Calculus Problem Book:
Form V and Form VI

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Contents

Preface

The following short booklet contains a collection of problems with solutions that roughly follow the topics typically covered in the calculus sequence for Form V and Form VI mathematics classes at Ethical Culture Fieldston School. The hope is to provide a book of problems of varying difficulty that will provide non-routine exercises as well as to record general themes and approaches to tackling these types of problems in general. Ideally these problems can be useful to all courses that cover these topics. Problems are roughly categorized by difficulty level and notated via *, **, or *** in increasing difficulty. We have adopted problems from memory, and from years of working with the material. The following references were consulted for problems as well as other considerations such as narrative structure in introducing various topics and thematic development of the material. [FGI96], [Pat04], [Pis14], [more later].

CHAPTER 1

Functions and Other Miscellaneous Content

1. Number Systems

Exercise 1   Sets and Types of Numbers
Set $A \subset B$ iff every element of A in also in B. Set $A \supset B$ iff every element of B is also in A and two sets are equal iff both $A \supset B$ and $A \subset B$.

2. Working with Inequalities

3. Polynomials

Exercise 2   Polynomials I
Let $f(x) = x^2 - 3$ and $g(x) = x^3 - 2x$. Both f and g are polynomials. You can form new polynomials by adding, subtracting, multiplying, and composing.

(1) Find $(f + g)(x)$
(2) Find $(f - 2g)(x)$
(3) Find $(f \cdot g)(x)$
(4) Find $f \circ g)(x)$
(5) Explain why $(f/g)(x)$ is not a polynomial.

solution 2

* *Exercise 3   Defining Polynomials*
A polynomial f has the general form $f(x) = a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+a_nx^n$, where $n \in \mathbb{Z}$ and $a_i \in \mathbb{R}$. Show that adding, subtracting, or multiplying two polynomials always results in a polynomial. Explain why this is not the case for division.

solution 3

Exercise 4   Polynomials II
A function f is even if and only if $f(x) = f(-x)$ for all x in its domain. A function is odd if and only if $f(-x) = -f(x)$ for all x in its domain.

(1) Give an example of an even polynomial.
(2) Give an example of an odd polynomial.

4. Trigonometric Functions

5. Miscellaneous

CHAPTER 2

Limits and Continuity

1. Intuitive Notion

2. Epsilon and Delta

* *Exercise 5 Dirichlet Function*

The Dirichlet function D is define as $D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$. Two sets of numbers are *not seperable* if it is impossible to find an open interval around an element from one set that contains only elements from that set and not the other. Rational numbers and irrational numbers are not separable. Use this to argue that $\lim_{x \to a} D(x)$ does not exist for any value of a. *solution* 5

3. Continuity

* *Exercise 6 A Bounded Function around Zero*

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|$ for all x in the domain of f. Prove that f is continuous at $x = 0$.

solution 6

* *Exercise 7 Some Points Don't Move That Much*

Let $f : [0, 1] \to [0, 1]$ be a function such that $|f(x)| \leq |x|$ for all x in the domain of f. Prove that f is continuous at $x = 0$.

solution 7

4. Intermediate Value Theorem

CHAPTER 3

The Derivative

1. Limit Definition and Properties of the Derivative

Exercise 8 Introducing a Discontinuity

Let $f(x) = |x|$. Use the limit definition of the derivative to show that $f'(x) = \dfrac{|x|}{x}$. Show using the definition of continuity how the derivative

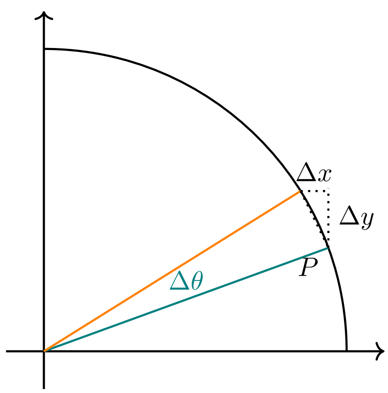
2. Tangent Line Problem

3. Higher Order Derivatives

4. Power Rule and others

* *Exercise 9 Unit Circle and Sine*

Let $P = (\cos\theta, \sin\theta)$. Use similar triangles to calculate an approximation for $\dfrac{\Delta y}{\Delta\theta}$ and explain how this shows what the derivative of $f(x) = \sin(x)$ is. Do the same for $g(x) = \cos(x)$ by looking at $\dfrac{\Delta x}{\Delta\theta}$.



solution 9

5. Implicit Differentiation

CHAPTER 4

Applications of the Derivative

CHAPTER 5

The Integral

CHAPTER 6

The Fundamental Theorem of Calculus

CHAPTER 7

Logarithms and Exponentials

CHAPTER 8

Methods of Integration

1. U Substitution

2. Integration by Parts

3. Method of Partial Fractions

4. Trigonometric Substitutions

5. Improper Integrals

CHAPTER 9

Differential Equations

CHAPTER 10

Applications of Integration

CHAPTER 11

Infinite Series

CHAPTER 12

Conics, Parametric Equations, Polar Coordinates

CHAPTER 13

Vectors and the Geometry of $\mathbb{R}^3$

CHAPTER 14

Vector Valued Functions

CHAPTER 15

The Partial Derivative and Function of Several Variables

CHAPTER 16

The Gradient and the Method of Lagrange Multipliers

CHAPTER 17

The Derivative of maps from $\mathbb{R}^n \to \mathbb{R}^m$

CHAPTER 18

Longer Problem Sets

Working with Parameters and the Derivative

In this exploration we will work with derivative to find tangent lines to a curve. We will also introduce working with parameters and see how this gives us a set of tools to solve more complex problems. We will end by using our skill with parameters to explore a little bit of the early history of Calculus by taking a look at Descartes' Method of Normals and Fermat's Method of Adequality.

First Let's Look at Tangent Lines to a Parabola.

(1) Let $f(x) = x^2 + 1$ and point $C(0, -2)$. Find the equation of the two lines that pass through C and are tangent to f.

(2) Let C vary its position along the y-axis. Let $C(0, k)$. Find the equations of the two lines that pass through C and are tangent to f. Notice your solutions will be in terms of the parameter k.

(3) Lastly, let's generalize and let C be any point in the plane $C(h, k)$. What are the equations of the two lines that pass through C and are tangent to f, in terms of the parameters h and k?

(4) Notice this gives you the equations for the lines through any point in the plane that are tangent to the given parabola. Demonstrate the ease with which you can find these lines that are tangent to the parabola, by find them for $P(5, 3)$ and $Q(3, -5)$.

(5) What do these equations tell you about the family of lines tangent to f? How is this different than just looking at $f'(x)$?

Descartes' Method Of Normals.

1. An Algorithm to Find Roots

2. Efficient Foraging

3. When is Venus Bright in the Sky?

4. Rainbows

5. Constructing the Demand Curve

6. The Shape of Bee Hive Cells

7. The Art Gallery Problem

8. Getting to π using Integration by Parts

9. Predicting Peak Oil

10. SIR: A Model for the Spread of Infectious Disease

11. Modeling Air Resistance

12. A Model for Combat

13. A Model for Relationships

CHAPTER 19

Answers

Exercise 2

blah blah

Exercise 3

The expression $A \subset B$ is read, the set A is contained in the set B. Whereas the expression $A \supset B$ is read the set A contains the set B.

(1) $\mathbb{Z} \supset \mathbb{N}$
(2) $\mathbb{R} \supset \mathbb{Q}$
(3) $\mathbb{N} \subset \mathbb{Q}$

Exercise 6

blah blah

Exercise 6

blah blah

Exercise 7

blah blah

Exercise 8

Exercise 9

blah blah

Bibliography

[FGI96] Fomin, Genkin, and Itenberg. *Mathematical Circles (Russian Experience).* Vol. 7. Mathematical World. American Mathematical Society, 1996.

[Pat04] Patrick. *Introduction to Counting and Probability.* 2nd. AoPS Inc., 2004.

[Pis14] Hossein Pishro-Nik. *Introduction to Probability, Statistics, and Random Processes.* Kappa Research, 2014.