness criterion known as the independence-of-irrelevant-alternatives criterion.

The Independence-of-Irrelevant-Alternatives Criterion

If candidate X is a winner of an election and in a recount one of the non-winning candidates withdraws or is disqualified, then X should still be a winner of the election.

How Many Pairwise Comparisons?

One practical difficulty with the method of pairwise comparisons is that as the number of candidates grows, the number of pairwise comparison grows even faster.

Example:

With 5 candidates we have a total of 10 pairwise comparisons.

With 10 candidates we have a total of 45 pairwise comparisons.

Ranking

Quite often it is important not only to know who wins the election but also to know who comes in second, third, and so on.

We need a voting method that gives us not just a winner but also a second place, a third place, and so on - in other words, a ranking of the candidates.

Extended Ranking Methods

Each of the four voting methods we discussed earlier has a natural extension that can be used to produce a ranking of the candidates.

They are:

1) Extended Plurality

2). Extended Borda Count

3). Extended Plurality-with-Elimination

4). Extended Pairwise Comparisons

Extended Plurality

Example:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

The number of first-place votes is:

A: 14 first-place votes, C: 11 first-place votes,
B: 4 first-place votes, D: 8 first-place votes.

Under the plurality method, A with the most first-place votes, is the winner.

If we extend the same logic to rank the remaining candidates, then C is second, D is third, and B is last.

Table: Ranking Under Extended Plurality

Place	1st	2nd	3rd	4th
Candidate	A	C	D	B
First-place votes	14	11	8	4

Extended Borda Count

Under the extended Borda count method candidates are ranked according to their Bordar point totals.

Example:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Borda point totals are:

A: 79 Borda points, C: 104 Borda points,

B: 106 Borda points, D: 81 Borda points.

Table: Ranking Under Extended Borda count

Place	1st	2nd	3rd	4th
Candidate	B	C	D	A
Borda points	106	104	81	79

Extended Plurality-with-Elimination

Under the extended plurality-with-elimination method we rank the candidates in reverse order of elimination.

Example:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

In the plurality-with-elimination method,
B is eliminated in the first-round,
C is eliminated in the second round,
A is eliminated in the third round,
D takes the first place.

Table: Ranking Under Extended Plurality-with-Elimination

Place	1st	2nd	3rd	4th
Candidate	D	A	C	B
Eliminated in round		3	2	1

Extended Pairwise Comparisons

Under the extended method of pairwise comparisons we rank the candidates according to the total number of points in their comparisons with the other candidates.

Example:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Table: Ranking Under Extended Pairwise Comparisons

Place	1st	2nd	3rd	4th
Candidate	C	B	D	A
Points	3	2	1	0

Recursive Ranking Method

The idea behind a recursive process is similar to that of a feedback loop: At each step of the process the output of the process determines the input to the next step of the process.

In the case of an election the process is to find the winner and remove the winner's name from the preference schedule, thus creating a new preference schedule. We then start the process all over again.

Recursive Plurality

Example:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Step 1: Choose the winner and remove.

Number of voters	14	10	8	4	1
1st choice	B	C	D	B	C
2nd choice	C	B	C	D	D
3rd choice	D	D	B	C	B

Step 2: Choose second place and remove.

Number of voters	14	10	8	4	1
1st choice	C	C	D	D	C
2nd choice	D	D	C	C	D

Step 3: Choose third and fourth places.

Table: Ranking Under Recursive Plurality

Place	1st	2nd	3rd	4th
Candidate	A	B	C	D

Recursive Borda Count Method

Example:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Step 1: Borda point totals are:

A: 79 Borda points, C: 104 Borda points,

B: 106 Borda points, D: 81 Borda points.

Step 2: Remove B.

Number of voters	14	10	8	4	1
1st choice	A	C	D	D	C
2nd choice	C	D	C	C	D
3rd choice	D	A	A	A	A

A, C, and D has 65, 85 and 72 Borda points respectively.

C is winner with 85 points. So second place goes to B.

Recursive Borda Count Method: Example Cont'd

Step 3: Remove C.

Number of voters	14	10	8	4	1
1st choice	A	D	D	D	D
2nd choice	D	A	A	A	A

A and D has 51 and 60 Borda points respectively. Thus third place goes to D, and last place goes to A.

Table: Ranking Under Recursive Borda Count

Place	1st	2nd	3rd	4th
Candidate	B	C	D	A

Recursive Plurality-with-Elimination

Example:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Step 1: Applying the plurality-with-elimination method produces D as the winner, so we remove D.

Number of voters	14	10	8	4	1
1st choice	A	C	C	B	C
2nd choice	B	B	B	C	B
3rd choice	C	A	A	A	A

Step 2: Applying the plurality-with-elimination method produces C as the winner. Thus second place goes to C.
Next remove C from the preference schedule.

Recursive Plurality-with-Elimination: Example Cont'd

Number of voters	14	10	8	4	1
1st choice	A	B	B	B	B
2nd choice	B	A	A	A	A

Step 3: (Choose third and fourth places.) B wins with 23 votes. Thus, third place goes to B and last place goes to A.

Table: Ranking Under Recursive Plurality-with-Elimination

Place	1st	2nd	3rd	4th
Candidate	D	C	B	A

Recursive Pairwise – Comparisons Method

Exercise:

Table: Preference schedule

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Rank the candidates in the election using the recursive pairwise-comparisons method.

Exercise:

Table: Preference schedule

Number of voters	10	7	5	5	4
1st choice	A	D	B	C	B
2nd choice	C	B	C	D	C
3rd choice	B	A	A	A	D
4th choice	D	C	D	B	A

1) Rank the candidates in the election using the recursive plurality method.
2) Rank the candidates in the election using the recursive Borda count method.
3) Rank the candidates in the election using the recursive pairwise-comparisons method.
4) Rank the candidates in the election using the recursive plurality-with-elimination method.

Exercise:

Table: Preference schedule

Number of voters	10	8	8	6	4	1
1st choice	B	D	C	A	D	A
2nd choice	A	A	B	C	B	B
3rd choice	C	B	A	D	C	D
4th choice	D	C	D	B	A	C

1) Rank the candidates using the extended plurality method.
2) Rank the candidates using the extended Borda count method.
3) Rank the candidates using the extended pairwise-comparisons method.
4) Rank the candidates using the extended plurality-with-elimination method.